

Overdetermined PDE Systems on Some Classes of Riemannian Manifolds

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Abstract

We study several classes of Riemannian manifolds which are defined by imposing a certain condition on the Ricci tensor. We consider the following cases: Ricci recurrent, Cotton, quasi Einstein, and pseudo Ricci symmetric condition. Such conditions can be interpreted as overdetermined PDE systems whose unknowns are the components of the Riemannian metric, and perhaps, in addition, some auxiliary functions. Hence even if the dimension of the manifold is small it is not easy to compute interesting examples by hand, and indeed very few examples appear in the literature. We will present large families of nontrivial examples of such manifolds. The relevant PDE systems are first transformed into an involutive form. After that in many cases, one can actually solve the resulting system explicitly. However, the involutive form itself already gives a lot of information about the possible solutions to the given problem. We will also discuss some relationships between the relevant classes.

Keywords Cotton tensor, conformally conservative manifold, pseudo Ricci symmetric manifold, quasi Einstein manifold, Ricci recurrent manifold, overdetermined PDE

MSC: 34A34, 53C21, 58A17

1 Introduction

In the following we are going to analyze and present examples of several classes of Riemannian manifolds: Ricci recurrent, pseudo Ricci symmetric, Cotton and quasi Einstein manifolds. Sometimes these classes are understood in a generalized sense without requiring the positive definiteness of the metric. In the following we will on the other hand only consider the positive definite case. However, in the actual construction of examples the positive definiteness does not play a role, so that using the same approach one can also construct examples which are not positive definite.

Ricci recurrent manifolds were first considered in [12] (even earlier recurrent Riemannian manifolds were introduced in [18]). Since then this class has been analyzed by many authors, see for example [7] and references therein, where also various generalizations and extensions of this notion are considered.

The concept of pseudo Ricci symmetric manifold was perhaps explicitly first introduced in [3]. However, earlier in [20] authors had obtained the characterizing condition when analyzing the existence of another structure on Riemannian manifolds.

As a term the Cotton manifold or Cotton metric is not very common; it is used in [10]. However, the Cotton tensor, introduced in [5], which gives the defining condition for such manifolds appears in wide variety of questions. We will explain below how Cotton manifolds are related to other classes of Riemannian manifolds.

There are in fact at least 3 different definitions for quasi Einstein manifolds. The one we are interested in was apparently introduced in [1]. The quasi Einstein property was then a special case of larger class of Riemannian manifolds. Other definitions, neither directly related to the present article nor to each other can be found for example in [4] and [2]. As the name suggest these spaces are typically related to problems in general relativity and the hence the metric in that case is typically not positive definite.

We will analyze some connections of the above classes of manifolds. However, the main part of our paper is devoted to the construction of large families examples of these different spaces. In the papers where these types of Riemannian manifolds are considered there are very few actual examples. In some sense this is natural since producing an example implies solving a relatively big system of PDE. Below we will show how to use the theory of overdetermined PDE (also called formal theory of PDE) [9, 14, 19] to produce solutions. The conclusion is in fact that with appropriate tools it is not particularly hard to find examples.

Below we have chosen examples more or less randomly with no particular application in mind. However, the reader who wants solutions of some specific form can easily adapt our approach to other contexts. Of course this approach does not always lead to explicit solutions, but the analysis can still give important information about the nature of solutions. In fact the special form of the system that is obtained in the analysis is even suitable for numerical computations, if one wants to explore numerically different possibilities.

The paper is organized as follows. In section 2 we recall some notions which are needed in the analysis. In section 3 the classes of Riemannian manifolds are introduced, and the relationships between them are analyzed. Then in section 4 we formulate our computational problems precisely. Finally in section 5 we present and discuss the examples and in section 6 there are some concluding remarks.

2 Preliminaries

2.1 Geometry

Let M be a smooth n dimensional manifold with Riemannian metric g . The pointwise norm of a tensor T is denoted by $|T|$. The covariant derivative is denoted by ∇ . We say that a tensor T is *parallel*, if $\nabla T = 0$. The curvature tensor is denoted by R and the *Ricci tensor* is $\text{Ri}_{jk} = R^i_{ijk}$ and the scalar curvature is $\text{sc} = \text{Ri}^k_k$. There are several conventions regarding the signs and indices of curvature tensors. We will follow [13].

In several places we will need the *Ricci identity* which for general tensors A of type (m, n) has the form

$$A^{j_1 \dots j_m}_{i_1 \dots i_n; ij} - A^{j_1 \dots j_m}_{i_1 \dots i_n; ji} = \sum_{q=1}^n A^{j_1 \dots j_m}_{i_1 \dots i_{q-1} \ell i_{q+1} \dots i_n} R^{\ell}_{ij i_q} - \sum_{p=1}^m A^{j_1 \dots j_{p-1} \ell j_{p+1} \dots j_m}_{i_1 \dots i_n} R^{j_p}_{ij \ell} \quad (2.1)$$

The Bianchi identity is

$$R_{hijk;\ell} + R_{hikl;j} + R_{hilj;k} = 0 \quad (2.2)$$

By multiplying above equation on g^{hk} , we have

$$\text{Ri}_{ij;\ell} - \text{Ri}_{i\ell;j} = R^h_{\ell ji;h}$$

which then implies that

$$\text{sc}_{;k} = 2 \text{div}(\text{Ri}) = 2 \text{Ri}^j_{k;j} \quad (2.3)$$

Let us then define some classical tensors which are needed in the sequel.

DEFINITION 2.1 Let M be a n dimensional Riemannian manifold with metric g . Then

(i) Schouten tensor is

$$S = \text{Ri} - \frac{\text{sc}}{2n-2} g$$

(ii) Cotton tensor is

$$C_{ijk} = S_{ij;k} - S_{ik;j}$$

(iii) Weyl tensor is

$$W_{hijk} = R_{hijk} + \frac{sc}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}) - \frac{1}{n-2}(Ri_{hk}g_{ij} - Ri_{hj}g_{ik} + Ri_{ij}g_{hk} - Ri_{ik}g_{hj}).$$

In some references Schouten tensor is some constant multiple of S given above. Note that $W = 0$ when $n \leq 3$ and $S = 0$ when $n = 2$. Let us also recall

THEOREM 2.1 Let M be a n dimensional Riemannian manifold. Then

- (1) M is conformally flat if and only if $C = 0$ when $n = 3$ or $W = 0$ when $n \geq 4$.
- (2) $\text{div}(W) = \frac{n-3}{n-2}C$ when $n \geq 4$.

2.2 Determinantal varieties

Let $\mathbb{R}^{m \times n}$ be the vector space of real $m \times n$ matrices and let V_r be the subvariety of matrices of rank at most r . This is a determinantal variety, defined by setting to zero all minors of size $(r+1) \times (r+1)$. There are thus $\binom{m}{r+1}\binom{n}{r+1}$ polynomials which generate the ideal defining V_r . However, not all generators are algebraically independent and one can show that

$$\text{codim}(V_r) = (m-r)(n-r)$$

Let then \mathcal{S}_n be the vector space of real $n \times n$ symmetric matrices and let V_r^s be the subvariety of symmetric matrices of rank at most r . The relevant ideal is now generated by $\binom{n}{r+1}^2$ polynomials, but due to symmetry we have much less independent generators, and one can show that in this case

$$\text{codim}(V_r^s) = \binom{n-r+1}{2} \tag{2.4}$$

2.3 Overdetermined PDE

The existence of various manifolds considered below depends on the solvability of certain systems of overdetermined PDE. For a general overview of overdetermined PDE we refer to [9, 14, 19] and references therein. Both books contain also historical comments on the development of the subject which started at the end of 19th century.

