





















$\phi_t^{N,i,i}, \phi_t^1 = \phi_t^{N,i,i+1}, \dots, \phi_t^j = \phi_t^{N,i,i+j}$  for large  $N$  and so on. It indicates that the Nash equilibrium converges to a limit independent of the boundary condition. Therefore, it is natural to study a similar game with infinite players. We conjecture that in general, as the number  $N$  of players goes to infinity, the limit of the Nash equilibrium of the finite-player, linear-quadratic stochastic differential game under the directed chain graphs gives us the Nash equilibrium of the infinite-player game, and moreover,  $\{\phi_t^j, i \in \mathbb{N}\}$  is the solution to the Riccati equation system of the infinite-player game. This will be discussed in section 3. Next, two particular examples are discussed to better illustrate the effect of the special boundary.

### 2.2.2 Boundary Condition 1: $X^N$ is attracted to 0

Here, we discuss the case when  $X^N$  is attracted to 0 which is also the common mean  $\mathbb{E}[\xi_i] = 0$  of the initial condition. It is equivalent to the general boundary condition (5)-(6) with  $m = 0$ . Without loss of generality, we can take constants:  $a_1 = \epsilon, c_1 = c$  and  $a_2 = c_2 = 0$ . Then the cost functional for player  $N$  is given by:

$$J^N(\alpha^N) := \mathbb{E} \left\{ \int_0^T \left( \frac{1}{2}(\alpha_t^N)^2 + \frac{\epsilon}{2}(X_t^N)^2 \right) dt + \frac{c}{2}(X_T^N)^2 \right\}.$$

The running cost function is defined by  $f^N(x, \alpha^N) = \frac{1}{2}(\alpha^N)^2 + \frac{\epsilon}{2}x^2$  and the terminal cost function is defined by  $g^N(x) = \frac{c}{2}x^2$ . Then,  $X^N$  is independent of the other players and is the solution of a self-controlled problem. We then make the same ansatz as (11) with  $\psi_t^{N,i} = 0$  for all  $i, 0 \leq t \leq T$ . As a result,  $Z^{i,i,k}$  and  $\phi^{N,i,j}$  are as (14) and (15), respectively. Consequently, we have the same conclusion: the functions  $\phi_t^{N,i,i+k} = \phi_t^{N,j,j+k}$  for all  $i, j \geq 1, k \geq 1$  and  $i+k < N, j+k < N$ ; and functions  $\phi_t^{N,i,j} (j < N)$  are independent of the boundary condition.

**Remark 1** (Shift invariance). *Notice that in this case  $\phi_t^{N,N,N}$  has the same solution as  $\phi_t^{N,i,i}$  ( $i < N$ ). Thus, in the ansatz (11), we can actually assume the solution  $\phi^{N,i,j}$  depends only on the difference  $j - i$  for  $j \geq i$ .*

### 2.2.3 Boundary Condition 2: $\alpha^N = 0$

We study the case when there is no control for the last player  $X^N$ , i.e. the dynamics of the state is given by:

$$dX_t^N = \sigma dW_t^N, \quad 0 \leq t \leq T; \quad X_0^N = \xi_N, \quad \mathbb{E}(\xi_N) = 0.$$

Player  $i$  chooses the strategy  $\alpha_t^i$  ( $i < N$ ) to minimize  $J^i$  given in (2) and the last player does not control, i.e.,  $\alpha^N \equiv 0$ . We make the same ansatz as in (11) with  $\psi_t^{N,i} = 0$  for all  $i$ . Then  $Z^{i,i,k}$  are the same as in (14) for  $i \leq N$ ,  $k \leq N$ ,  $0 \leq t \leq T$ , and  $\phi^{N,i,j}$ ,  $i \leq N-1$ ,  $j \leq N$  satisfy (15), however, for  $i = N$ ,  $\dot{\phi}_t^{N,N,N} = -\epsilon$  for  $0 \leq t \leq T$  with  $\phi_T^{N,N,N} = c$ . Thus, it is demonstrated again that the boundary condition does not affect the solutions  $\phi^{N,i,j}$  ( $j < N$ ), however, the functions  $\phi^{N,i,N}$  for all  $i$  are different from those in section 2.2.2, which depends on the boundary.

### 2.3 Closed-loop Nash Equilibrium

In search for closed-loop Nash equilibria, the controls are of the form  $\alpha^k(t, x)$ . When computing  $\partial_{x^j} H^i$  in the derivation of the BSDE for  $Y^{i,j}$ , one needs to pay attention in taking derivatives with respect to  $x^j$  in  $\hat{\alpha}^k$  for  $k \neq i$ , using  $\hat{\alpha}^k = -y^{k,k}$  and the ansatz (22). This is a tedious but straightforward computation which leads to the fact that the obtained closed-loop equilibrium coincides with the open-loop equilibrium identified before. We omit the details here as well as repeating this remark in the following sections. The only place where closed-loop and open-loop equilibria will be different is in section 5 when we will look at a mixture of directed chain and mean field interactions for finite player games, as it is already the case for pure mean field interaction studied in [4]. However, they will coincide again for the infinite-player games in section 5.2.

## 3 Infinite-Player Game Model

Motivated by the limit of the finite-player game discussed in section 2, we define the game with infinite players on a directed chain structure as shown in fig. 1. In remark 2 in section 3.1, we will see that the Hamiltonian only depends on finite players, which will make it well-defined. We assume that the state dynamics of all players are given by the stochastic differential equations of the form: for  $i \geq 1$ ,

$$dX_t^i = \alpha_t^i dt + \sigma dW_t^i, \quad 0 \leq t \leq T, \quad (17)$$

where  $(W_t^i)_{0 \leq t \leq T}$ ,  $i \geq 1$  are one-dimensional, independent Brownian motions on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$ . Similar to the setup for the finite-player games in section 2, we assume that the

drift coefficients  $\alpha^i$  are adapted to the filtration of the Brownian motions and satisfy  $\mathbb{E}[\int_0^T |\alpha_t^i|^2 dt] < \infty$ . We also assume that the diffusion coefficients are constant and identically denoted by  $\sigma > 0$ . The system starts at time  $t = 0$  from *i.i.d.* square-integrable random variables  $X_0^i = \xi_i$  with  $\mathbb{E}(\xi_i) = 0$ , independent of the Brownian motions. In this model, player  $i$  chooses its own strategy  $\alpha^i$  in order to minimize its expected cost function of the form:

$$J^i(\boldsymbol{\alpha}) := \mathbb{E} \left[ \int_0^T f^i(X_s, \alpha_s^i) ds + g^i(X_T) \right], \quad (18)$$

where the running and terminal cost functions  $f^i(x, \alpha^i)$ ,  $g^i(x)$  are the same as in (3).

### 3.1 Open-Loop Nash Equilibrium

We search for an open-loop Nash equilibrium of the infinite system (17) among admissible strategies  $\{\alpha_t^i, i = 1, 2, \dots, 0 \leq t \leq T\}$ .

**Definition** (Open-loop Nash equilibrium). *We call  $\boldsymbol{\alpha} := \{\alpha^i, i \geq 1\}$  an open-loop Nash equilibrium if for every player  $i$  and for any other  $(\mathcal{F}_t)_{0 \leq t \leq T}$  adapted and square-integrable control  $\beta$ . we have*

$$J^i(\alpha^1, \dots, \alpha^{i-1}, \beta, \alpha^{i+1}, \dots) \geq J^i(\alpha^1, \dots, \alpha^{i-1}, \alpha^i, \alpha^{i+1}, \dots). \quad (19)$$

First, we define the Hamiltonian  $H^i$  of the form:

$$\begin{aligned} H^i(x^1, x^2, \dots, y^{i,1}, \dots, y^{i,n_i}, \alpha^1, \alpha^2, \dots) \\ := \sum_{k=1}^{n_i} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2, \end{aligned} \quad (20)$$

assuming it is defined on real numbers  $x^i$ ,  $y^{i,k}$ ,  $\alpha^i$ ,  $i \geq 1$ ,  $k \geq 1$ , where only finitely many  $y^{i,k}$  are non-zero for every given  $i$ . Here,  $n_i$  is a finite number depending on  $i$  with  $n_i > i$ . This assumption is checked in remark 2 below. Thus, the Hamiltonian  $H^i$  is well defined for  $i \geq 1$ .

The adjoint processes  $Y_t^i = (Y_t^{i,j}; j \leq n_i)$  and  $Z_t^i = (Z_t^{i,j,k}; j \leq n_i, k \geq 1)$  for  $i \geq 1$  are the solutions of the following BSDEs for  $0 \leq t \leq T$ ,  $i \geq 1$ ,  $1 \leq j \leq n_i$ ,

$$\begin{cases} dY_t^{i,j} &= -\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j})dt + \sum_{k=1}^{\infty} Z_t^{i,j,k} dW_t^k, \\ Y_T^{i,j} &= \partial_{x^j} g_i(X_T) = c(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}). \end{cases} \quad (21)$$

**Remark 2.** For every  $j \neq i$  or  $i + 1$ ,  $dY_t^{i,j} = \sum_{k=1}^{\infty} Z_t^{i,j,k} dW_t^k$  and  $Y_T^{i,j} = 0$  implies  $Z_t^{i,j,k} = 0$  for all  $k$ . This observation is consistent with (10) in the finite player game case. Note also that  $Y^{i,i+1} = Y^{i,i}$ . There must be finitely many non-zero  $Y^{i,j}$ 's for every  $i$ . Hence, the Hamiltonian  $H^i$  in (20) can be rewritten as

$$H^i(x^1, x^2, \dots, y^{i,i}, y^{i,i+1}, \alpha^1, \alpha^2, \dots) = \alpha^i y^{i,i} + \alpha^{i+1} y^{i,i+1} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2.$$

Since each  $H^i$  is minimized at  $\hat{\alpha}^i = -y^{i,i}$ , inspired by the conclusion from the finite-player game (see also remark 1), we then make the ansatz of the form:

$$Y_t^{i,i} = \sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j, \quad 0 \leq t \leq T \quad (22)$$

for some deterministic scalar functions  $\phi_t^i$  satisfying the terminal conditions:  $\phi_T^0 = c, \phi_T^1 = -c, \phi_T^i = 0$  for  $i \geq 2$ . Substituting the ansatz (22), the optimal strategy  $\hat{\alpha}^i$  and the forward equation for  $X^i$  in (17) are

$$\hat{\alpha}_t^i = -Y_t^{i,i} = -\sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j, \quad dX_t^i = -\sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j dt + \sigma dW_t^i \quad (23)$$

for  $i \geq 1, 0 \leq t \leq T$ . Differentiating the ansatz (22), we obtain

$$dY_t^{i,i} = \sum_{\ell=0}^{\infty} \dot{\phi}_t^{\ell} X_t^{i+\ell} dt - \sum_{\ell=0}^{\infty} \left( \sum_{j=0}^{\ell} \phi_t^j \phi_t^{\ell-j} \right) X_t^{i+\ell} dt + \sigma \sum_{\ell=i}^{\infty} \phi_t^{\ell-i} dW_t^{\ell}. \quad (24)$$

Now by comparing the two Itô's decompositions (24) and (21) of  $Y_t^{i,i}$ , we obtain

$$Z_t^{i,i,k} = 0 \quad \text{for } k < i \quad \text{and} \quad Z_t^{i,i,k} = \sigma \phi_t^{k-i} \quad \text{for } k \geq i$$

and the system of Riccati equations: for  $i \geq 0, 0 \leq t \leq T$

$$\dot{\phi}_t^i = \sum_{j=0}^i \phi_t^j \phi_t^{i-j} + \epsilon(-\delta_{0,i} + \delta_{1,i}), \quad \phi_T^i = c(\delta_{0,i} - \delta_{1,i}). \quad (25)$$

The solutions to this Riccati system coincide with the limit of the solutions to the ODE system (15) of the N-player directed chain game in section 2, i.e.,  $\phi^i = \lim_{N \rightarrow \infty} \phi^{N,i,i+j}$  in the supremum norm. The Riccati system (25) is solvable.

