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Subdirect Products of Rings without $*-$Reversible Elements

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Abstract

A left (right) zero divisor $a \in R$ is called right (left) $*-$reversible if $ax = 0 (xa = 0)$, for $x \in R, x \neq 0$, then $xa^* = 0 (a^*x = 0)$. In this note we prove that a $*-$prime involution ring is a $*-$compressible if and only if it has no $*-$reversible element. Moreover, we show that semiprime ring with involution R is a subdirect product of ring without $*-$reversible elements if and only if R is $*-$compressible. Several results related to $*-$compressible ring are obtained.

Keywords: $*-$reversible element, $*-$compressible ring

1 Introduction and Preliminaries

Throughout the paper R denotes an associative ring with identity. Recall that an involution on a ring R is an additional binary operation $*$, such that $(a+b)^* = a^* + b^*, (ab)^* = b^*a^*, (a^*)^* = a$, for all $a, b \in R$. Let S, T, N, K denote the set of all symmetric elements $(x = x^*)$, the set of all traces $(x + x^*)$, the set of all norms (xx^*) and the set of skew-symmetric elements $(x^* = -x)$ in R respectively. Not that $T \cup N \subseteq S$. A ring R is called $*-$prime ring if $AB = 0$ implies $A = 0$ or $B = 0$ where A and B are $*-$ideals of R (e.g., $I^* \subseteq I$) [2]. It is obvious that prime rings are $*-$prime. Some characterizations of $*-$prime rings can be found in [2]. A ring R is called semiprime if $A^2 = 0$ implies $A = 0$ where A is an ideal of R. Let R be a ring with involution, a left zero divisor $a \in R$ is called right $*-$reversible if $ax = 0$ implies $xa^* = 0$ for $x \in R$. Right $*-$reversible element and $*-$reversible element are defined analogously [3]. W.Fakieh and S.K.Nauman proved that if R has no $*-$reversible element then R has no nonzero symmetric divisors of zero [7]. Further work on $*-$reversible elements appears in [4], [3], [7].

A ring R is called $*-$compressible if for any $y \in T \cup N, ay^nb = 0$ implies $ayb = 0$, where n is a power of 2 [1]. In early work, Andrunakievic and Rjabuhin have shown that a ring R is without nilpotents if and only if R is a subdirect product of skew domains [5]. In [1, Proposition 2 and Theorem 2], Tao-Cheng Yit has proved that if R

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is $*-$prime then R is $*-$compressible if and only if every nonzero symmetric element in $T \cup N$ is a nondivisor of zero, and if R is a semiprime ring with involution then R is $*-$compressible if and only if R is a subdirect product of rings without nonzero symmetric divisors of zero.

In this note, we shall prove T.C.Yit results by considering the $*-$reversible elements instead of the symmetric divisors of zero in $T \cup N$. Moreover, we study some properties of $*-$reversible elements in $*-$compressible ring; In particular, we show that if R is $*-$compressible ring then the set of $*-$reversible elements is closed under multiplication and we prove that if the involution $*$ is proper involution (e.g. $aa^* = 0$ implies $a = 0$) and $s_1 s_2 s_3 = 0$, where s_i is $*-$reversible elements of $*-$compressible ring R, then the products of the s_i's is zero in any order.

2 Results

First, we recall some properties of $*-$compressible ring in [1].

Proposition 2.1. [1, *proposition 1*] *Let R be a $*-$compressible ring, $x \in R$.*

1. *If $s \in T \cup N$ with $s^n = 0$, then $s = 0$.*
2. *If $xx^* = 0$, then $x^* x = 0$.*

Proposition 2.2. [1, *proposition 2*] *Let R be a $*-$prime ring. Then R is $*-$compressible if and only if every nonzero symmetric elements in $T \cup N$ is a nondivisior of zero.*

The following proposition shall be useful in the proof of the results of this note.