Very early it was realized that before one could actually prove any existence results for overdetermined PDE in some definite function space one should first analyze the structural properties of the system. The main difficulty of the analysis of overdetermined systems is related to integrability conditions: in other words by differentiating the equations one may find new equations which are algebraically independent of the original equations. The process of finding the integrability conditions is called *completion*, and the goal was to find all integrability conditions.

Analysis of the completion process lead to two complementary approaches: geometric and algebraic. The geometric approach is based on interpreting PDE as submanifolds of jet spaces. The algebraic approach requires that the nonlinearities are polynomial and hence the equations can be interpreted as differential polynomials and the systems themselves are differential ideals generated by the given differential polynomials.

It turns out that proving that the system is complete, or that the system can actually be completed, is quite tricky and we simply refer again to [9, 14, 19] for details. In spite of this heavy machinery which is required for the theory the end result is perhaps surprisingly constructive: there are actual algorithms for computing the completed system, i.e. the system which contains all integrability conditions. The completed system is called the *involution system*, and the completion algorithm is usually known as Cartan-Kuranishi algorithm.

The analysis of structural properties of overdetermined PDE is also called formal theory of PDE. The word formal appears because one can say that the involutive form of the system has solutions as formal power series. One can say that in the involutive system all relevant information about the system is explicit while in the initial system it was only implicit.

An analogous situation arises in polynomial algebra. A polynomial system generates an ideal which in turn defines the corresponding variety. Now computing the Gröbner basis of the ideal gives a lot of information about the variety [6]. Intuitively one may think about computing the involutive form of a system of PDE like computing the Gröbner basis of an ideal.

The idea of Gröbner bases can be generalized to differential equations, where equations are interpreted as differential polynomials [11]. However, not all properties of Gröbner bases of the algebraic case carry over to the differential case. Anyway the ideas related to Gröbner bases and ideal theory in general are present in the actual implementations of completion algorithms.

One final comparison to polynomial case is perhaps helpful. In the polynomial case any variety can be decomposed to a finite number of irreducible varieties which means that any polynomial ideal is an intersection of finite number of primary ideals. This property is still valid in the differential context in the following form: any radical differential ideal is a finite intersection of prime differential ideals. Hence if the nonlinearities are polynomial, and they will be in all systems considered below, one may also try to find the decomposition of the involutive form. Evidently finding this decomposition greatly facilitates any further analysis of the system.

In what follows we will use the algorithm `rifsimp` which is described in detail in [16], see also [15]. The acronym `rif` means *reduced involutive form*. This algorithm assumes that the nonlinearities are polynomial, and that the implied differential field is the field of rational functions. It can also handle inequations and it can compute the decomposition of the system.

The algorithm is implemented as the command `rifsimp` in Maple.¹ In setting up the systems of equations the Differential Geometry package of Maple was also very useful.

Finally we note that the word "overdetermined" is a bit misleading. This term is traditionally used, but actually one simply means the analysis of general PDE systems. The important concept is the involutivity (or some other canonical form), and in many (or even most) cases it is not necessary to define precisely what is meant by the term overdetermined (or underdetermined). In particular below this definition is not needed. Also one can find (at least) two different definitions in the literature which are both reasonable in their ways; see [9] and [19] for these different definitions.

3 Some properties and relationships between various classes

In the following we will consider several classes of Riemannian manifolds. These classes are defined by requiring that the corresponding Ricci tensor satisfies some condition P. In this case we can also say that the manifold or the Riemannian metric is of the type P. Of course we will always assume that $Ri \neq 0$.

DEFINITION 3.1 Ricci tensor is

- *Ricci recurrent*, RR, if there is a nonzero one form β such that

$$Ri_{ij;\ell} = \beta_\ell Ri_{ij} . \tag{3.1}$$

- *pseudo Ricci symmetric*, PRS, if there is a nonzero one form α such that

$$Ri_{ij;\ell} = 2\alpha_\ell Ri_{ij} + \alpha_i Ri_{\ell j} + \alpha_j Ri_{i\ell} \tag{3.2}$$

- *quasi Einstein*, QE, if there are functions a and b , and one form ω such that

$$Ri = ag + b \frac{\omega \otimes \omega}{|\omega|^2} \tag{3.3}$$

If $b = 0$ the Ricci tensor is Einstein.

- *Cotton*, CO, if the Cotton tensor is zero.

In dimension two we have $Ri = \kappa g$ where κ is the Gaussian curvature. Hence any manifold is Einstein, Cotton and Ricci recurrent. On the other hand no manifold is pseudo Ricci symmetric. Hence from now on we suppose that the dimension $n \geq 3$.

¹<https://www.maplesoft.com/>

It is clear that if Ri is parallel the RR condition cannot be satisfied, and on the other hand the CO condition is always satisfied. Below we will see that PRS condition is incompatible with parallelism.

If the metric satisfies the condition $\text{div}(W) = 0$ it is sometimes said to be *conformally conservative*. Hence by Theorem 2.1 Cotton manifolds are conformally conservative when $n \geq 4$ and conformally flat when $n = 3$. Finally recall that if M is an Einstein manifold then $a = \text{sc}/n = \text{constant}$.

3.1 Ricci recurrent

Let us then start with the RR case. Multiplying (3.1) by Ri^{ij} we obtain

$$\beta = \frac{\text{Ri}^{ij} \text{Ri}_{ij;k}}{|\text{Ri}|^2} = \frac{1}{2} \nabla \ln(|\text{Ri}|^2)$$

From this we get the following characterization.

LEMMA 3.1 Let $\text{NRi} = \frac{\text{Ri}}{|\text{Ri}|}$. Then Ri is RR if and only if NRi is parallel.

Proof. Simply taking the covariant derivative of NRi we see that it is zero precisely when Ri is RR with β as given above. \square

However, it turns out that the RR case can be characterized purely in an algebraic way. In [17] it is shown that actually

$$\text{Ri}_i^k \text{Ri}_\ell^i = \frac{1}{2} \text{sc} \text{Ri}_\ell^k \quad (3.4)$$

Note that this result crucially depends on the fact that g is positive definite. But this leads easily to the following characterization of the Ricci tensor.

THEOREM 3.2 Suppose that Ri is recurrent. Then it has a double eigenvalue $\frac{\text{sc}}{2}$ and eigenvalue zero of multiplicity $n - 2$. Moreover

$$\beta = \nabla \ln(\text{sc}) \quad , \quad \text{sc}^2 = 2 |\text{Ri}|^2 \quad \text{and} \quad \text{Ri} \beta = \frac{\text{sc}}{2} \beta .$$

Proof. Let λ_j be the eigenvalues of Ri . Then the formula (3.4) implies that

$$\lambda_k^2 = \frac{1}{2} (\lambda_1 + \dots + \lambda_n) \lambda_k .$$

This gives the first statement. Since the scalar curvature cannot be zero the formula for β is obtained multiplying (3.1) by g^{ij} . Taking the trace in (3.4) gives the second formula. Then by formula (2.3) we have

$$\text{sc} \beta_\ell = \text{sc}_{;\ell} = 2 g^{hk} \text{Ri}_{h\ell;k} = 2 g^{hk} \beta_k \text{Ri}_{h\ell} = 2 \text{Ri} \beta$$

\square

From this we immediately get

COROLLARY 3.1 Ricci tensor cannot be at the same time RR and CO.