**Proposition 2** (Catalan functions). *With  $c > 0$ ,  $\varepsilon > 0$ , the solution to (25) satisfies*

$$\sum_{j=0}^{\infty} \phi_t^j = 0, \quad \phi_t^0 = \frac{(-\varepsilon - c\sqrt{\varepsilon})e^{2\sqrt{\varepsilon}(T-t)} + \varepsilon - c\sqrt{\varepsilon}}{(-\sqrt{\varepsilon} - c)e^{2\sqrt{\varepsilon}(T-t)} - \sqrt{\varepsilon} + c} > 0, \quad (26)$$

for  $0 \leq t \leq T$ . Moreover, the functions  $\phi_t^k$ 's are obtained by a series expansion of the generating function  $S_t(z) = \sum_{k=0}^{\infty} z^k \cdot \phi_t^k$ ,  $z \leq 1$  of  $\{\phi_t^k\}$  given by  $S_t(1) \equiv 0$ , and

$$S_t(z) = \frac{(-\varepsilon(1-z) - c\sqrt{\varepsilon(1-z)}(1-z))e^{2\sqrt{\varepsilon(1-z)}(T-t)} + \varepsilon(1-z) - c\sqrt{\varepsilon(1-z)}(1-z)}{(-\sqrt{\varepsilon(1-z)} - c(1-z))e^{2\sqrt{\varepsilon(1-z)}(T-t)} - \sqrt{\varepsilon(1-z)} + c(1-z)} \quad (27)$$

for  $0 \leq t \leq T$ ,  $z < 1$ . We call  $\phi_t^k$ 's Catalan functions.

*Proof.* Given in appendix A.1. □

**Remark 3.** *It follows from (2) that the forward dynamics (23) can be written as:*

$$dX_t^i = - \sum_{j=0}^{\infty} \phi_t^j X_t^{i+j} dt + \sigma dW_t^i = \phi_t^0 \cdot \left( \sum_{j=1}^{\infty} \frac{-\phi_t^j}{\phi_t^0} X_t^{i+j} - X_t^i \right) dt + \sigma dW_t^i \quad (28)$$

for  $i \geq 1$ ,  $0 \leq t \leq T$ . This is a mean-reverting type process with  $\phi_t^0 > 0$ . We also see that this system is invariant under the shift of indices of individuals, i.e., the law of  $X^i$  is the same as that of  $X^1$  for every  $i$  and also  $X^i$  is independent of  $(W^1, \dots, W^{i-1})$ .

We end with a summary of this section on the infinite player game.

**Proposition 3.** *An open-loop Nash equilibrium for the infinite-player stochastic game with cost functionals (18) with (3) is determined by (28), where  $\{\phi^j, j \geq 0\}$  are the unique solution to the infinite system (25) of Riccati equations.*

## 4 Catalan Markov Chain

In order to simplify our analysis, we look at the stationary solution  $\{\phi^j, j \geq 0\}$  of (25) and the corresponding dynamics of (28), as  $T \rightarrow \infty$ . For simplicity,

we assume  $\epsilon = 1$ . By taking  $T \rightarrow \infty$ , we obtain the stationary long-time behavior satisfying  $\dot{\phi}^j = 0$  for all  $j$ . Then, (25) gives the recurrence relation for the stationary solution  $\{\phi^j, j \geq 0\}$ :

$$\phi^0 = 1 \quad \text{and} \quad \sum_{j=0}^n \phi^j \phi^{n-j} = \delta_{0,n} - \delta_{1,n}; \quad n \geq 0. \quad (29)$$

This is closely related to the recurrence relation of *Catalan* numbers. By using a moment generating function method as in appendix A.1, we get the stationary solutions

$$\phi^0 := 1, \quad \phi^1 := -\frac{1}{2}, \quad \phi^j := -\frac{(2j-3)!}{(j-2)!j!2^{2j-2}} \quad \text{for } j \geq 2. \quad (30)$$

We consider the continuous-time Markov chain  $M(\cdot)$  with state space  $\mathbb{N}$  and generator matrix  $\mathbf{Q} = (q_{i,j})$ , where  $(i,j)$  element  $q_{i,j}$  of  $\mathbf{Q}$  is given by  $q_{i,j} := p_{j-i} \cdot \mathbf{1}_{\{j \geq i\}}$  with  $p_k := -\phi^k$ ,  $k \geq 0$ ,  $i, j \geq 1$ . Note that the transition probabilities of the continuous-time Markov chain  $M(\cdot)$ , called a Catalan Markov chain, are  $p_{i,j}(t) = \mathbb{P}(M(t) = j | M(0) = i) = (e^{t\mathbf{Q}})_{i,j}$ ,  $i, j \geq 1$ ,  $t \geq 0$ . For simplicity, we assume  $\sigma = 1$ . Then with replacement of  $\phi_t^j$ ,  $t \geq 0$  by the stationary solution  $\phi^j$  in (30), the infinite particle system  $(X^i, i \geq 1)$  in (28) can be represented formally as a linear stochastic evolution equation:

$$d\mathbf{X}_t = \mathbf{Q} \mathbf{X}_t dt + d\mathbf{W}_t; \quad t \geq 0, \quad (31)$$

where  $\mathbf{X}_\cdot = (X^i, i \geq 1)$  with  $\mathbf{X}_0 = \mathbf{x}_0$  and  $\mathbf{W}_\cdot = (W^k, k \geq 1)$ . Its solution is

$$\mathbf{X}_t = e^{t\mathbf{Q}} \mathbf{x}_0 + \int_0^t e^{(t-s)\mathbf{Q}} d\mathbf{W}_s; \quad t \geq 0. \quad (32)$$

Without loss of generality, let us assume  $\mathbf{X}_0 = \mathbf{0}$ . Then,

$$\begin{aligned} X_t^i &= \int_0^t \sum_{j=i}^{\infty} p_{i,j}(t-s) dW_s^j = \int_0^t \sum_{j=i}^{\infty} \mathbb{P}(M(t-s) = j | M(0) = i) dW_s^j \\ &= \mathbb{E}^M \left[ \int_0^t \sum_{j=i}^{\infty} \mathbf{1}_{(M(t-s)=j)} dW_s^j | M(0) = i \right]; \quad t \geq 0, \end{aligned} \quad (33)$$

where the expectation is taken with respect to the probability induced by the Catalan Markov chain  $M(\cdot)$ , independent of the Brownian motions  $(W^j, j \in \mathbb{N}_0)$ . This is a Feynman–Kac representation formula for the infinite particle system  $\mathbf{X}$  in (33) associated with the continuous-time Markov chain  $M(\cdot)$  with the generator  $\mathbf{Q}$ . Interestingly, we may compute quite explicitly the corresponding transition probability.

**Proposition 4.** *With  $\mathbf{x}_0 = \mathbf{0}$ , the Gaussian process  $X_t^i$ ,  $i \geq 1$ ,  $t \geq 0$  in (33), corresponding to the Catalan Markov chain with the generator  $\mathbf{Q}$ , is*

$$X_t^i = \sum_{j=i}^{\infty} \int_0^t \frac{(t-s)^{2(j-i)}}{(j-i)!} \cdot \rho_{j-i}(-(t-s)^2) e^{-(t-s)} \cdot dW_s^j, \quad (34)$$

where  $W^j$ ,  $j \in \mathbb{N}$  are independent standard Brownian motions and  $\rho_i(\cdot)$  is defined by

$$\rho_i(x) := \frac{1}{2^i} \sum_{j=i}^{2i-1} \frac{(i-1)!}{(2j-2i)!!(2i-j-1)!} \cdot (-x)^{-\frac{j}{2}}, \quad (35)$$

for  $i \geq 1$ , and  $\rho_0(x) = 1$  for  $x \leq 0$ .

*Proof.* Given in appendix A.2. □

**Remark 4.** *To evaluate the asymptotic properties, it can be shown that*

$$\rho_j(-\nu^2) = \frac{1}{2^j \nu^j} \cdot \sqrt{\frac{2\nu}{\pi}} \cdot e^\nu \cdot K_{j-(1/2)}(\nu); \quad j \geq 1, \quad (36)$$

where  $K_n(x)$  is the modified Bessel function of the second kind defined by

$$K_n(x) = \int_0^\infty e^{-x \cosh t} \cosh(nt) dt (> 0); \quad n > -1, x > 0.$$

The asymptotic behaviors of  $X^i$  in (34) are derived rather straightforwardly from its explicit expression and are summarized in the following with proofs in appendix A.3 - appendix A.5.



## 4.1 Asymptotic Behavior of the Variances as $t \rightarrow \infty$

It follows from (34) that for  $t \geq 0$ , the variance of the Gaussian process  $X^i$ ,  $i \geq 1$ , in (33) is given by

$$\begin{aligned} \text{Var}(X_t^i) &= \text{Var}(X_t^1) = \sum_{j=0}^{\infty} \int_0^t \frac{(t-s)^{4j}}{(j!)^2} |\rho_j(-(t-s)^2)|^2 e^{-2(t-s)} ds \\ &= \sum_{k=1}^{\infty} \int_0^t \frac{2}{\pi} \frac{\nu^{2k+1}}{(k!)^2 4^k} (K_{k-(1/2)}(\nu))^2 d\nu + \frac{1 - e^{-2t}}{2} \end{aligned} \quad (37)$$

**Proposition 5.** *The asymptotic variance is  $\lim_{t \rightarrow \infty} \text{Var}(X_t^1) = 1/\sqrt{2}$ .*

## 4.2 Asymptotic Independence

The auto-covariance and cross-covariance are given respectively by: for  $s \leq t$

$$\begin{aligned} \mathbb{E}[X_s^1 X_t^1] &= \sum_{j=0}^{\infty} \int_0^s \frac{((t-s+u)u)^{j+1/2}}{\pi(j!)^2 2^{2j-1}} K_{j-1/2}(t-s+u) K_{j-1/2}(u) du, \\ \mathbb{E}[X_t^1 X_t^{j+1}] &= \sum_{\ell=0}^{\infty} \int_0^t \frac{s^{j+2\ell+1}}{\pi(j+\ell)! \ell! 2^{j+2\ell-1}} K_{j+\ell-1/2}(s) K_{\ell-1/2}(s) ds, \quad t \geq 0. \end{aligned} \quad (38)$$

The following propositions give two results about these covariances and the details of the proofs are given in Appendix appendix A.5.

**Proposition 6** (Ergodicity). *The auto-covariance  $\mathbb{E}[X_s^1 X_t^1]$  is positive. For every  $s > 0$ , as  $t \rightarrow \infty$ , it converges to 0, i.e., the process is ergodic.*

**Proposition 7** (Asymptotic behavior of the cross-covariance). *Similarly, for every  $k \geq 0$  and for any  $t > 0$  the cross-covariance  $\mathbb{E}[X_t^1 X_t^{k+1}]$  is positive, and  $0 < \lim_{t \rightarrow \infty} \mathbb{E}[X_t^1 X_t^{k+1}] \leq 1/\sqrt{2}$ . The asymptotic cross-covariance is positive and bounded above, which means the states are asymptotically dependent.*

## 5 Mixture of Directed Chain and Mean Field Interaction

In the spirit of the paper, we shall look at the game on a mixed system, including the directed chain interaction and the mean field interaction for finite players. This section repeats the same steps as before to analyze the mixed system game. The state dynamics of all the payers are of the form:  $dX_t^i = \alpha_t^i dt + \sigma dW_t^i$  for  $i \geq 1$  as in the previous sections.