Proof. If the Ricci tensor satisfies the condition RR then

$$S_{ij;k} = \frac{\text{sc}_{;k}}{\text{sc}} \left(\text{Ri} - \frac{1}{2(n-1)} \text{sc} g_{ij} \right) = \frac{\text{sc}_{;k}}{\text{sc}} S_{ij}$$

Note that Schouten tensor is also "recurrent" in this case. Now by previous Theorem we know that Ri has an eigenvector v such that $\text{Ri} v = 0$ and $g(\text{grad}(\text{sc}), v) = 0$. This implies that

$$C_{ijk} v^j = S_{ij;k} v^j - S_{ik;j} v^j = -\frac{1}{2(n-1)} \text{sc}_{;k} v_i \neq 0 .$$

\square

3.2 Pseudo Ricci symmetric

It turns out that the associated one form is almost the same in the PRS case.

LEMMA 3.3 Let the Ricci tensor be PRS with associated one form α . Then $\text{Ri } \alpha = 0$ and

$$\alpha = \frac{1}{4} \nabla \ln(|\text{Ri}|^2)$$

If the scalar curvature is not zero then

$$\alpha = \frac{1}{2} \nabla \ln(\text{sc}) \quad \text{and} \quad \text{sc}^2 = c |\text{Ri}|^2$$

for some constant c .

It is easy to see that if the scalar curvature is constant then it has to be zero. However, it seems to be an open problem if the case $\text{sc} = 0$ can actually occur. If one does not assume the positive definiteness of the metric then it is easy to construct examples with $\text{sc} = 0$. However, we were unable to find an example with a positive definite metric.

Proof. Multiplying (3.2) with $g^{i\ell}$ and using the formula (2.3) we get

$$\text{sc}_{;j} = 2g^{i\ell} \text{Ri}_{ij;\ell} = 4\alpha_\ell g^{i\ell} \text{Ri}_{ij} + 2g^{i\ell} \alpha_i \text{Ri}_{\ell j} + 2\alpha_j \text{sc} = 6\alpha^i \text{Ri}_{ij} + 2\alpha_j \text{sc}$$

On the other hand multiplying (3.2) with g^{ij} we get

$$\text{sc}_{;\ell} = 2\alpha_\ell \text{sc} + 2\alpha^i \text{Ri}_{i\ell}$$

Hence $\text{Ri } \alpha = 0$. The expression for α is then obtained by multiplying (3.2) with Ri^{ij} . If $\text{sc} \neq 0$ we have $\text{sc}_{;\ell} = 2\alpha_\ell \text{sc}$ which gives the other expression for α . \square

From this we get immediately

COROLLARY 3.2 (i) The Ricci tensor cannot be both RR and PRS.

(ii) If Ri is parallel then the PRS condition cannot be satisfied.

Proof. (i) The previous Lemmas imply that the associated one forms in the two cases satisfy $\beta = 2\alpha$. Hence

$$\beta_\ell \text{Ri}_{ij} = \text{Ri}_{ij;\ell} = 2\alpha_\ell \text{Ri}_{ij} + \alpha_i \text{Ri}_{\ell j} + \alpha_j \text{Ri}_{i\ell}$$

implies that $\alpha_i \text{Ri}_{\ell j} + \alpha_j \text{Ri}_{i\ell} = 0$. Multiplying this with α^i gives $|\alpha|^2 \text{Ri} = 0$ which is impossible.

(ii) If Ri is parallel, then multiplying (3.2) with α^ℓ gives $2|\alpha|^2 \text{Ri} = 0$ which is impossible. \square

3.3 Quasi Einstein

Let us then analyze quasi Einstein structure. To this end it is convenient to formulate the condition (3.3) differently. Let us introduce the tensor $T = \text{Ri}_{ij} - a g_{ij}$. Then we can say

(i) M is an Einstein manifold, if $T = 0$ and $a = \text{sc}/n$.

(ii) M is a quasi Einstein manifold, if the matrix rank of T is one.

In this way we see that the one form ω and the function b are actually quite irrelevant in the analysis of the existence of quasi Einstein structure.

Now if we want that the symmetric tensor T is of matrix rank one then according to the formula (2.4) there are only $\binom{n}{2}$ algebraically independent differential equations. On the other hand there are $1 + \binom{n+1}{2}$ unknowns, namely a and the components of g , so it should not be too difficult to find solutions. Note that no derivatives of a appear in the equations, so it is natural to first eliminate a from the equations and then consider the system obtained for the metric g . This is the approach we will follow below.

Of course when we suppose that g is of specific form it is not a priori clear how many independent differential equations are obtained in this way. Note that as a PDE system QE system is essentially different from RR and PRS systems. QE is a fully nonlinear system, i.e. nonlinear in highest derivatives while RR and PRS systems are quasilinear.

Once the appropriate T is found b and ω can easily be computed. By taking traces in the formula (3.3) one sees immediately that $b = \text{sc} - na$ and then ω can be solved from the linear system

$$T_{ij} \omega^j = (\text{sc} - na) \omega_i$$

Still another way to characterize the QE case which is actually useful when considering examples is that Ri has a simple eigenvalue $\text{sc} - (n-1)a$ corresponding to the eigenvector ω , and all vectors orthogonal to ω are eigenvectors with eigenvalue a whose multiplicity is thus $n-1$. But this formulation immediately gives

THEOREM 3.4 If the Ricci tensor is recurrent and $n = 3$ then it is automatically quasi Einstein. If $n > 3$ the Ricci tensor cannot be both recurrent and quasi Einstein.

Proof. By Theorem 3.2 the eigenvalue structure of Ri can satisfy both RR and QE conditions only if $n = 3$. In this case if ω is the eigenvector corresponding to the zero eigenvalue we can write the Ricci tensor as follows

$$\text{Ri}_{ij} = \frac{\text{sc}}{2} \left(g_{ij} - \frac{\omega_i \omega_j}{|\omega|^2} \right).$$

□

On the other hand PRS and QE conditions can be satisfied in any dimension. In these cases the form of the Ricci tensor is as follows.

THEOREM 3.5 Let us suppose that both PRS and QE conditions are satisfied and let α be the associated one form. If $a \neq 0$ we have

$$\text{Ri}_{ij} = \frac{\text{sc}}{n-1} \left(g_{ij} - \frac{\alpha_i \alpha_j}{|\alpha|^2} \right)$$

If $a = 0$ then

$$g(\omega, \alpha) = 0 \quad , \quad \text{Ri}_{ij} = \text{sc} \frac{\omega_i \omega_j}{|\omega|^2} \quad \text{and} \quad \nabla_\alpha \omega = |\alpha|^2 \omega .$$

Proof. By multiplying formula (3.3) with α^i and applying Lemma 3.3, we have

$$a\alpha_j + b \frac{g(\omega, \alpha)}{|\omega|^2} \omega_j = 0 \tag{3.5}$$

If $a \neq 0$ and $g(\omega, \alpha) \neq 0$ then α and ω are linearly dependent and we may choose $\omega = \alpha$ which gives

$$a = \frac{\text{sc}}{n-1} \quad \text{and} \quad b = -a .$$

When $a = 0$ the first two statements are obvious. To get the third we take the covariant derivative of the formula $\text{Ri}_{ij} = \text{sc} \frac{\omega_i \omega_j}{|\omega|^2}$, and then multiply it with α^i . Then using the formula (3.2) and simplifying we get the result. □

LEMMA 3.6 Suppose that Ri satisfies the PRS condition and that $\text{sc} \neq 0$; then

- (i) if Ri is also QE with $a \neq 0$ then it is CO
- (ii) if Ri is also CO then it is QE

Proof. If the PRS condition is satisfied and $\text{sc} \neq 0$ then the Cotton tensor is

$$C_{ijk} = \frac{\text{sc}_{;k}}{2\text{sc}} \left(\text{Ri}_{ij} - \frac{\text{sc}}{n-1} g_{ij} \right) - \frac{\text{sc}_{;j}}{2\text{sc}} \left(\text{Ri}_{ik} - \frac{\text{sc}}{n-1} g_{ik} \right)$$

Now simply substituting the expression for Ri given in Theorem 3.5 shows that $C = 0$ which proves the statement (i).

On the other hand it is easy to check that the Cotton tensor of the above form can be zero only if the matrix rank of $\text{Ri}_{ij} - \frac{\text{sc}}{n-1} g_{ij}$ is one which is precisely the QE condition. □

There is a completely different way to construct QE metrics which we now describe. The form of the condition makes one think about conformal equivalence. Let g be a given metric and let $\hat{g} = \exp(2\lambda)g$ be a metric that is conformally equivalent to it. If now $\hat{\text{Ri}}$ is the Ricci tensor associated to \hat{g} then

$$\hat{\text{Ri}}_{ij} = \text{Ri}_{ij} - (n-2)(\lambda_{;ij} - \lambda_{;i}\lambda_{;j}) - (\Delta\lambda + (n-2)|\nabla\lambda|^2)g_{ij}$$

Hence if we can find a metric g and a function λ such that $\text{Ri}_{ij} = (n-2)\lambda_{;ij}$ then $\hat{\text{Ri}}$ is quasi Einstein:

$$\hat{\text{Ri}}_{ij} = -(\Delta\lambda + (n-2)|\nabla\lambda|^2) \exp(-2\lambda)\hat{g}_{ij} + (n-2)\lambda_{;i}\lambda_{;j}$$

Note that the equations are trivially satisfied if $\text{Ri} = \nabla\nabla\lambda = 0$. However, if $\nabla\lambda \neq 0$ there is still a nontrivial solution

$$\hat{\text{Ri}}_{ij} = -(n-2)|\nabla\lambda|^2 \exp(-2\lambda)\hat{g}_{ij} + (n-2)\lambda_{;i}\lambda_{;j}$$

The existence of solutions to PDE system $\text{Ri} = T$ where T is a given symmetric tensor is analyzed in [8]. It is instructive to consider this system from the point of view of overdetermined systems.

At the outset there are thus $\frac{1}{2}n(n+1)$ second order quasilinear PDE for the components of the metric so that it seems the system is determined. However, we have the Bianchi identity (2.3) which should be taken into account. Let us then define the Bianchi operator B by the formula

$$B(T) = 2 \operatorname{div}(T) - \nabla \operatorname{tr}(T) = 2g^{ij}T_{ik;j} - g^{ij}T_{ij;k}$$

Hence if the system $\text{Ri} = T$ has solutions then necessarily the condition $B(T) = 0$ must also be satisfied. Note that this is a system of n first order equations in metric g . Let us then define the modified Bianchi operator

$$\tilde{B}(T) = 2g^{ij}T_{ik,j} - g^{ij}T_{ij,k}$$

Note that the standard and covariant derivatives agree up to lower order corrections. Hence the components of $\tilde{B}(\text{Ri})$ are second order differential operators. This leads to the following system:

$$\begin{cases} \text{Ri} = T \\ \tilde{B}(\text{Ri}) = \tilde{B}(T) \\ B(T) = 0 \end{cases}$$

Interestingly the initial system $\text{Ri} = T$ is not elliptic while the above completed system is, provided that Ri is of full (matrix) rank. The ellipticity can then be used to show the existence of local solutions. In our case the system is

$$\begin{cases} \text{Ri} = (n-2)\lambda_{;ij} \\ \tilde{B}(\text{Ri}) = (n-2)\tilde{B}(\lambda_{;ij}) \\ B(\lambda_{;ij}) = 0 \end{cases} \quad (3.6)$$

However, from the point of view of existence of solutions this is a very different system since λ is also unknown, and moreover this is a third order system.

4 Setting up the computational problem

We will choose some families of metrics and try to construct metrics which satisfy one or more of the conditions in Definition 3.1. All conditions lead to systems of PDE whose nonlinearities are polynomial, and hence we can use `rifsimp` to analyse them.

It is known that the complexity of computing Gröbner basis is very bad (doubly exponential) in the worst case. Of course computing the involutive form is even more difficult. However, typically the time required for these computations is far from the worst case. On the other hand since there is no reasonable probability measure in the "space of all problems" there are no rigorous results on "average" complexity. Anyway all the solutions given below were obtained usually in few seconds and in any case in less than a minute with standard PC. Note that the decomposition of the system may take a lot more time than computing just the "most general" solution.

In all cases there actually were several components in the system, so that in the language of differential algebra the initial system was never prime differential ideal. It is not quite clear how to interpret this geometrically. Of course as components they provide essentially different solutions to PDE systems. From this it does not necessarily follow that the corresponding Riemannian manifolds are essentially different (i.e. not isometric). We did not attempt to study this problem. Since there were so many different components in all we will mostly give below only the most general one.

Let us now formulate more precisely the PDE systems that we are trying to solve.

Problem 4.1 (RR problem) Find a metric g such that $P_{ijk} = 0$ where

$$P_{ijk} = \text{sc Ri}_{ij;k} - \text{sc}_{;k} \text{Ri}_{ij} \quad (4.1)$$

This is a third order quasilinear system of $\frac{1}{2}n^2(n+1)$ PDE.

Problem 4.2 (PRS problem) Find a metric g such that $Q_{ijk} = 0$ where

$$Q_{ijk} = 2\text{sc Ri}_{ij;k} - 2\text{sc}_{;k} \text{Ri}_{ij} - \text{sc}_{;i} \text{Ri}_{kj} - \text{sc}_{;j} \text{Ri}_{ik} \quad (4.2)$$

This is a third order quasilinear system of $\frac{1}{2}n^2(n+1)$ PDE.

Problem 4.3 (CO problem) Find a metric g such that the Cotton tensor $C = 0$. This is a third order quasilinear system of $n^2(n-1)$ PDE.

Problem 4.4 (First QE problem) Find a metric g and a function a such that the matrix rank of $T = \text{Ri} - ag$ is one. According to (2.4) there are $\binom{n}{2}$ algebraically independent fully nonlinear second order equations.