### 5.1 Finite-Player Game

In this  $N$ -player model, player  $i$  chooses its own strategy  $\alpha^i$  in order to minimize its objective function of the mixed form:  $i \leq N$

$$J^i(\alpha^1, \dots, \alpha^N) := \mathbb{E} \left\{ \int_0^T f^i(X_t, \alpha^i) dt + g^i(X_T) \right\}, \quad (39)$$

where the running cost and terminal cost functions are defined by

$$f^i(x, \alpha^i) := \frac{1}{2}(\alpha^i)^2 + u \cdot \frac{\epsilon}{2}(x^{i+1} - x^i)^2 + (1 - u) \cdot \frac{\epsilon}{2}(\bar{x} - x^i)^2, \quad (40)$$

$$g^i(x) := u \cdot \frac{c}{2}(x^{i+1} - x^i)^2 + (1 - u) \cdot \frac{c}{2}(\bar{x} - x^i)^2, \quad (41)$$

for some positive constants  $\epsilon, c$  and a weight  $u \in [0, 1]$ . Here,  $\bar{x}$  is defined by  $\bar{x} = (x_1 + \dots + x_N)/N$  and we use the convention  $x^{N+1} \equiv 0$  for notational simplicity.

Each player optimizes the cost determined by the mixture of two criteria: distance from the neighbor in the directed chain with weight  $u$  and distance from the empirical mean  $\bar{X}$  with weight  $1 - u$ . The system is again completed by describing the behavior of player  $N$ . For simplicity, we consider the boundary condition of the system where  $X^N$  is attracted to 0 (cf. section 2.2.2). If  $u = 1$ , the system becomes the directed chain system discussed before. If  $u = 0$ , it becomes a mean-field system where each player is attracted towards the mean.

#### 5.1.1 Open-Loop Nash Equilibrium

As before, we find an open-loop Nash equilibrium of the system among strategies  $\{\alpha_t^i, i = 1, \dots, N\}$  by reiterating the previous procedure and solving the

corresponding BSDE system. It is described by

$$dX_t^i = \left[ -u \sum_{k=i}^N \phi_t^{N,i,k} X_t^k + (1-u)(\bar{X}_t - X_t^i)\theta_t \right] dt + \sigma dW_t^i, \quad (42)$$

where  $\bar{X} := (X^1 + \dots + X^N)/N$ ,  $\phi_t^{N,i,j}$  and  $\theta_t$  are determined by the ODE system

$$\begin{aligned} & u\dot{\phi}_t^{N,i,\ell} - u^2 \sum_{j=i}^{\ell} \phi_t^{N,i,j} \phi_t^{N,j,\ell} - 2u(1-u)\theta_t \phi_t^{N,i,\ell} \\ & + u\epsilon(\delta_{i,\ell} - \delta_{i+1,\ell}) + [(1-u)\dot{\theta}_t - (1-u)^2\theta_t^2]\delta_{i,\ell} \\ & + \frac{1}{N}(1-u) \left[ -\dot{\theta}_t + (1-u)\theta_t^2 + u\theta_t \left( \sum_{j=1}^{\ell} \phi_t^{N,j,\ell} + \sum_{k=i}^N \phi_t^{N,i,k} \right) \right] = 0, \quad (43) \\ & u\theta_t \sum_{k=i}^N \phi_t^{N,i,k} - \dot{\theta}_t + (1-u)\theta_t^2 - \epsilon \left( 1 - \frac{1}{N} \right) = 0 \end{aligned}$$

with terminal condition  $\phi_T^{N,i,\ell} = c(-\delta_{i,\ell} + \delta_{i+1,\ell})$ ,  $\theta_T = c(1 - N^{-1})$  for  $\ell \geq i$ ,  $0 \leq t \leq T$  and for fixed  $u \in (0, 1)$ .

The Hamiltonian is denoted by  $H^i(x^1, \dots, x^N, y^{i,1}, \dots, y^{i,N}, \alpha^1, \dots, \alpha^N) :=$

$$\sum_{k=1}^N \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + u \frac{\epsilon}{2}(x^{i+1} - x^i)^2 + (1-u) \frac{\epsilon}{2}(\bar{x} - x^i)^2,$$

for player  $i \leq N-1$ , and  $H^N(x^1, \dots, x^N, y^{i,1}, \dots, y^{i,N}, \alpha^1, \dots, \alpha^N) :=$

$$\sum_{k=1}^N \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + u \frac{\epsilon}{2}(x^N)^2 + (1-u) \frac{\epsilon}{2}(\bar{x} - x^i)^2$$

for player  $N$ . Minimizing the Hamiltonian with respect to  $\alpha^i$ ,

$$\partial_{\alpha^i} H^i = y^{i,i} + \alpha^i = 0 \quad \text{leading to the choice: } \hat{\alpha}^i = -y^{i,i}.$$

The adjoint processes  $Y_t^i = (Y_t^{i,j}; j = 1, \dots, N)$  and  $Z_t^i = (Z_t^{i,j,k}; j = 1, \dots, N, k = 1, \dots, N)$  for  $i = 1, \dots, N$  are defined as the solutions of

the backward stochastic differential equations (BSDEs):

$$i < N : \left\{ \begin{array}{l} dY_t^{i,j} = -\partial_{x^j} H^i(X_t, Y_t^i, \alpha_t) dt + \sum_{k=0}^N Z_t^{i,j,k} dW_t^k \\ = - \left[ u\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j}) \right. \\ \quad \left. + (1-u)\epsilon(\bar{X}_t - X_t^i) \left( \frac{1}{N} - \delta_{i,j} \right) \right] dt + \sum_{k=0}^N Z_t^{i,j,k} dW_t^k, \\ Y_T^{i,j} = \partial_{x^j} g_i(X_T) = uc(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}) \\ \quad + (1-u)c(\bar{X}_T - X_T^i) \left( \frac{1}{N} - \delta_{i,j} \right). \end{array} \right. \quad (44)$$

$$i = N : \left\{ \begin{array}{l} dY_t^{N,j} = - \left[ u\epsilon X_t^N \delta_{N,j} \right. \\ \quad \left. + (1-u)\epsilon(\bar{X}_t - X_t^N) \left( \frac{1}{N} - \delta_{N,j} \right) \right] dt + \sum_{k=0}^N Z_t^{N,j,k} dW_t^k, \\ Y_T^{N,j} = ucX_T^N \delta_{N,j} + (1-u)c(\bar{X}_T - X_T^N) \left( \frac{1}{N} - \delta_{N,j} \right). \end{array} \right. \quad (45)$$

When  $j = i$ , it becomes:

$$\left\{ \begin{array}{l} dY_t^{i,i} = \left[ u\epsilon(X_t^{i+1} - X_t^i) + (1-u)\epsilon(\bar{X}_t - X_t^i) \left( 1 - \frac{1}{N} \right) \right] dt \\ \quad + \sum_{k=0}^N Z_t^{i,i,k} dW_t^k, \\ Y_T^{i,i} = -uc(X_T^{i+1} - X_T^i) - (1-u)c(\bar{X}_T - X_T^i) \left( 1 - \frac{1}{N} \right), \quad i < N \end{array} \right. \quad (46)$$

$$\left\{ \begin{array}{l} dY_t^{N,N} = \left[ -u\epsilon X_t^N + (1-u)\epsilon(\bar{X}_t - X_t^N) \left( 1 - \frac{1}{N} \right) \right] dt \\ \quad + \sum_{k=0}^N Z_t^{N,N,k} dW_t^k, \\ Y_T^{N,N} = ucX_T^N - (1-u)c(\bar{X}_T - X_T^N) \left( 1 - \frac{1}{N} \right). \end{array} \right. \quad (47)$$

Considering the BSDE system and the initial condition, we then make the following ansatz with function parameters depending on  $N$ :

$$Y_t^{i,i} = u \sum_{j=i}^N \phi_t^{N,i,j} X_t^j - (1-u)(\bar{X}_t - X_t^i) \theta_t^N, \quad (48)$$

for some deterministic scalar functions  $\phi_t, \theta_t$  satisfying the terminal condition: when  $i < N$ ,  $\phi_T^{N,i,i} = c$ ,  $\phi_T^{N,i,i+1} = -c$ ,  $\phi_T^{N,i,j} = 0$  for  $N \geq j \geq i+2$ ;  $\phi_T^{N,N,N} = c$  and  $\theta_T^N = c(1 - \frac{1}{N})$ . For simplicity of notation, we denote  $\theta_t = \theta_t^N$ . Using the ansatz (48), the optimal strategy and forward equation become:

$$\left\{ \begin{array}{l} \hat{\alpha}^i = -Y_t^{i,i} = -u \sum_{j=i}^N \phi_t^{N,i,j} X_t^j + (1-u)(\bar{X}_t - X_t^i)\theta_t, \\ dX_t^j = \left[ -u \sum_{k=j}^N \phi_t^{N,j,k} X_t^k + (1-u)(\bar{X}_t - X_t^j)\theta_t \right] dt + \sigma dW_t^j. \end{array} \right. \quad (49)$$

By taking the averages, we obtain

$$\begin{aligned} d\bar{X}_t &= -u \cdot \frac{1}{N} \sum_{j=1}^N \sum_{k=j}^N \phi_t^{N,j,k} X_t^k dt + \sigma \cdot \frac{1}{N} \sum_{j=1}^N dW_t^j \\ &= -u \cdot \frac{1}{N} \sum_{k=1}^N \left( \sum_{j=1}^k \phi_t^{N,j,k} \right) X_t^k dt + \sigma \cdot \frac{1}{N} \sum_{k=1}^N dW_t^k \end{aligned}$$

and then

$$\begin{aligned} d(\bar{X}_t - X_t^i) &= -u \cdot \frac{1}{N} \sum_{k=1}^{i-1} \left( \sum_{j=1}^k \phi_t^{N,j,k} \right) X_t^k dt + u \sum_{k=i+1}^N \left( \phi_t^{N,i,k} - \frac{1}{N} \sum_{j=1}^k \phi_t^{N,j,k} \right) X_t^k dt \\ &\quad + \left( u\phi_t^{N,i,i} - u \frac{1}{N} \sum_{j=1}^i \phi_t^{N,j,i} + (1-u)\theta_t \right) X_t^i dt - (1-u)\bar{X}_t \theta_t dt \\ &\quad + \sigma \left( \frac{1}{N} \sum_{k=1}^N dW_t^k - u dW_t^i \right). \end{aligned} \quad (50)$$

Differentiating the ansatz eq. (48) and using eq. (50), we obtain

$$\begin{aligned} dY_t^{i,i} &= u \cdot \sum_{j=i}^N [X_t^j \dot{\phi}_t^{N,i,j} dt + \phi_t^{N,i,j} dX_t^j] \\ &\quad - (1-u) \cdot \left( \dot{\theta}_t (\bar{X}_t - X_t^i) dt + \theta_t d(\bar{X}_t - X_t^i) \right) \\ &\stackrel{\text{def}}{=} u \cdot \text{I} - (1-u) \cdot \text{II} \end{aligned} \quad (51)$$

For the first term, we have

$$\begin{aligned}
\text{I} &= \sum_{j=i}^N [X_t^j \dot{\phi}_t^{N,i,j} dt + \phi_t^{N,i,j} dX_t^j] \\
&= \sum_{k=i}^N (\dot{\phi}_t^{N,i,k} - u \sum_{j=i}^k \phi_t^{N,i,j} \phi_t^{N,j,k} - (1-u)\theta_t \phi_t^{N,i,k}) X_t^k dt \\
&\quad + (1-u)\theta_t \sum_{k=i}^N \phi_t^{N,i,k} \cdot \bar{X}_t dt + \sigma \sum_{k=i}^N \phi_t^{N,i,k} dW_t^k.
\end{aligned}$$

Then, for the second term, we have

$$\begin{aligned}
\text{II} &= \dot{\theta}_t (\bar{X}_t - X_t^i) dt + \theta_t d(\bar{X}_t - X_t^i) \tag{52} \\
&= -u\theta_t \frac{1}{N} \sum_{k=1}^{i-1} (\sum_{j=1}^k \phi_t^{N,j,k}) X_t^k dt + u\theta_t \sum_{k=i+1}^N (\phi_t^{N,i,k} - \frac{1}{N} \sum_{j=1}^k \phi_t^{N,j,k}) X_t^k dt \\
&\quad - [\dot{\theta}_t - u\theta_t (\phi_t^{N,i,i} - \frac{1}{N} \sum_{j=1}^i \phi_t^{N,j,i}) - (1-u)\theta_t^2] X_t^i dt \\
&\quad + (\dot{\theta}_t - (1-u)\theta_t^2) \bar{X}_t dt + \sigma (\frac{1}{N} \sum_{k=1}^N dW_t^k - dW_t^i).
\end{aligned}$$

Thus  $dY_t^{i,i} = u \cdot \text{I} - (1-u) \cdot \text{II}$  in (51) can be written as:

$$\begin{aligned}
&\sum_{k=1}^{i-1} \left( u(1-u)\theta_t \frac{1}{N} \sum_{j=1}^k \phi_t^{N,j,k} \right) X_t^k dt \\
&+ \sum_{k=i+1}^N \left[ u\dot{\phi}_t^{N,i,k} - u^2 \sum_{j=i}^k \phi_t^{N,i,j} \phi_t^{N,j,k} \right. \tag{53} \\
&\quad \left. - u(1-u)\theta_t \phi_t^{N,i,k} - u(1-u)\theta_t (\phi_t^{N,i,k} - \frac{1}{N} \sum_{j=1}^k \phi_t^{N,j,k}) \right] X_t^k dt \\
&+ \left[ u\dot{\phi}_t^{N,i,i} - u^2 (\phi_t^{N,i,i})^2 - 2u(1-u)\theta_t \phi_t^{N,i,i} \right. \\
&\quad \left. + (1-u)\dot{\theta}_t + u(1-u)\theta_t \frac{1}{N} \sum_{j=1}^i \phi_t^{N,j,i} - (1-u)^2 \theta_t^2 \right] X_t^i dt
\end{aligned}$$

$$\begin{aligned}
& + \left[ u(1-u)\theta_t \sum_{k=i}^N \phi_t^{N,i,k} - (1-u)\dot{\theta}_t + (1-u)^2\theta_t^2 \right] \bar{X}_t dt \\
& + u\sigma \sum_{k=i}^N \phi_t^{N,i,k} dW_t^k - (1-u)\sigma\theta_t \left( \frac{1}{N} \sum_{k=1}^N dW_t^k - dW_t^i \right).
\end{aligned}$$

Now we compare the two Itô's decompositions (46) and (53). The martingale terms give the processes  $Z_t^{i,j,k}$ :

$$\begin{aligned}
Z_t^{i,i,k} &= -(1-u)\sigma\theta_t \frac{1}{N} \quad \text{for } k < i, \\
Z_t^{i,i,i} &= u\sigma\phi_t^{N,i,i} + (1-u)\sigma\theta_t \left(1 - \frac{1}{N}\right) \quad \text{and} \quad Z_t^{i,i,k} = u\sigma\phi_t^{N,i,k} \quad \text{for } k > i.
\end{aligned}$$

And from the drift terms, we get the following system of ordinary differential equations for  $\phi^{N,i,k}$ :

when  $i < N$ ,  $k = i$

$$\begin{aligned}
& u\dot{\phi}_t^{N,i,i} - u^2(\phi_t^{N,i,i})^2 - 2u(1-u)\theta_t\phi_t^{N,i,i} \\
& + (1-u)\dot{\theta}_t \left(1 - \frac{1}{N}\right) - (1-u)^2\theta_t^2 \left(1 - \frac{1}{N}\right) \\
& + u(1-u)\theta_t \frac{1}{N} \left( \sum_{j=1}^i \phi_t^{N,j,i} + \sum_{\ell=i}^N \phi_t^{N,i,\ell} \right) \\
& = -u\epsilon - (1-u)\epsilon \left(1 - \frac{1}{N}\right)^2, \quad \phi_T^{N,i,i} = c,
\end{aligned} \tag{54}$$

for  $k = i + 1$

$$\begin{aligned}
& u\dot{\phi}_t^{N,i,i+1} - u^2(\phi_t^{N,i,i}\phi_t^{N,i,i+1} + \phi_t^{N,i,i+1}\phi_t^{N,i+1,i+1}) \\
& - 2u(1-u)\theta_t\phi_t^{N,i,i+1} - (1-u)\dot{\theta}_t \frac{1}{N} + (1-u)^2\theta_t^2 \frac{1}{N} \\
& + u(1-u)\theta_t \frac{1}{N} \left( \sum_{j=1}^{i+1} \phi_t^{N,j,i+1} + \sum_{\ell=i}^N \phi_t^{N,i,\ell} \right) \\
& = u\epsilon + (1-u)\epsilon \left(1 - \frac{1}{N}\right) \frac{1}{N}, \quad \phi_T^{N,i,i+1} = -c,
\end{aligned} \tag{55}$$

for  $k \geq i + 2$

$$\begin{aligned}
& u\dot{\phi}_t^{N,i,k} - u^2 \sum_{j=i}^l \dot{\phi}_t^{N,i,j} \dot{\phi}_t^{N,j,k} - 2u(1-u)\theta_t \dot{\phi}_t^{N,i,k} \\
& - (1-u)\dot{\theta}_t \frac{1}{N} + (1-u)^2 \theta_t^2 \frac{1}{N} + u(1-u)\theta_t \frac{1}{N} \left( \sum_{j=1}^k \dot{\phi}_t^{N,j,k} + \sum_{\ell=i}^N \dot{\phi}_t^{N,i,\ell} \right) \quad (56) \\
& = (1-u)\epsilon \left(1 - \frac{1}{N}\right) \frac{1}{N}, \quad \phi_T^{N,i,k} = 0,
\end{aligned}$$

and

$$\begin{aligned}
& u(1-u)\theta_t \sum_{k=i}^N \dot{\phi}_t^{N,i,k} - (1-u)\dot{\theta}_t + (1-u)^2 \theta_t^2 = (1-u)\epsilon \left(1 - \frac{1}{N}\right), \quad (57) \\
& \theta_T = c \left(1 - \frac{1}{N}\right);
\end{aligned}$$

When  $i = k = N$ ,

$$\begin{aligned}
& u\dot{\phi}_t^{N,N,N} - u^2 (\dot{\phi}_t^{N,N,N})^2 - 2u(1-u)\theta_t \dot{\phi}_t^{N,N,N} \\
& + (1-u)\dot{\theta}_t \left(1 - \frac{1}{N}\right) - (1-u)^2 \theta_t^2 \left(1 - \frac{1}{N}\right) \\
& + u(1-u)\theta_t \frac{1}{N} \left( \sum_{j=1}^N \dot{\phi}_t^{N,j,N} + \dot{\phi}_t^{N,N,N} \right) \quad (58) \\
& = -u\epsilon - (1-u)\epsilon \left(1 - \frac{1}{N}\right)^2, \quad \phi_T^{N,N,N} = c,
\end{aligned}$$

and

$$\begin{aligned}
& u(1-u)\theta_t \dot{\phi}_t^{N,N,N} - (1-u)\dot{\theta}_t + (1-u)^2 \theta_t^2 \\
& = (1-u)\epsilon \left(1 - \frac{1}{N}\right), \quad \theta_T = c \left(1 - \frac{1}{N}\right). \quad (59)
\end{aligned}$$

**Proposition 8.** *An open-loop Nash equilibrium for the finite-player stochastic game with cost functionals (39) with (40)-(41) is determined by (49), where  $\{\phi^{N,i,j}, \theta\}$  are the unique solution to the finite system (54)-(57) of Riccati equations with (58)-(59).*



When  $u = 1$ , the systems are exactly what we obtained for finite-player directed chain game in section 2. We have the similar conclusion that the boundary condition does not affect the functions  $\phi_t^{N,i,j}$  ( $j < N$ ) for all  $i < N$ . We can also compare the system with the system (63) we introduce later. Under suitable assumptions, the system may converge, as the number  $N$  of players goes to infinity.

## 5.2 Infinite-Player Game Model with Mean-Field Interaction

Motivated by section 5.1 and following section 3, we can define a game with infinite players on a mixed system, including the directed chain interaction and the mean field interaction. This section searches for an open-loop Nash equilibrium and repeats the same steps as before to analyse the infinite mixed system game. We have a more general Catalan Markov chain and table 1 below shows the asymptotic behaviors of the variances and covariances as  $t \rightarrow \infty$  for the process with different types of interactions. Comparing it with Table 1 in [6], we have similar conclusions except that our asymptotic variance of purely directed chain does not explode.

We assume the same drift and diffusion coefficients and the initial conditions for  $X^i$  as the finite-player game. By choosing  $\alpha_t^i$ , player  $i$  tries to minimize:

$$\begin{aligned}
J^i(\alpha^1, \alpha^2, \dots) := & \mathbb{E} \left\{ \int_0^T \left( \frac{1}{2}(\alpha_t^i)^2 + u \cdot \frac{\epsilon}{2}(X_t^{i+1} - X_t^i)^2 \right. \right. \\
& \left. \left. + (1-u) \cdot \frac{\epsilon}{2}(m_t - X_t^i)^2 \right) dt \right. \\
& \left. + u \cdot \frac{c}{2}(X_T^{i+1} - X_T^i)^2 + (1-u) \cdot \frac{c}{2}(m_T - X_T^i)^2 \right\}, \tag{60}
\end{aligned}$$

for some positive constants  $\epsilon$ ,  $c$  and  $u \in [0, 1]$ . Here, there is an issue in the choice of  $m_t$ . Intuitively, it should come from the finite-player mixed game described in section 5.1 as the limit of  $\bar{X}$  as  $N \rightarrow \infty$ . Combined with the fact that we had  $\mathbb{E}\{X_t^i\}$  independent of  $i$ , it is natural to set  $m_t = \mathbb{E}\{X_t^i\}$  and check afterwards that this mean value does not depend on  $i$  de facto after solving the fixed point step. Note that the case  $u = 0$  is very particular, and consists in solving the same mean field game problem for every  $i$ . The case  $u = 1$  has already been studied in section 3, and therefore, in what follows, we concentrate on the case  $u \in (0, 1)$ .

### 5.2.1 Open-Loop Nash Equilibrium

We search for a Nash equilibria of the system among strategies  $\{\alpha_t^i, i \geq 1\}$ . For  $i \geq 1$ , minimizing the Hamiltonian

$$\sum_{k=1}^{\infty} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + u \frac{\epsilon}{2}(x^{i+1} - x^i)^2 + (1-u) \frac{\epsilon}{2}(m_t - x^i)^2, \quad (61)$$

with respect to  $\alpha^i$ , and following closely to Carmona, Fouque, and Sun [4], we obtain

$$dX_t^i = \left( -u \sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j + (1-u)(m_t - X_t^i)\psi_t \right) dt + \sigma dW_t^i, \quad (62)$$

where  $\phi_t^k$  and  $\psi_t$  are determined by the following system of Riccati equation:  $k \geq 0$

$$\begin{aligned} \dot{\phi}_t^k &= u \sum_{j=0}^k \phi_t^j \phi_t^{k-j} + 2(1-u)\psi_t \phi_t^k + \epsilon(-\delta_{0,k} + \delta_{1,k}), \\ \phi_T^k &= c(\delta_{0,k} - \delta_{1,k}), \\ \dot{\psi}_t &= u\psi_t \sum_{j=0}^{\infty} \phi_t^j + (1-u)(\psi_t)^2 - \epsilon, \quad \psi_T = c. \end{aligned} \quad (63)$$

In appendix A.6 we show the following result which simplifies it considerably.