Problem 4.5 (Second QE problem) Find a metric g and a function λ such that $\text{Ri} = (n-2)\nabla\nabla\lambda$. Here we have $\frac{1}{2}n(n+1)$ quasilinear second order equations in $\frac{1}{2}n(n+1)+1$ unknowns. Note that in the completed system (3.6) there are integrability conditions which are expressed in terms of (modified) Bianchi operator. However, there is no need to compute them explicitly since `rifsimp` takes care of them automatically.

Note that the numbers of equations given above is the maximum number of algebraically independent PDE in the initial system. This number is achieved, if the metric is assumed to be completely general. However, if we assume that the metric is of specific form we may initially have less equations.

But the number of equations is in fact not really important in the present context. We recall that also with polynomial ideals the number of generators of the ideal does not matter, and even the number of generators of the Gröbner basis does not give any useful information. In the same way it is very convenient when using the algorithm `rifsimp` that it is not necessary to check beforehand if the equations are actually algebraically independent; `rifsimp` takes care of that automatically. Hence in practice we simply compute the relevant tensor and then require that all of its components are zero; if there are some redundant equations in the system then `rifsimp` simply discards them.

In the following we will look for the solutions of these systems. Since our analysis is local we will always consider only the situation in a single coordinate system. All our examples are of the form "separation of variables"; in other words all unknown functions are of functions of one variable only. In this way our PDE systems reduce to ODE systems, and it is thus easier to find solutions. The actual form of the initial guess of the metrics is not very critical. Experimenting with different choices showed that it is not particularly hard to find nontrivial solutions. Hence the reader can easily modify our examples and find other solutions.

In the actual computations the use of inequations was quite convenient. We may just look for those solutions where some function or a more complicated expression is nonzero. This is very natural in the problems below if for example we are not interested in cases where some unknown functions vanishes, or that the differential of the scalar curvature vanishes, or that $\text{Ri} = 0$. This can be speed up significantly the computations because then `rifsimp` does not need to worry about irrelevant subcases.

Let us now briefly describe the output of `rifsimp`. The algorithm tries to express highest ranking derivatives in terms of lower ranking derivatives. Let $f = (f_1, \dots, f_k)$ be the unknown functions with $x = (x^1, \dots, x^n)$ as independent variables. Let α^j be some multiindices. Then the first part of output is as follows:

$$\partial^{\alpha^j} f_j = F_j(x, f, \dots) \quad , \quad 1 \leq j \leq m \quad (4.3)$$

In the arguments of F_j there are only derivatives of lower ranking than $\partial^{\alpha^j} f_j$. The number m is not known a priori. `rifsimp` also tries to eliminate the components f_j from equations as far as possible. In linear algebra the Gaussian elimination reduces the problem to upper triangular form. Of course "differential nonlinear upper triangular form" does not exist in general, but `rifsimp` tries to compute a representation which is as close to it as possible. In the examples below it is seen clearly how this works. Note that the representation may depend heavily on the ranking chosen: the number m , multiindices α^j and functions F_j are not intrinsic.

When the problems are nonlinear all the relevant information about the system cannot always be expressed as in (4.3). In these cases there are additional equations, called *constraints* in `rifsimp`, of the form

$$H_\ell(x, f, \dots) = 0 \quad , \quad 1 \leq \ell \leq s$$

where the highest ranking derivatives of the arguments of H_ℓ are present nonlinearly. Below we will see an example of this case also.

In addition to this there may be certain inequations in the output. When computing F_j and H_ℓ sometimes one has to make some decisions if certain expressions are zero or not. This is how the system decomposes: in the generic case one assumes that "typically" any expression is nonzero. The algorithm keeps track of these assumptions and gives them in the output. But of course assuming that some expression is zero can produce solutions which are not contained in the "general" solution. Potentially there can be a lot of these branch points so that computing the decomposition, called *casesplit* in `rifsimp`, can take much more time than computing the generic solution.

Note that the output of `rifsimp` has a lot of structure and contains a lot of information. So even if one is unable to actually explicitly solve the equations given in the output one typically can immediately obtain some important facts which characterize the possible solution set. In many cases considered below the output can even be easily used for numerical computations while it is not at all clear how a numerical solution could be computed using the initial system.

Now the fact that `rifsimp` tries to approach the "upper triangular form" makes also the explicit solution easier. One can first solve equations with fewer variables, and then substitute these solutions to equations which contain more variables, like back substitution in Gaussian elimination. In solving the equations we often used the command `dsolve` in MAPLE.

5 Results

5.1 3 dimensional case

Let us consider the following metric:

$$g = f_1(x^1)(dx^1)^2 + f_2(x^1)h_2(x^2)(dx^2)^2 + f_3(x^1)h_3(x^2)q(x^3)(dx^3)^2 \quad (5.1)$$

EXAMPLE 5.1 Problem 4.1 with (5.1).

In this case our PDE system has a priori 18 independent equations but actually we have only 14 (not necessarily independent) nonzero equations. The system splits into seven subsystems. However, six systems either require that some unknown functions are constants, or the corresponding solutions give only the trivial solution where β reduces to zero. Note that it is anyway possible that in those cases there are solutions which are not special cases of the one given below.

The remaining component of the system has three differential equations; first two equations for f_j :

$$f_2'' = \frac{f_2'(f_1'f_2 + f_2'f_1)}{2f_1f_2}$$

$$f_3'' = \frac{f_2f_3f_1'f_3' + 2f_1f_2(f_3')^2 - f_1f_3f_2'f_3'}{2f_1f_2f_3}$$

Evidently now one can give f_1 arbitrarily and then solve the remaining functions. However, one can actually eliminate one of the functions by solving f_1 and f_3 in terms of f_2 which gives the following family of solutions:

$$f_1 = \frac{c_2(f_2')^2}{f_2} \quad \text{and} \quad f_3 = c_1f_2^m.$$

Note that m need not be an integer. Then we have the third differential equation which contain h_j and f_j . However, when we substitute the above formulas the functions f_j disappear and we are left with

$$h_3'' = \frac{(2m-1)c_2h_2(h_3')^2 + mc_2h_3h_2'h_3' - m^2h_2^2h_3^2}{2mc_2h_2h_3}$$

Solving this for h_2 yields

$$h_2 = \frac{c_2h_3^{1/m}(h_3')^2}{h_3^2(c_2c_3 - m^2h_3^{1/m})} = \frac{c_2m^2(h')^2}{h(c_2c_3 - m^2h)}$$

where we have introduced a new function $h_3 = h^m$. Then writing f instead of f_2 we can write our final metric as

$$g = \frac{c_2(f')^2}{f}(dx^1)^2 + \frac{m^2c_2f(h')^2}{h(c_2c_3 - m^2h)}(dx^2)^2 + c_1f^mh^mq(dx^3)^2$$

Clearly one can choose constants and functions such that g is positive definite. For scalar curvature we get $sc = (1-m)c_3/(2mfh)$ and thus $\beta = -\nabla \ln(fh)$. Note that $m \neq 1$ because otherwise also $Ri = 0$.

EXAMPLE 5.2 Problem 4.2 with (5.1).