**Proposition 9.**  $\phi_t^j$  satisfies  $\sum_{j=0}^{\infty} \phi_t^j = 0$  for  $0 \leq t \leq T$  and thus,  $\psi_t$  is the unique solution to  $\dot{\psi}_t = (1-u)(\psi_t)^2 - \epsilon$ ,  $0 \leq t \leq T$ ,  $\psi_T = c$  in eq. (63).

**Proposition 10.** An open-loop Nash equilibrium for the infinite-player stochastic game with cost functionals (60) is determined by (62), where  $\{\phi_t^{N,i,j}, \psi_t\}$  are the unique solution to the infinite system (63) of Riccati equations.

Looking at the stationary solution in the limit ( $T \rightarrow \infty$ ), and without loss of generality assuming  $\epsilon = 1$  again, the recurrence relation can be solved by the method of moment generating function to obtain:

$$\begin{cases} \psi = \sqrt{\frac{1}{1-u}}, & \phi^0 = \frac{1-\sqrt{1-u}}{u}, \\ \phi^1 = -\frac{1}{2}, & \phi^k = -\frac{(2k-3)!}{(k-2)!k!2^{2k-2}} u^{k-1}, \quad \text{for } k \geq 2. \end{cases} \quad (64)$$

### 5.2.2 Catalan Markov Chain for the Mixed Model

As in section 4, letting  $T \rightarrow \infty$ , with replacement of  $\phi_t^j$  by the stationary solution  $\phi^j$  in (64),  $X^i$ ,  $i \geq 1$  defined by eq. (62) can be represented by the form of eq. (31) but now with a new matrix  $\mathbf{Q}^{(u)}$ , where its  $(i, j)$  element  $q_{i,j}$  is given by  $q_{i,i} = -1$ ,  $q_{i,j} = -u\phi^{j-i} \cdot \mathbf{1}_{\{j>i\}}$  with  $\phi^j$  in (64) for  $i, j \geq 1$ . Since  $u^2 \sum_{i=1}^{k-1} \phi^i \phi^{k-i} = -2u\phi^k$ , we have  $(\mathbf{Q}^{(u)})^2 = I - uB$  with  $B$  having 1's on the upper second diagonal and 0's elsewhere.

With a smooth function  $F(x) := \exp(-\sqrt{-x})$ ,  $x \in \mathbb{C}$ , the matrix exponential of  $\mathbf{Q}^{(u)}t$  can be written formally  $\exp(\mathbf{Q}^{(u)}t) = F((-I + uB)t^2)$ . With a slight modification of proof of proposition 4 in appendix A.2, we may compute it explicitly. We can summarize our finding on the limiting process as  $T \rightarrow \infty$ :

**Proposition 11.** *With  $\mathbf{x}_0 = \mathbf{0}$ , the Gaussian process  $X_t^i$ ,  $i \in \mathbb{N}$ ,  $t \geq 0$ , corresponding to the (Catalan) general Markov chain with generator  $\mathbf{Q}^{(u)}$ , is*

$$X_t^i = \sum_{j=i}^{\infty} \int_0^t \frac{u^{j-i}(t-s)^{2(j-i)}}{(j-i)!} \cdot \rho_{j-i}(-(t-s)^2) e^{-(t-s)} \cdot dW_s^j, \quad t \geq 0. \quad (65)$$

where  $\rho_j(\cdot)$  is defined in eq. (35).

### 5.2.3 Asymptotic Behavior

Table 1 exhibits the asymptotic behaviors of their variances and covariances as  $t \rightarrow \infty$ . The calculation is given in appendix A.7. We find that only when  $u = 0$  (i.e., pure mean field game), the asymptotic cross-covariance is zero, which means the states are asymptotically independent. Otherwise, they are dependent and their covariance is finite. Note in the purely nearest neighbor interaction studied in Detering, Fouque, and Ichiba [6], i.e., in the case  $u = 0$ , the variance is not stabilized as in our ‘‘Catalan’’ interaction equilibrium dynamics.

## 6 Periodic Directed Chain Game

We consider a stochastic game with finite players on a periodic ring structure in fig. 1 (b). Assume the dynamics of the states of the individual players are

$u$	Interaction Type	Asymptotic Variance	Asymptotic Independence between two players
$u = 0$	Purely mean-field	Stabilized	Independent
$u \in (0, 1)$	Mixed interaction	Stabilized	Dependent
$u = 1$	Purely directed chain	Stabilized	Dependent

Table 1: Asymptotic behaviors as  $t \rightarrow \infty$

given by  $N$  stochastic differential equations of the form:

$$dX_t^i = \alpha_t^i dt + \sigma dW_t^i, \quad i = 1, \dots, N, \quad 0 \leq t \leq T, \quad (66)$$

where  $(W_t^i)_{0 \leq t \leq T}$ ,  $i = 1, \dots, N$  are one-dimensional independent standard Brownian motions. The drift coefficient function, the diffusion coefficient and the initial conditions are assumed to be the same as those in section 2. In this model, player  $i$  chooses its own strategy  $\alpha^i$  in order to minimize its objective function of the form:

$$J^i(\alpha^1, \dots, \alpha^N) := \mathbb{E} \left\{ \int_0^T \left[ \frac{1}{2}(\alpha_t^i)^2 + \frac{\epsilon}{2}(X_t^{i+1} - X_t^i)^2 \right] dt + \frac{c}{2}(X_T^{i+1} - X_T^i)^2 \right\}, \quad (67)$$

with some constants  $\epsilon > 0$ ,  $c \geq 0$ , where we define  $X^{N+1} = X^1$  or more generally,  $X_t^{i+j} = X_t^{(i+j) \bmod N}$ , because of the periodic ring structure, for  $i, j = 1, \dots, N$ .

## 6.1 Construction of an Open-Loop Nash Equilibrium

We construct an open-loop Nash equilibria of the system among strategies  $\{\alpha_t^i, 1 \leq i \leq N\}$  by the Pontryagin stochastic maximum principle. The Hamiltonian  $H^i$  for player  $i$  is

$$\sum_{k=1}^N \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2. \quad (68)$$

The adjoint processes  $Y_t^i = (Y_t^{i,j}; j = 1, \dots, N)$  and  $Z_t^i = (Z_t^{i,j,k}; j, k = 1, \dots, N)$  for  $i = 1, \dots, N$  are defined as the solutions of the system of

BSDEs:

$$\begin{cases} dY_t^{i,j} &= -\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j})dt + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k, \\ Y_T^{i,j} &= c(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}). \end{cases} \quad (69)$$

Based on the sufficiency part of the Pontryagin stochastic maximum principle, we can get an open-loop Nash equilibrium by minimizing the Hamiltonian  $H^i$  with respect to  $\alpha^i$ :  $\partial_{\alpha^i} H^i = y^{i,i} + \alpha^i = 0$  leadings to the choice  $\hat{\alpha}^i = -y^{i,i}$  for each  $i$ . With this choice for the controls  $\alpha^i$ 's, the forward equation (66) becomes coupled with the backward equation (69). We make the ansatz: for  $t \geq 0$ ,  $i \geq 1$ ,

$$Y_t^{i,i} = \sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j}, \quad (70)$$

for some deterministic scalar functions  $\phi_t^{N,j}$  satisfying the terminal conditions:  $\phi_T^{N,0} = c$ ,  $\phi_T^{N,1} = -c$ ,  $\phi_T^{N,k} = 0$  for  $k \geq 2$  and  $X_t^{i+j} := X_t^{(i+j) \bmod N}$ . Using the ansatz, the optimal strategy  $\hat{\alpha}^i$  and the forward equation (66) become:

$$\hat{\alpha}^i = -Y_t^{i,i} = -\sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j}, \quad dX_t^i = -\sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j} dt + \sigma dW_t^i. \quad (71)$$

Using the equations (71), we can differentiate the ansatz (70): for  $1 \leq i \leq N$ ,  $t \leq 0$ ,

$$dY_t^{i,i} = \sum_{j=0}^{N-1} X_t^{i+j} \dot{\phi}_t^{N,j} dt - \sum_{j=0}^{N-1} \phi_t^{N,j} \sum_{k=0}^{N-1} \phi_t^{N,k} X_t^{i+j+k} dt + \sum_{j=0}^{N-1} \sigma \phi_t^{N,j} dW_t^{i+j} \quad (72)$$

Now comparing the two Itô's decompositions (72) and (69) of  $Y_t^{i,i}$ , we obtain

$$\begin{aligned} \dot{\phi}_t^{N,i} &= \sum_{j=0}^{N-1} \phi_t^{N,j} \dot{\phi}_t^{N,N+i-j} - \epsilon(\delta_{0,i} - \delta_{1,i}), \quad \phi^{N,i}(T) = c(\delta_{0,i} - \delta_{1,i}), \\ Z_t^{i,i,k} &= \sigma \phi_t^{N,N+k-i}, \quad 0 \leq t \leq T, \quad 1 \leq i \leq N \end{aligned} \quad (73)$$

(cf. (25)), where we use a convention  $\phi_t^{N,N+i-j} = \phi_t^{N,i-j}$ , if  $i - j \geq 0$ .

It can be written as a matrix Riccati equation:

$$\dot{\Phi}^N(t) = \Phi^N(t)\Phi^N(t) - \mathbf{E}, \quad \Phi^N(T) := \mathbf{C}, \quad (74)$$

where we denote by  $\Phi^N(\cdot)$  the  $N \times N$  matrix-valued function with  $(i, j)$  element being  $\phi_t^{N,N+i-j}$  for  $1 \leq i, j \leq N$  by using the convention  $\phi_t^{N,N+i-j} = \phi_t^{N,i-j}$ , if  $i \geq j$ , and by  $\mathbf{C} = (c_{i,j})$  and  $\mathbf{E} = (\epsilon_{i,j})$ , respectively the  $N \times N$  matrices with  $(i, j)$  element being  $c_{i,j} := c(\delta_{i,j} - \delta_{i,j+1})$  and  $\epsilon_{i,j} := \epsilon(\delta_{i,j} - \delta_{i,j+1})$  for  $1 \leq i, j \leq N$ .

**Proposition 12.** *The solution  $\phi_t^{N,k}$ ,  $k = 1, \dots, N$  to the system of Riccati equations (73) satisfies the relation  $\sum_{k=0}^{N-1} \phi_t^{N,k} = 0$  for  $0 \leq t \leq T$ .*

*Proof.* Given in appendix A.8. □

**Proposition 13.** *An open-loop Nash equilibrium for the linear quadratic stochastic game with cost functionals (67) for the  $N$  players with a periodic boundary condition  $X^{N+1} = X^1$  is given by (71), where  $\{\phi_t^{N,k}\}$  are uniquely determined by the system (73) of Riccati equations.*

With finite  $N$ , these equations (73) are not easy to solve explicitly. If we let  $N \rightarrow \infty$ , we expect that the system converges to the Riccati system of the infinite-player game studied in section 3.

**Conjecture 1.** *The limit of each element in  $\Phi^N(\cdot)$  in (74) exists as  $N \rightarrow \infty$ , i.e.,  $\Phi^N(t) \rightarrow \Phi^\infty(t)$  and the limit  $\Phi^\infty(t)$  is an infinite dimensional, lower triangular, matrix-valued function of  $t \geq 0$  given by  $\Phi^\infty(t) = (\Phi^{\infty,i,j}(t))_{i,j \in \mathbb{N}}$  with  $\Phi^{\infty,i,j}(\cdot) \equiv 0$  if  $i < j$ ;  $\Phi^{\infty,i,j}(\cdot) \equiv \phi^{i-j}$  if  $i \geq j$ , where  $\phi^k$ 's are given in proposition 2.*

**Remark 5.** *Proving this conjecture is equivalent to verify that for every  $j$ ,  $\sum_{k=j+1}^{N-1} \phi_t^{N,k} \phi_t^{N,N+j-k} \rightarrow 0$  as  $N \rightarrow \infty$ . As of now, this remains an open problem.*

Our conjecture is substantiated by numerical evidences presented below.