Now we have our PDE system which is very similar to RR case, but of course the solutions are different, by Theorem 3.2. Again we have 14 PDE, and computing with `rifsimp` get three cases where $\alpha \neq 0$. One case is

the following:

$$\begin{aligned} f_2' &= \frac{f_2 f_3'}{f_3} \\ f_3'' &= \frac{f_1 (f_3')^2 + f_3 f_1' f_3'}{2f_1 f_3} \\ h_3''' &= \frac{h_2 h_3^2 (h_2'' h_3' + 3h_2' h_3'') + 4h_2^2 h_3 h_3' h_3'' - h_2^2 (h_3')^2 - 2h_2 h_3 h_2' (h_3')^2 - 2h_3^2 (h_2')^2 h_3'}{2h_2^2 h_3^2} \end{aligned}$$

It turns out that in the first two equations we can solve f_1 and f_2 in terms of f_3 , and in the last one we get h_2 in terms of h_3 :

$$\begin{aligned} f_1 &= \frac{c_1 (f_3')^2}{f_3} \\ f_2 &= c_2 f_3 \\ h_2 &= \frac{(h_3')^2}{h_3 (c_3 h_3 + c_4)} \end{aligned}$$

Then writing $f_3 = f$ and $h_3 = h$ we get

$$\begin{aligned} g &= \frac{c_1 (f')^2}{f} (dx^1)^2 + \frac{c_2 f (h')^2}{h (c_3 h + c_4)} (dx^2)^2 + f h q (dx^3)^2 \quad \text{and} \\ \alpha &= -\frac{f'}{2f} dx^1 \quad , \quad \text{Ri} = -\frac{c_1 c_3 + c_2}{4c_1 c_2 h} \left(\frac{c_2 (h')^2}{c_3 h + c_4} (dx^2)^2 + q h^2 (dx^3)^2 \right) . \end{aligned}$$

Note that the solution obtained satisfies also the QE condition with $a \neq 0$, and hence also the CO condition by Lemma 3.6.

EXAMPLE 5.3 Problem 4.3 with metric (5.1).

Now we have 8 equations in the system. By computing with `rifsimp` get three cases and in the most general case we have

$$\begin{aligned} f_2''' &= F(f_1, f_2, f_3) \\ h_3'' &= H(f_1, f_2, f_3, h_2, h_3) \end{aligned}$$

where F and H are very complicated expressions, involving also the derivatives of its arguments, so that we do not write them down explicitly. The function H at the outset depends on f_j but if f_j satisfy the first equation then actually H does not depend on x^1 . The dependence of H on f_j is only through initial conditions of the first equation, and consequently by standard theorems we have the local solution, and the above system can even be used for numerical computations.

However, it turns out that one can describe the solution in a more explicit way. One can actually solve the first equation for f_3 using quadratures which yields

$$f_3(x^1) = \exp(\hat{F}(f_1, f_2, f_1', f_2', f_2''))$$

where in the expression \hat{F} there are also some integrals whose integrands depend on the variables indicated. Now substituting this expression to the second equation gives

$$2h_2 h_3 h_3'' - h_3 h_2' h_3' - 2h_2 (h_3')^2 + c_0 h_2^2 h_3^2 = 0$$

where c_0 is constant. Solving this gives

$$h_2 = \frac{(h_3')^2}{(c_1 - c_0 \ln(h_3)) h_3^2}$$

Hence we can choose f_1 , f_2 , h_3 and q freely and it is clear that this choice, and the choice of constants c_0 and c_1 , can be done in such a way that the metric is positive definite.

EXAMPLE 5.4 Problem 4.4 with metric (5.1).

Constructing the appropriate PDE system we obtain five nonzero equations. Since the function a appears algebraically and in some equations even linearly we can solve it and substitute back to the equations. Note

that one gets different families of solutions, depending on the choice of a . However, we will analyze only one particular family of solutions.

After choosing a we are thus left with one single PDE; `rifsimp` gives then us the following system:

$$\begin{aligned} f_2''' &= F_1(f_1, f_2, f_3) \\ f_3''' &= F_2(f_1, f_2, f_3) \\ h_3'' &= H(f_1, f_2, f_3, h_2, h_3) \end{aligned}$$

The expressions for F_j and H are again so big that we do not give them explicitly. Also the dependence of H on f_j is only through initial conditions as in the previous example, so that choosing f_1 and h_2 arbitrarily yields an ODE system in the standard form.

However, we can also solve the system explicitly; denoting $f_2 = f$ the first two equations give

$$\begin{aligned} f_1 &= \frac{c_1 c_3 (f')^2}{f_3 f} \\ f_3 &= c_3 m^{-m} f^{1-m} (c_2 f - 1)^m \end{aligned}$$

Substituting this into third equations yields

$$h_3'' = \frac{((3m-2)h_2 h_3' + 2(m-1)h_3 h_2')h_3'}{4(m-1)h_2 h_3}$$

Denoting $h_3 = h$ and solving for h_2 yields

$$h_2 = c_4 h^{(2-3m)/(2m-2)} (h')^2$$

After this it is straightforward to compute a , b and ω which gives

$$\begin{aligned} a &= \frac{m}{8(1-m)c_4 f} h^{(2-m)/(2m-2)} - \frac{c_2^2 f^2 + (m-2)c_2 f + (m-1)^2}{2c_1 m^m f^m} (c_2 f - 1)^{m-2} \\ b &= \frac{m}{8(m-1)c_4 f} h^{(2-m)/(2m-2)} + \frac{m(m-1)}{2c_1 m^m f^m} (c_2 f - 1)^{m-2} \\ \omega &= f h' (c_2 f - 1) \partial_{x^1} + (2-2m) h f' \partial_{x^2} \end{aligned}$$

EXAMPLE 5.5 Problem 4.5 with metric (5.1).

Here we see that λ_2 , h_3 and f_2 must be constants; for simplicity let us choose $\lambda_2 = h_3 = f_2 = 1$. Then for other functions we obtain

$$\begin{aligned} f_1 &= c_1 (\lambda_1')^2 \\ f_3 &= c_2 \lambda_1^2 \\ q_3 &= \frac{c_1 (\lambda_3')^2}{c_1 c_3 - c_2 \lambda_3^2} \end{aligned}$$

Note that there is no condition on h_2 , and also λ_1 and λ_3 can be freely chosen. This solution implies that $\text{Ri} = \nabla \nabla \lambda = 0$, but of course $\widehat{\text{Ri}}$ gives a nontrivial example of QE manifold.

5.2 First 4 dimensional case

Let us then consider a simple four dimensional case:

$$g = (dx^1)^2 + f(x^1)q(x^4)((dx^2)^2 + (dx^3)^2 + (dx^4)^2) \quad (5.2)$$

EXAMPLE 5.6 Problem 4.1 with metric (5.2).

It turns out that RR system has only solutions with $\text{Ri} = 0$ so there are no examples of this form. Here the use of inequations was very convenient. When one added to the PDE system the condition $\beta \neq 0$, `rifsimp` concluded that the system is inconsistent.

EXAMPLE 5.7 Problem 4.2 with metric (5.2).

This illustrates quite well how `rifsimp` handles the system and how the solutions can split into several (in this case two) families, so that we describe this in more detail. The equations of the PRS system give

$$\begin{aligned} f'' &= \frac{(f')^2}{2f} \\ q''' &= \frac{4qq'q'' - 3(q')^3}{q^2} \\ 4fq^2(q'')^2 - 10fq(q')^2q'' + 6f(q')^4 + 2q^4(f')^2q'' - 3q^3(f')^2(q')^2 &= 0 \end{aligned} \quad (5.3)$$

Note that the last equation depends on both variables x^1 and x^4 so it seems that there might not be solutions. The third equation is different from the other two in another way. In first two equations highest order derivative is explicitly given in terms of lower order derivatives. In the final equation the highest derivative, namely q'' appears non linearly and consequently cannot be given explicitly. Recall that `rifsimp` calls equations of this type constraints.