## 6.2 Numerical Results

By the methods given in [9], we get the numerical solution of the matrix Riccati equation (74). Taking  $\epsilon = 2$ ,  $c = 1$ ,  $T = 10$  (large terminal time), fig. 2

(a)-(b) shows the behaviors of the  $\phi$  functions defined by the system (25) for  $N = 4$  and  $N = 100$ . They converge to the constant solutions of the infinite game given in section 4, except in the tail close to maturity as  $T$  is large but not infinite. This result confirms our conjecture stated in the previous section. fig. 2 (c) shows the behavior of the function  $\sum_{k=1}^{N-1} \phi_t^{N,k} \phi_t^{N,N-k}$  for different values of  $N = 5, 10, 20, 50, 100$ . As we can see, the sum converges to 0 when  $N$  becomes larger, which supports the statement with  $j = 0$  in remark 5. Although these numerical results give us strong evidence and confidence that the conjecture is true, a mathematical proof is still needed and it is part of our ongoing research.

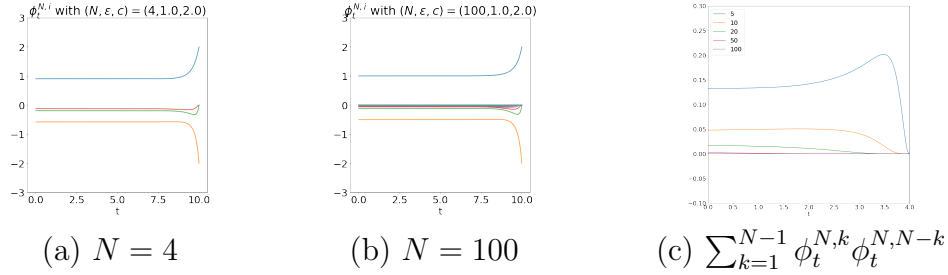


Figure 2: The blue line (top) is  $\phi_t^{N,0}$  and the orange line (bottom) is  $\phi_t^{N,1}$  in (a)-(b).  $\sum_{k=1}^{N-1} \phi_t^{N,k} \phi_t^{N,N-k}$  for different values of  $N = 5, 10, 20, 50, 100$  from top to bottom in (c).

## 7 Directed Infinite Tree Game

We describe a stochastic game on a directed tree structure with  $N \geq 2$  generations first. Starting with one player in the root node denoted by  $(1, 1)$  in the first generation, recursively each parent has a fixed, common number of descendants, denoted by  $d \geq 1$ , and there are  $d^{n-1}$  players in the  $n$ -th generation for  $n \geq 1$ . For  $1 \leq n \leq N, 1 \leq k \leq d^{n-1}$ ,  $X^{n,k}$  represents the state of the  $k$ -th individual of the  $n$ -th generation, and its direct descendants in the  $(n+1)$ st generation are labelled as  $\{X^{n+1,(k-1)d+1}, X^{n+1,(k-1)d+2}, \dots, X^{n+1,kd}\}$ . We consider the stochastic differential game of players in the  $N$  generations and then we generalize to a stochastic differential game in a directed infinite tree by considering its limit as  $N \rightarrow \infty$ . The network is shown in fig. 3.

We assume the dynamics of the states of the players are given by the

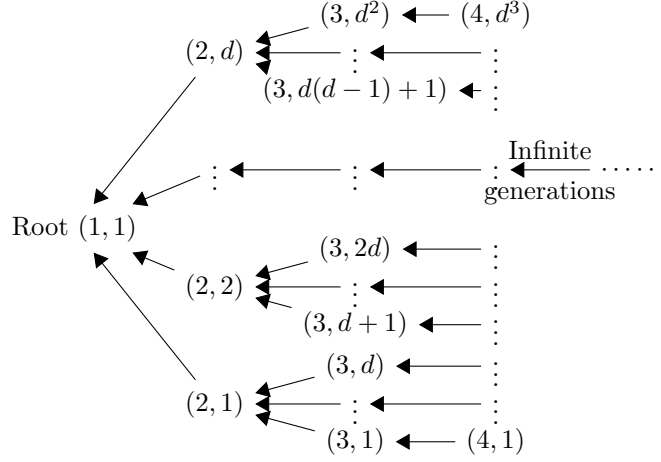


Figure 3: Directed Tree Network

stochastic differential equations of the form:

$$dX_t^{n,k} = \alpha_t^{n,k} dt + \sigma dW_t^{n,k}, \quad 0 \leq t \leq T, \quad (75)$$

where  $(W_t^{n,k})_{0 \leq t \leq T}$ ,  $1 \leq n \leq N$ ,  $1 \leq k \leq d^{n-1}$  are one-dimensional independent standard Brownian motions. Similarly, we assume that the diffusion is one-dimensional and the diffusion coefficients are constant and identical denoted by  $\sigma > 0$ . The drift coefficients  $\alpha^{n,k}$ 's are adapted to the filtration of the Brownian motions and satisfy  $\mathbb{E}[\int_0^T |\alpha_t^{n,k}|^2 dt] < \infty$ . The system starts at time  $t = 0$  from *i.i.d.* square-integrable random variables  $X_0^{n,k} = \xi_{n,k}$  independent of the Brownian motions and, without loss of generality, we assume  $\mathbb{E}(\xi_{n,k}) = 0$  for every pair of  $(n, k)$ .

In this model, among the first  $N-1$  generations, each player  $(n, k)$  chooses its own strategy  $\alpha^{n,k}$  in order to minimize its objective function of the form: for  $1 \leq n < N$

$$\begin{aligned} & J^{n,k}(\alpha^{m,\ell}; 1 \leq m \leq N, 1 \leq \ell \leq d^m) \\ & := \mathbb{E} \left\{ \int_0^T \left( \frac{1}{2} (\alpha_t^{n,k})^2 + \frac{\epsilon}{2} (\bar{X}_t^{n+1,k} - X_t^{n,k})^2 \right) dt + \frac{c}{2} (\bar{X}_T^{n+1,k} - X_T^{n,k})^2 \right\}, \end{aligned} \quad (76)$$

where  $\bar{X}_t^{n,k} := \sum_{i=(k-1)d+1}^{kd} X_t^{n,i}/d$  for some constants  $\epsilon > 0$  and  $c \geq 0$  and for  $n, k \geq 1$ . The running cost and the terminal cost functions are defined by



$f^{n,k}(x, \alpha^{n,k}) := \frac{1}{2}(\alpha^{n,k})^2 + \frac{\epsilon}{2}(\bar{x}^{n+1,k} - x^{n,k})^2$  and  $g^{n,k}(x) := \frac{c}{2}(\bar{x}^{n+1,k} - x^{n,k})^2$ , respectively with  $\bar{x}^{n,k} := \sum_{i=(k-1)d+1}^{kd} x^{n,i}/d$ . For simplicity, the behaviours of the  $N$ -th generation are described by the boundary condition where all the players  $\{X^{N,k}, 1 \leq k \leq d^{N-1}\}$  are attracted to 0. The cost functional for player  $(N, k)$  is given by:

$$J^{N,k}(\alpha^{N,k}) := \mathbb{E} \left\{ \int_0^T \left( \frac{1}{2}(\alpha_t^{N,k})^2 + \frac{\epsilon}{2}(X_t^{N,k})^2 \right) dt + \frac{c}{2}(X_T^{N,k})^2 \right\} \quad (77)$$

for  $k = 1, \dots, d^{N-1}$ . Since players of the last generation do not depend on the other players, the boundary condition defines a self-controlled problem.

Now, inspired by the conclusion in section 2, as the number  $N$  of generations goes to infinity, i.e.,  $N \rightarrow \infty$ , the effect of the boundary condition should vanish. Thus it is natural and reasonable that we decide to pass the  $N$ -generation finite tree to an infinite tree with infinite number of generations, and study the Nash equilibrium of the infinite-tree game. We still assume each parent has  $d$  direct descendants. The dynamics of the states and the costs are the same as (75) and (76) with  $n \geq 1$ .

## 7.1 Open-Loop Nash Equilibria

We search for an open-loop Nash equilibrium of the directed infinite-tree system among strategies  $\{\alpha^{n,k}; n \geq 1, 1 \leq k \leq d^{n-1}\}$ . The Hamiltonian  $H^{n,k}(x^{m,l}, y^{n,k;m,l}, \alpha^{m,l}; m \in \mathbb{N}, 1 \leq l \leq d^{m-1})$  for player  $(n, k)$  is

$$\sum_{m=1}^{M_n} \sum_{l=1}^{d^{m-1}} \alpha^{m,l} y^{n,k;m,l} + \frac{1}{2}(\alpha^{n,k})^2 + \frac{\epsilon}{2}(\bar{x}^{n+1,k} - x^{n,k})^2,$$

assuming it is defined on  $Y_t^{n,k}$ 's where *only finitely many*  $Y_t^{n,k;m,l}$ 's are non-zero for every given  $(n, k)$ . Here,  $M_n \in \mathbb{N}$  represents a depth of this finite dependence depending on  $n$  with  $M_n > n$  for  $n \geq 1$ . This assumption is checked in remark 6 below. Thus, the Hamiltonian  $H^{n,k}$  for player  $(n, k)$  is well defined for  $n, k \geq 1$ .

The adjoint processes  $Y_t^{n,k} = (Y_t^{n,k;m,l}; m \in \mathbb{N}, 1 \leq l \leq d^{m-1})$  and  $Z_t^{n,k} = (Z_t^{n,k;m,l;p,q}; m, p \in \mathbb{N}, 1 \leq l \leq d^{m-1}, 1 \leq q \leq d^{p-1})$  for  $n \in \mathbb{N}, 1 \leq k \leq d^{n-1}$

are defined as the solutions of BSDEs

$$\begin{aligned} dY_t^{n,k;m,l} &= -\epsilon(\bar{X}_t^{n+1,k} - X_t^{n,k})(\bar{\delta}_{m,\ell}^{n+1,k} - \delta_{m,\ell}^{n,k})dt + \sum_{p=1}^{\infty} \sum_{q=1}^{d^{p-1}} Z_t^{n,k;m,l;p,q} dW_t^{p,q}, \\ Y_T^{n,k;m,l} &= \partial_{x^{m,l}} g_{n,k}(X_T) = c(\bar{X}_T^{n+1,k} - X_T^{n,k})(\bar{\delta}_{m,\ell}^{n+1,k} - \delta_{m,\ell}^{n,k}), \end{aligned} \quad (78)$$

where  $\delta_{m,\ell}^{n,k} := 1$ , if  $(n, k) = (m, \ell)$ ; 0, otherwise, and  $\bar{\delta}_{m,\ell}^{n,k} := \sum_{i=(k-1)d+1}^{kd} \delta_{m,\ell}^{n,i}/d$ .