Anyway when solving the first equation we obtain $f = (c_1x^1 + c_0)^2$ and substituting this to the final equation gives

$$2f(2qq'' - 3(q')^2)(2c_1^2q^3 + qq'' - (q')^2) = 0$$

Now taking the first factor we have

$$\begin{aligned} 2qq'' - 3(q')^2 = 0 &\Rightarrow q = \frac{1}{(c_2x^4 + c_3)^2} \\ \alpha &= -\frac{c_1}{c_1x^1 + c_0} dx^1 \\ \text{Ri} &= -\frac{2(c_1^2 + c_2^2)}{(c_2x^4 + c_3)^2} \left((dx^2)^2 + (dx^3)^2 + (dx^4)^2 \right) \end{aligned} \quad (5.4)$$

while the second factor yields

$$\begin{aligned} 2c_1^2q^3 + qq'' - (q')^2 = 0 &\Rightarrow q = \frac{c_2^2}{c_1^2(\cosh(c_2x^4 + c_3))^2} \\ \alpha &= -\frac{c_1}{c_1x^1 + c_0} dx^1 + c_2 \tanh(c_2x^4 + c_3) dx^4 \\ \text{Ri} &= -c_2^2((dx^2)^2 + (dx^3)^2) \end{aligned} \quad (5.5)$$

Note that both families of solutions also satisfy the second equation of the system (5.3).

The solution (5.4) satisfies also the QE condition and hence by Lemma 3.6 also the condition CO. On the other hand the Ricci tensor corresponding to the solution (5.5) has two double eigenvalues and hence cannot be QE.

EXAMPLE 5.8 Problem 4.3 with metric (5.2).

The CO case is very easy: f is arbitrary and

$$q'' = \frac{3(q')^2}{2q} \Rightarrow q = \frac{1}{(c_1x^4 + c_0)^2}$$

Here we get the same q as in (5.4). Hence here PRS case is a subcase of CO case.

EXAMPLE 5.9 Problem 4.4 with metric (5.2).

After computing the minors we solve a from one of the equations which yields

$$a = -\frac{2f''q^3f + (f')^2q^3 + 2fq'q'' - (q')^2f}{4f^2q^3}$$

Substituting this expression to the system and then computing with `rifsimp` gives the following system for f and q :

$$\begin{aligned} f''' &= -\frac{-2ff''f' + (f')^3}{f^2} \\ q'' &= \frac{4f''q^3f - 4(f')^2q^3 + (q')^2f}{2qf} \end{aligned}$$

The first equation can be solved and then the second one is in the standard form:

$$f = 4c_3 \cosh(c_1 x^1 + c_2)^2$$

$$q'' = \frac{32c_1^2 c_3 q^3 + (q')^2}{2q}$$

Interestingly if $4c_1^2 c_3 = 1$ then q is a Weierstrass elliptic function.

EXAMPLE 5.10 Problem 4.5 with metric (5.2).

Here it is natural to suppose that $\lambda = \lambda_1(x^1)\lambda_4(x^4)$. The most general solution is now incompatible with the positive definiteness of the metric. However, we still have a nontrivial family solutions. First we set $\lambda_4 = 1$ and then compute $q = 1/(c_1 x^4 + c_0)^2$. So here again q must be the same as in CO case and in one of the PRS cases. Hence all metrics obtained in this way must satisfy also the CO condition.

Substituting the computed value of q to the system leaves us with the following equations:

$$2ff'' + ff'\lambda_1' + 8c_1^2 f + (f')^2 = 0$$

$$4f^2\lambda_1'' - 6ff'\lambda_1' - 12c_1^2 f - 3(f')^2 = 0$$

There is no explicit formula for solution but again by standard theorems the local solution exists. Note that the metrics obtained in this way satisfy the QE condition. Hence we have metrics g and \hat{g} which are both QE and which are conformally equivalent.

5.3 Second 4 dimensional case

Let us now consider

$$g = f_1(x^1)(dx^1)^2 + f_2(x^1)(dx^2)^2 + f_3(x^1)(dx^3)^2 + f_4(x^1)(dx^4)^2 \quad (5.6)$$

EXAMPLE 5.11 Problem 4.1 with metric (5.6).

It turns out that RR conditions force two of the functions f_2, f_3 and f_4 to be constants. Choosing for example $f_3 = f_4 = 1$ we have $\beta = \nabla \ln(\text{sc})$ where

$$\text{sc} = \frac{-2f_1 f_2 f_2'' + f_1 (f_2')^2 + f_2 f_1' f_2'}{2f_1^2 f_2^2}$$

In essence the problem reduces to the 2 dimensional case and of course in 2 dimensions any metric is RR.

EXAMPLE 5.12 Problem 4.2 with metric (5.6).

The system decomposes to 8 components and the most general one gives the following system:

$$f_2'' = \frac{(f_2 f_3 f_4 f_1' + f_1 f_3 f_4 f_2' - f_1 f_2 f_4 f_3' - f_1 f_2 f_3 f_4') f_2'}{2f_1 f_2 f_3 f_4}$$

$$f_3'' = \frac{(f_2 f_3 f_4 f_1' - f_1 f_3 f_4 f_2' + f_1 f_2 f_4 f_3' - f_1 f_2 f_3 f_4') f_3'}{2f_1 f_2 f_3 f_4}$$

$$f_4'' = \frac{f_2 f_3 f_4 f_1' f_4' + 2f_1 (f_4)^2 f_2' f_3' + f_1 f_3 f_4 f_2' f_4' + f_1 f_2 f_4 f_3' f_4' + f_1 f_2 f_3 (f_4')^2}{2f_1 f_2 f_3 f_4}$$

It turns out that one can solve this explicitly. Let us set $f_3 = f$; then the other functions are given by

$$f_1 = c_1 c_2 c_4 m^2 \exp(c_3 f^{m+1}) f^{(m^2 - m - 1)/(m+1)} (f')^2$$

$$f_2 = c_2 f^m$$

$$f_4 = c_4 \exp(c_3 f^{m+1}) f^{-m/(m+1)}$$

This yields

$$\alpha = \frac{(m - c_3(m+1)^2 f^{m+1}) f'}{2(m+1)f} dx^1$$

$$\text{Ri} = -\frac{c_3(m+1)^2}{2c_1 c_2 m^2} (dx^4)^2$$

Note that Ri also satisfies the QE condition with $a = 0$, see Theorem 3.5.

EXAMPLE 5.13 Problem 4.3 with metric (5.6).

In the CO case we have initially 6 equations, but `rifsimp` gives only the following 2 equations:

$$\begin{aligned} f_2''' &= F_2(f_1, f_2, f_3, f_4) \\ f_3''' &= F_3(f_1, f_2, f_3, f_4) \end{aligned}$$

Here the expressions of F_2 and F_3 are so big that we do not give them explicitly. Anyway choosing f_1 and f_4 arbitrarily we have a standard ODE system for f_2 and f_3 .

EXAMPLE 5.14 Problem 4.4 with metric (5.6).

There are three families of solutions and as usual we give the most general. First `rifsimp` gives:

$$\begin{aligned} f_2'' &= \frac{2f_1f_2^2f_3f_4f_4'' - f_1f_2^2f_3(f_4')^2 + f_1f_2^2f_4f_3f_4' - f_1f_2f_4^2f_2'f_3' + f_1f_3f_4^2(f_2')^2 - f_2^2f_3f_4f_1f_4' + f_2f_3f_4^2f_1f_2'}{2f_1f_2f_3f_4^2} \\ f_3'' &= \frac{2f_1f_2f_3^2f_4f_4'' - f_1f_2f_3^2(f_4')^2 + f_1f_2f_4^2(f_3')^2 - f_1f_3f_4^2f_2'f_3' + f_1f_3^2f_4f_2'f_4' - f_2f_3^2f_4f_1f_4' + f_2f_3f_4^2f_1f_3'}{2f_1f_2f_3f_4^2} \end{aligned}$$

If f_1 and f_4 are given arbitrarily this is in standard form so the local solution exists. Then we see that we can choose $\omega = dx^1$, and a and b are given by

$$\begin{aligned} a &= \frac{f_1f_2f_3(f_4')^2 + f_2f_3f_4f_1f_4' + f_1f_2f_4f_3f_4' - f_1f_3f_4f_2'f_4' - 2f_1f_2f_3f_4f_4''}{4f_1^2f_2f_3f_4^2} \\ b &= \frac{f_1f_2f_3(f_4')^2 + f_2f_3f_4f_1f_4' + f_1f_4^2f_2'f_3' - 2f_1f_2f_3f_4f_4''}{2f_1^2f_2f_3f_4^2} \end{aligned}$$

EXAMPLE 5.15 Problem 4.5 with metric (5.6).

Here we choose that λ also is only function of x^1 . This gives the system

$$\begin{aligned} f_2'' &= \frac{f_1f_3f_4(f_2')^2 + f_2f_3f_4f_1f_2' - f_1f_2f_4f_2'f_3' - f_1f_2f_3f_2'f_4' - 4f_1f_2f_3f_4f_2'\lambda'}{2f_1f_2f_3f_4} \\ f_3'' &= \frac{f_1f_2f_4(f_3')^2 + f_2f_3f_4f_1f_3' - f_1f_3f_4f_2'f_3' - f_1f_2f_3f_3'f_4' - 4f_1f_2f_3f_4f_3'\lambda'}{2f_1f_2f_3f_4} \\ f_4'' &= \frac{f_1f_2f_3(f_4')^2 + f_2f_3f_4f_1f_4' - f_1f_3f_4f_2'f_4' - f_1f_2f_4f_3'f_4' - 4f_1f_2f_3f_4f_4'\lambda'}{2f_1f_2f_3f_4} \\ \lambda'' &= \frac{f_1(f_2f_3f_4' + f_2'f_3f_4 + f_2'f_3'f_4) + 2\lambda'(f_1'f_2f_3f_4 + f_1f_2'f_3f_4 + f_1f_2f_3'f_4 + f_1f_2f_3f_4')}{4f_1f_2f_3f_4} \end{aligned}$$

This looks complicated but actually we can solve it in terms of f_2 . So denoting $f = f_2$ and $\nu = \sqrt{1 + n^2 + m^2}$ we obtain

$$\begin{aligned} f_1 &= c_1c_2c_3f^{-\nu-2}(f')^2 \\ f_3 &= c_2f^m \\ f_4 &= c_3f^n \\ \lambda &= -\frac{1 + n + m + \nu}{4} \ln(f) \end{aligned}$$

5.4 Third 4 dimensional case

Let us consider the following four dimensional case:

$$g = q(x^4)(dx^1)^2 + u(x^3)(dx^2)^2 + h(x^2)(dx^3)^2 + f(x^1)(dx^4)^2 \quad (5.7)$$

EXAMPLE 5.16 Problem 4.1 with metric (5.7).

In the RR case we have 16 equations. Again there are sub cases but the most general solution is the following. For h and u we have the equations

$$\begin{aligned} h''' &= \frac{h'(2hh'' - (h')^2)}{2h^2} \\ u'' &= \frac{u(h')^2 + h(u')^2 - 2huh''}{2hu} \end{aligned}$$

Again the second equation depends on h only through initial conditions. Hence by standard theorems local solutions exist for these equations. Then β is of the form $\beta = \beta_1 dx^1 + \beta_4 dx^4$ where β_j depend only on f and q .

EXAMPLE 5.17 Problem 4.2 with metric (5.7).

The system is incompatible with the requirement $\alpha \neq 0$, so there are no PRS metrics of this form.

EXAMPLE 5.18 Problem 4.3 with metric (5.7).

In this case the equations yield only solutions where $\nabla \text{Ri} = 0$, so there are no interesting examples of this form.

EXAMPLE 5.19 Problem 4.4 with metric (5.7).

Computing the minors and solving for a we obtain a system which requires that two of the functions f , h , u and q must be constants. This leads to a solution with nontrivial Ricci tensor, but this Ricci tensor is necessarily Einstein so there are no nontrivial quasi Einstein metrics of this form.

EXAMPLE 5.20 Problem 4.5 with metric (5.7).

Here the most general solution to the differential equations is incompatible with positive definiteness of the metric. For the next general system one has $\lambda_2 = \lambda_3 = f = 1$ and for other functions we obtain

$$\begin{aligned} 2h^2 h''' - h'(2hh'' - (h')^2) &= 0 \\ 2qq'' - (q')^2 &= 0 \\ 2huu'' + 2huh'' - h(u')^2 - u(h')^2 &= 0 \\ 4q\lambda_1'' + (q')^2 \lambda_1 &= 0 \\ 2q\lambda_4' - \lambda_4 q' &= 0 \end{aligned}$$

This has the following family of solutions

$$\begin{aligned} q &= (c_1 x^4 + c_0)^2 \\ u &= (c_2 x^3 + c_3)^2 \\ h &= (c_4 x^2 + c_5)^2 \\ \lambda_4 &= c_6 (c_1 x^4 + c_0) \\ \lambda_1 &= c_7 \cos(c_1 x^1) + c_8 \sin(c_1 x^1) \end{aligned}$$

Here $\text{Ri} = \nabla \nabla \lambda = 0$, but of course $\widehat{\text{Ri}}$ is nontrivial.

6 Conclusion

We have seen above that with appropriate methods one can readily get large classes of nontrivial examples of various classes of Riemannian manifolds. The particular initial form of the Riemannian metric was not critical; testing different choices revealed that typically one could always find solutions. The formal theory of PDE can be also useful for exploring other classes of Riemannian manifolds, and in general related questions in differential geometry. Spivak writes in [21, p. 189] that many problems in differential geometry are in fact problems of overdetermined PDE whose solution require that one knows all the integrability conditions. Since manipulating PDE systems by hand is typically extremely tedious, one has invented "incredibly concise and elegant ways to state the integrability conditions [...] without ever even mentioning partial derivatives." However, the tools which were used above to compute the involutive form of various systems can perhaps be helpful in many other contexts in differential geometry, at least in producing relevant (counter)examples in various situations.

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