**Remark 6.** For every  $(m, l) \neq (n, k)$  or  $(n+1, i)$  where  $(k-1)d+1 \leq i \leq kd$ ,  $dY_t^{n,k;m,l} = \sum_{p=1}^{\infty} \sum_{q=1}^{d^{p-1}} Z_t^{n,k;m,l;p,q} dW_t^{p,q}$  and  $Y_T^{n,k;m,l} = 0$  implies  $Z_t^{n,k;m,l;p,q} = 0$  for all  $(p, q)$ . Thus, there must be finitely many non-zero  $Y^{n,k;m,l}$ 's for every  $(n, k)$ . Hence, the Hamiltonian can be rewritten as

$$\begin{aligned} H^{n,k}(x^{m,l}, y^{n,k;n,k}, y^{n,k;n+1,i}, \alpha^{m,l}; m \in N, 1 \leq l \leq d^{m-1}, (k-1)d+1 \leq i \leq kd) \\ = \alpha^{n,k} y^{n,k;n,k} + \sum_{i=(k-1)d+1}^{kd} \alpha^{n+1,i} y^{n,k;n+1,i} + \frac{1}{2}(\alpha^{n,k})^2 + \frac{\epsilon}{2}(\bar{x}^{n+1,k} - x^{n,k})^2. \end{aligned}$$

By minimizing the Hamiltonian  $H^{n,k}$  at  $\hat{\alpha}^{n,k} = -y^{n,k;n,k}$  with respect to  $\alpha^{n,k}$  for all  $(n, k)$ , we can get an open-loop Nash equilibrium. We make the ansatz:

$$Y_t^{n,k;n,k} = \sum_{i=0}^{\infty} \phi_t^i \sum_{j=0}^{d^i-1} X_t^{n+i,d^i k-j} = \sum_{m=n}^{\infty} \phi_t^{m-n} \sum_{j=0}^{d^{m-n}-1} X_t^{m,d^{m-n}k-j}, \quad (79)$$

for some deterministic scalar function  $\phi^k$  satisfying the terminal conditions:  $\phi_T^0 = c$ ,  $\phi_T^1 = -\frac{c}{d}$ ,  $\phi_T^k = 0$  for  $k \geq 2$ . Using the ansatz, the optimal strategy  $\hat{\alpha}^{n,k}$  and the forward equation for  $X^{n,k}$  in (75) for an open-loop Nash equilibrium become:

$$\begin{aligned} \hat{\alpha}_t^{n,k} &= -Y_t^{n,k;n,k} = -\sum_{m=n}^{\infty} \phi_t^{m-n} \sum_{j=0}^{d^{m-n}-1} X_t^{m,d^{m-n}k-j}, \\ dX_t^{n,k} &= -\sum_{m=n}^{\infty} \phi_t^{m-n} \sum_{j=0}^{d^{m-n}-1} X_t^{m,d^{m-n}k-j} dt + \sigma dW_t^{n,k}. \end{aligned} \quad (80)$$

Now comparing the two Itô's decompositions of  $dY^{n,k;n,k}$  from (78) and (79)-(80), we obtain from the martingale terms:

$$Z_t^{n,k;n,k;p,q} = \sigma \phi_t^{p-n} \text{ for } p \geq n \text{ and } 1 \leq q \leq d^{p-n}; \quad Z_t^{n,k;n,k;p,q} = 0, \text{ otherwise,}$$

for  $0 \leq t \leq T$ , and we obtain from the drift terms:  $k \geq 0, 0 \leq t \leq T$

$$\dot{\phi}_t^k = \sum_{j=0}^k \phi_t^j \phi_t^{k-j} - \epsilon \left( \delta_{0,k} - \frac{1}{d} \cdot \delta_{1,k} \right), \quad \phi_T^k = c \left( \delta_{0,k} - \frac{1}{d} \cdot \delta_{1,k} \right). \quad (81)$$

This Riccati system is closely related to the one in (25) for the infinite-player directed chain game and we can have a similar result.

**Proposition 14.** *Let  $\phi^{(k)} := \phi^k$  in (81) to avoid confusion from the power. We have  $\sum_{k=0}^{\infty} d^k \phi^{(k)} = 0$ , and the functions  $\phi^k$ 's can be obtained by a series expansion.*

*Proof.* Given in appendix A.9. □

**Proposition 15.** *An open-loop Nash equilibrium for the linear quadratic stochastic game with cost functionals (76) for the infinite players on the directed tree in fig. 3 is given by (80), where  $\{\phi^i\}$  are uniquely determined by the system (81) of Riccati equations.*

Without loss of generality, we assume  $\epsilon = 1$  and  $\sigma = 1$ . Following section 4, by taking  $T \rightarrow \infty$ , we look at the stationary long-time behavior of the Riccati system (81) satisfying  $\dot{\phi}^k = 0$  for all  $k$ . Then the system gives the recurrence relation:  $\phi^0 = 1$ ,  $\phi^1 = -1/(2d)$  and  $\sum_{j=0}^k \phi^j \phi^{k-j} = 0$  for  $k \geq 0$ . By using a moment generating function method as in appendix A.10, we obtain the stationary solution (cf. (30)):

$$\phi^0 = 1, \quad \phi^1 = -\frac{1}{2d}, \quad \text{and} \quad \phi^k = -\frac{(2k-3)!}{(k-2)!k!2^{2k-2}} \cdot \frac{1}{d^k} \quad \text{for } k \geq 2. \quad (82)$$

## 7.2 Catalan Markov Chain for the Directed Tree Model

As  $T \rightarrow \infty$ , the limit of average of the infinite particle system (80) can be rewritten as

$$d\bar{\mathbf{X}}_t = \mathbf{Q}_{\text{d-tree}} \bar{\mathbf{X}}_t dt + d\bar{\mathbf{W}}_t, \quad t \geq 0 \quad (83)$$

a linear stochastic evolution equation of Ornstein-Uhlenbeck type, where  $\bar{\mathbf{X}} := (\bar{X}^k \equiv \sum_{i=1}^{d^{k-1}} X^{k,i}/d^{k-1}, k \in \mathbb{N})$  with  $\bar{\mathbf{X}}_0 = \bar{\mathbf{x}}_0$ , and  $\bar{\mathbf{W}} := (\bar{W}_t^k = \sum_{i=1}^{d^{k-1}} W^{k,i}/d^{k-1}, k \in \mathbb{N})$  is a vector of averaged Brownian motions with mean 0 and variance  $t/d^k$  in each generation  $k \in \mathbb{N}$ , and  $\mathbf{Q}_{d\text{-tree}}$  is exactly the same as  $\mathbf{Q}$  in section 4. Its solution is

$$\bar{\mathbf{X}}_t = e^{t\mathbf{Q}_{d\text{-tree}}}\bar{\mathbf{x}}_0 + \int_0^t e^{(t-s)\mathbf{Q}_{d\text{-tree}}}\mathrm{d}\bar{\mathbf{W}}_s; \quad t \geq 0. \quad (84)$$

Similarly to proposition 4 we can find the following formulas for  $X_t^{1,1} \equiv \bar{X}_t^1$  and its asymptotic variance. Proof is in appendix A.11.

**Proposition 16.** *With  $\mathbf{x}_0 = \mathbf{0}$ , the formula for the root node  $X_t^{1,1}$  in (83) is:*

$$X_t^{1,1} = \sum_{j=1}^{\infty} \int_0^t \frac{(t-s)^{2(j-1)}}{(j-1)!} \cdot \rho_{j-1}(-(t-s)^2) e^{-(t-s)} \cdot \mathrm{d}\bar{W}_s^j, \quad t \geq 0, \quad (85)$$

where  $\rho_i(\cdot)$  is in (35). Moreover, the asymptotic variance of  $X_t^{1,1}$  is finite, i.e.,

$$\lim_{t \rightarrow \infty} \mathrm{Var}(X_t^{1,1}) = \frac{\sqrt{2}}{2} \cdot \left(1 + \left(\frac{d-1}{d}\right)^{1/2}\right)^{-1/2} \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right]. \quad (86)$$

**Remark 7** (Connection to the mean-field game). *When  $d$  goes to infinity, we are in the regime of the mean field game. The asymptotic variance is  $\frac{1}{2}$  which is consistent with the variance of an Ornstein-Uhlenbeck process where the particle is attracted to 0 and the volatility and the mean reversion constant are both 1.*

## 8 Conclusion

We studied a linear-quadratic stochastic differential game on a directed chain network. We were able to identify Nash equilibria in the case of finite chain with various boundary conditions and in the case of an infinite chain. This last case allows for more explicit computation in terms of Catalan functions and Catalan Markov chain. The Catalan open-loop Nash equilibrium that we obtained is characterized by interactions with all the neighbors in one

direction of the chain weighted by Catalan functions, even though the interaction in the objective functions is only with the nearest neighbor. Under equilibrium the variance of a state converges in the infinite time limit as opposed to the diverging behavior observed in the nearest neighbor dynamics studied in Detering, Fouque & Ichiba [6]. Our analysis is extended to mixed games with directed chain and mean field interaction so that our game model includes the two extreme network interactions, fully connected and only one neighbor connection. It is also extended to game on a deterministic tree structure. Our ongoing and future research concerns games with interactions on directed tree-like stochastic networks modeled as branching processes.

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## A Appendix

### A.1 Proof of Proposition 2

Define  $S_t(z) := \sum_{k=0}^{\infty} z^k \phi_t^{(k)}$  where  $0 \leq z < 1$  with  $\phi_t^{(k)} = \phi_t^k$  in (25) to avoid confusion for  $t \geq 0, k \geq 0$ . Then substituting (25) into  $S_t(z)$ , we obtain the one-dimensional Riccati equation

$$\dot{S}_t(z) = \sum_{k=0}^{\infty} z^k \dot{\phi}_t^{(k)} = (S_t(z))^2 - \epsilon(1 - z), \quad S_T(z) = c(1 - z) \quad (87)$$

and its solution is given by (27). One needs to be careful when taking  $z = 1$  because the series defining  $S_t(1)$  may not converge a priori. Instead, we take a sequence  $\{z_n\}$  converging to 1, the limit of  $S_t(z_n)$  converges to the ODE  $\dot{S}_t(1) = (S_t(1))^2, S_T(1) = 0$ , and hence, solving this limiting ODE, we obtain (26).

### A.2 Catalan Markov Chain and Proposition 4

We have the Catalan probabilities  $\{p_k > 0, k \geq 1\}$ :  $\sum_{k=1}^{\infty} p_k = 1$  and  $p_k = \frac{1}{2} \sum_{i=1}^{k-1} p_i p_{k-i}$ . It is easily seen then that  $-\mathbf{Q}^2 = -I + B$  is an infinite Jordan block matrix with diagonal components  $-1$ , where  $B$  having 1's on the upper second diagonal and 0's elsewhere. Then as a smooth function  $F(x) := \exp(-\sqrt{-x}), x \in \mathbb{C}$  of Jordan block matrix, we have

$$\exp(\mathbf{Q}t) = F((-I + B)t^2) = \sum_{k=0}^{\infty} \frac{t^{2k} F^{(k)}(-t^2)}{k!} B^k,$$

where the  $k$ -th derivative  $F^{(k)}$  is given by  $F^{(k)}(x) = \rho_k(x)F(x)$ ,  $x \in \mathbb{C}$  with  $\rho_k$  in (35) from direct calculations and mathematical induction. Therefore, substituting them into (34), we obtain proposition 4.

### A.3 Proof of Remark 4 in section 4

By  $\rho_k$ 's formula in (36), we have for  $\nu \geq 0$ ,  $k \geq 1$ ,

$$\rho_k(-\nu^2) = \frac{1}{2^k} \sum_{j=k}^{2k-1} \frac{(j-1)!}{(2j-2k)!!(2k-j-1)!} = \frac{1}{2^k \nu^k} \cdot \sqrt{\frac{2\nu}{\pi}} \cdot e^\nu \cdot K_{k-(1/2)}(\nu),$$

where  $K_n(x)$  is the modified Bessel function of the second kind, i.e.,

$$K_n(x) = \int_0^\infty e^{-x \cosh t} \cosh(nt) dt; \quad n > -1, \quad x > 0.$$

Then, by the change of variables, we obtain

$$\begin{aligned} \text{Var}(X_t^1) &= \sum_{k=0}^{\infty} \int_0^t \frac{(t-s)^{4k}}{(k!)^2} |\rho_k(-(t-s)^2)|^2 e^{-2(t-s)} ds \\ &= \sum_{k=1}^{\infty} \int_0^t \frac{2}{\pi} \frac{\nu^{2k+1}}{(k!)^2 4^k} (K_{k-(1/2)}(\nu))^2 d\nu + \frac{1-e^{-2t}}{2}; \quad t \geq 0. \end{aligned}$$

### A.4 Proof of Proposition 5 in section 4

Using the following identities from the special functions

$$\int_0^\infty t^{\alpha-1} (K_\nu(t))^2 dt = \frac{\sqrt{\pi}}{4\Gamma((\alpha+1)/2)} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2} - \nu\right) \Gamma\left(\frac{\alpha}{2} + \nu\right),$$

$$\frac{\sqrt{2}}{4} x \sqrt{x^2 - \sqrt{x^4 - 16}} = \sum_{k=0}^{\infty} \binom{4k}{2k} \frac{1}{2k+1} \frac{1}{x^{4k}}, \quad \text{for } x \geq 2,$$

based on remark 4, we obtain the limit of variance of  $X_t^1$ , as  $t \rightarrow \infty$ , i.e.,

$$\begin{aligned}
\lim_{t \rightarrow \infty} \text{Var}(X_t^1) &= \frac{1}{2} + \sum_{k=1}^{\infty} \int_0^{\infty} \frac{2 s^{2k+1}}{\pi(k!)^2 4^k} \cdot [K_{k-(1/2)}(s)]^2 ds \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi(k!)^2 4^k} \cdot \frac{\pi \Gamma(k+1) \Gamma(2k+(1/2))}{8 \Gamma(k+(3/2))} \\
&= \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \binom{4k}{2k} \frac{1}{2k+1} \frac{1}{2^{4k}} = \frac{1}{2} \sum_{k=0}^{\infty} \binom{4k}{2k} \frac{1}{2k+1} \frac{1}{2^{4k}} \\
&= \frac{1}{2} \cdot \frac{\sqrt{2}}{4} 2\sqrt{2^2-0} = \frac{1}{\sqrt{2}}.
\end{aligned}$$

## A.5 Proofs of Propositions 6-7 in section 4

From the expression (34) for  $X_t^1$ , the auto-covariance  $\mathbb{E}[X_s^1 X_t^1]$  and the cross covariance  $\mathbb{E}[X_t^1 X_t^{j+1}]$  are

$$\begin{aligned}
\mathbb{E}[X_s^1 X_t^1] &= \sum_{i=0}^{\infty} \frac{1}{\pi(i!)^2 2^{2i-1}} \int_0^s (t-v)^{i+1/2} (s-v)^{i+1/2} K_{i-1/2}(t-v) K_{i-1/2}(s-v) dv \\
&= \sum_{i=0}^{\infty} \frac{1}{\pi(i!)^2 2^{2i-1}} \int_0^s ((t-s+v)v)^{i+1/2} K_{i-1/2}(t-s+v) K_{i-1/2}(v) dv > 0;
\end{aligned} \tag{88}$$

$$\begin{aligned}
\mathbb{E}[X_t^1 X_t^{j+1}] &= \sum_{i=j}^{\infty} \int_0^t \frac{1}{\pi i! (i-j)!} \frac{(t-\nu)^{2i-j+1}}{2^{2i-j-1}} K_{i-1/2}(t-\nu) K_{i-j-1/2}(t-\nu) d\nu \\
&= \sum_{i=0}^{\infty} \frac{1}{\pi(j+i)! j!} \frac{1}{2^{j+2i-1}} \int_0^t s^{j+2i+1} K_{j+i-1/2}(s) K_{i-1/2}(s) ds \\
&\xrightarrow{t \rightarrow \infty} \sum_{i=0}^{\infty} \frac{1}{\pi(j+i)! j!} \frac{1}{2^{j+2i-1}} \int_0^{\infty} s^{j+2i+1} K_{j+i-1/2}(s) K_{i-1/2}(s) ds > 0.
\end{aligned} \tag{89}$$

By the Cauchy-Schwarz inequality, as  $t \rightarrow \infty$ , the asymptotic cross co-



variance between  $X_t^1$  and  $X_t^{j+1}$  is bounded by

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}[X_t^1 X_t^{j+1}] &\leq \lim_{t \rightarrow \infty} (\mathbb{E}[(X_t^1)^2])^{1/2} \cdot (\mathbb{E}[(X_t^{j+1})^2])^{1/2} \\ &= \lim_{t \rightarrow \infty} \text{Var}(X_t^1) = \frac{1}{\sqrt{2}} \end{aligned} \quad (90)$$

for  $j \geq 0$ , because  $X^1$  and  $X^{j+1}$  have the same distribution.

To compute the asymptotic auto-covariance, fix  $s > 0$  and let  $t \rightarrow \infty$ . By the asymptotic expansion of the modified Bessel function  $K_\alpha(z)$ ,  $z > 0$ , there exists a positive constant  $c > 0$  such that for every sufficiently large  $t (> s)$

$$\sup_{i \geq 0} \frac{1}{i! t^{i+1}} \int_0^s ((t-s+v)v)^{i+1/2} K_{i-1/2}(t-s+v) K_{i-1/2}(v) dv \leq c \cdot e^{-(t-s)}.$$

Then combining this estimate with (88), we obtain

$$\mathbb{E}[X_s^1 X_t^1] \leq \sum_{i=0}^{\infty} \frac{4ct^{i+1} e^{-(t-s)}}{\pi i! 4^i} \leq \frac{4ct}{\pi} e^{-(t-s)+(t/4)} \xrightarrow{t \rightarrow \infty} 0.$$

## A.6 Proof of Proposition 9

Define  $S_t(z) := \sum_{k=0}^{\infty} z^k \phi_t^{(k)}$  for  $0 \leq z < 1$  and  $\dot{\phi}_t^{(k)} := \dot{\phi}_t^k$  again to avoid confusion from the power. Then

$$\begin{aligned} u \dot{S}_t(z) &= \sum_{k=0}^{\infty} z^k u \dot{\phi}_t^{(k)} = u^2 (S_t(z))^2 + u(1-u) \psi_t S_t(z) - u(1-z) \epsilon, \\ u S_T(z) &= u(1-z)c \end{aligned} \quad (91)$$

as in appendix A.1. For  $z \rightarrow 1$ , we obtain the ODE:  $u \dot{S}_t(1) = u^2 (S_t(1))^2 + u(1-u) \psi_t S_t(1)$ ,  $u S_T(1) = 0$ , and hence,  $S_t(1) \equiv 0$  and conclude the proof.

## A.7 Some details on Table 1

It follows from Proposition 11 that

$$\text{Var}(X_t^1) = \text{Var}\left(\sum_{k=0}^{\infty} \int_0^t \frac{u^k (t-s)^{2k}}{k!} F^{(k)}(-(t-s)^2) dW_k(s)\right) \quad (92)$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \int_0^t \frac{u^{2k} (t-s)^{4k}}{(k!)^2} |\rho_k(-(t-s)^2)|^2 e^{-2(t-s)} ds \\
&= \sum_{k=1}^{\infty} \int_0^t \frac{2u^{2k}}{\pi(k!)^2 4^k} \nu^{2k+1} (K_{k-\frac{1}{2}}(\nu))^2 d\nu + \frac{1-e^{-2t}}{2}
\end{aligned}$$

for  $t \geq 0$ . As  $t \rightarrow \infty$ , we obtain

$$\begin{aligned}
&\lim_{t \rightarrow \infty} \text{Var}(X_t^1) \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \int_0^{\infty} \frac{2u^{2k} s^{2k+1}}{\pi(k!)^2 4^k} \cdot [K_{k-(1/2)}(s)]^2 ds \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} u^{2k} \cdot \frac{\Gamma(2k + (1/2))}{4^{k+1} k! \Gamma(k + (3/2))} = \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \binom{2k}{k} \frac{u^{2k}}{8^k} \\
&= \frac{1}{2} + \frac{1}{2} \left( \left(1 - 4 \frac{u^2}{8}\right)^{-\frac{1}{2}} - 1 \right) = \frac{1}{2} \left(1 - \frac{u^2}{2}\right)^{-\frac{1}{2}} < \infty.
\end{aligned}$$

## A.8 Proof of Proposition 12

Define  $S_t^N(z) = \sum_{k=0}^{N-1} z^k \phi_t^{N,k}$ , then, by (73),

$$\dot{S}_t^N(z) = (S_t^N(z))^2 + (1-z^N) \left[ \sum_{j=0}^{N-2} z^j \cdot \sum_{k=j+1}^{N-1} \phi_t^{N,k} \phi_t^{N,N+j-k} \right] - (1-z)\epsilon, \quad (93)$$

for  $0 \leq t \leq T$  with  $S_T^N(z) = (1-z)c$  for  $0 \leq z < 1$ . As  $z \rightarrow 1$ ,  $\dot{S}_t^N(1) = (S_t^N(1))^2$ ,  $S_T^N(1) = 0$ , and hence,  $S_t^N(1) = \sum_{k=0}^{N-1} \phi_t^{N,k} = 0$ .

## A.9 Proof of Proposition 14 in section 7

Similar to the proof of lemma proposition 2 in appendix A.1, define  $S_t(z) = \sum_{k=0}^{\infty} z^k \psi_t^{(k)}$  where  $0 \leq z < 1$  and  $\psi_t^{(k)} := d^k \phi_t^{(k)}$  in (81). The Riccati system for  $\psi^{(k)}$  functions is now the same as (87). The conclusion follows directly from the Riccati equation.

## A.10 Stationary Solution of (81) in section 7

Define  $R_t(z) := \sum_{k=0}^{\infty} z^k \phi_t^{(k)}$  where  $0 \leq z < 1$  and  $\phi_t^{(k)} := \phi_t^k$  in (81) to avoid confusion. Without loss of generality, we assume  $\epsilon = 1$ . Then

$R_T(z) = c(1 - d^{-1}z)$  and for  $0 \leq t \leq T$

$$\dot{R}_t(z) = \sum_{k=0}^{\infty} z^k \dot{\phi}_t^{(k)} = \sum_{k=0}^{\infty} z^k \sum_{j=0}^k \phi_t^{(j)} \phi_t^{(k-j)} - 1 + \frac{z}{d} = (R_t(z))^2 - \left(1 - \frac{z}{d}\right). \quad (94)$$

Thus, the stationary solution  $\phi^{(k)}$  of (81), as  $T \rightarrow \infty$ , is obtained by the Taylor expansion of  $\sqrt{1 - (z/d)}$ .

### A.11 Proof of Proposition 16

(85) follows directly from (84) and proposition 4. Taking the limit  $t \rightarrow \infty$  in the variance formula

$$\text{Var}(\bar{X}_t^1) = \sum_{k=1}^{\infty} \int_0^t \frac{2}{\pi} \frac{s^{2k+1}}{(k!)^2 4^k} (K_{k-(1/2)}(s))^2 \cdot \frac{1}{d^k} ds + \frac{1 - e^{-2t}}{2},$$

we obtain (86):

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Var}(\bar{X}_t^1) &= \frac{1}{2} + \sum_{k=1}^{\infty} \int_0^{\infty} \frac{2 s^{2k+1}}{\pi (k!)^2 4^k} \cdot [K_{k-(1/2)}(s)]^2 \cdot \frac{1}{d^k} ds \\ &= \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \binom{4k}{2k} \frac{1}{2k+1} \frac{1}{2^{4k} d^k} = \frac{1}{2} \sum_{k=0}^{\infty} \binom{4k}{2k} \frac{1}{2k+1} \frac{1}{(2d^{1/4})^{4k}} \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{4} 2d^{1/4} \sqrt{4d^{1/2} - \sqrt{16d - 16}} \\ &= \frac{\sqrt{2}}{2} d^{1/4} \sqrt{\sqrt{d} - \sqrt{d-1}} = \frac{\sqrt{2}}{2} \frac{d^{1/4}}{\sqrt{\sqrt{d} + \sqrt{d-1}}} \\ &= \frac{\sqrt{2}}{2} \left(1 + \sqrt{\frac{d-1}{d}}\right)^{-1/2} \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right]. \end{aligned}$$

The limit is monotone in  $d$  with maximum of  $1/\sqrt{2}$  at  $d = 1$ .