Proposition 2.3. *Let a be a square zero element in $*-$compressible ring, then a is a skew symmetric element.*

Proof. Let $a^2 = 0$, then

$$a^* a(a+a^*)^2 aa^* = 0.$$

Since R is $*-$compressible, we have

$$a^* a(a+a^*)aa^* = 0.$$

Thus $a^* aa^* aa^* = 0$ and so, $(aa^*)^3 = 0$. Hence $aa^* = 0$ by proposition 2.1. Consequently, $(a+a^*)^2 = 0$ and so $(a+a^*) = 0$ since R is $*-$compressible, hence $a = -a^*$ □

Proposition 2.4 and Theorem 2.5 are analogous to [1, proposition 2] and [1, theorem 1] respectively, by considering $*-$reversible elements instead of symmetric zero divisors under the same conditions.

Proposition 2.4. *Let R be a $*-$prime ring. Then R is $*-$compressible if and only if R has no $*-$reversible elements.*

Proof. Let $x \in R$ be a nonzero $*-$reversible element of R, then there exists $0 \neq y$, such that $xy = 0$ and this implies that $yx^* = 0$.
Then we have the following equality for $r \in R$:

$$x(x^*ry^* + yr^*x)^2y^* = x[(x^*ry^*)^2 + (yr^*x)^2 + x^*ry^*yr^*x + yr^*xx^*ry^*]y^* = 0.$$

Since R is $*-$compressible, it follows,

$$x[x^*ry^* + yr^*x]y^* = 0$$

Therefore, $xx^*ry^*y^* = 0$. Since R is $*-$prime, we have $xx^* = 0$ or $(y^*)^2 = 0$. If $xx^* = 0$, then $rxx^* = 0$. Since x is $*-$reversible element we have, $xrx = 0$ then $x^*x^* = 0 \Rightarrow x^*(x^*r) = 0$, so $x^*rx = 0$. Again, R is $*-$prime ring, it follows $x = 0$.

If $(y^*)^2 = 0$ then $(yr^*ry^*)^2 = (yr^*ry)^2 = 0$ as $y^* = -y$ by proposition 2.3, then we have $yr^*ry^* = yr^*ry = 0$ since R is $*-$compressible. Hence $y = 0$ or $yr^* = 0$ by [2, Theorem 5.4], thus $y = 0$ and R has no $*-$reversible element.
Conversely, let R has no $*-$reversible element. By [7, theorem 1] R has no nonzero symmetric divisor of zero. Thus R is $*-$compressible ring by preposition 2.2. □

Theorem 2.5. *Let R be $*-$compressible and Q is the prime radical of R, then $Q = \{\cap P' : P' a * -$prime ideal such that R/P' has no $*-$reversible element$\}$.*

Proof. Let P be a prime ideal, then $N = C(P)$ is an m-system. By [1, Proposition 3] there exist a $*-$prime ideal P' such that R/P' is $*-$compressible and $P' \cap N = \phi$. Thus for such a $*$-prime ideal P', we have $P' \subseteq C(N) = P$ and R/P' is $*$-prime ring. So R/P' has no $*$-reversible element by proposition 2.4.

Therefore $Q \supseteq \cap\{P'|P'$ is $*-$prime ideal and R/P' has no $*-$reversible elements$\}$ and since Q is the minimal semiprime ideal, it follows $Q = \cap\{P'|P'$ is $*-$prime ideal and R/P' has no $*-$reversible elements$\}$. □

Now, we can derive our main theorem.

Theorem 2.6. *Let R be semiprime with involution, then R is $*-$compressible if and only if R is a subdirect product of rings without $*-$reversible elements.*

Proof. Let R be $*-$compressible ring then by theorem 2.5, R is a subdirect product of rings without $*-$reversible elements. Conversely, let R be a subdirect product of rings without $*-$reversible elements. Therefore, R has no nonzero symmetric zero divisor by [7, Theorem 1]. So R is $*-$compressible by [1, theorem 2]. □

Recall that the involution $*$ is proper involution if $aa^* = 0$ implies $a = 0$ [6]. Lemma 2.2 and lemma 2.8 below give us some properties of the set of $*-$reversible elements of $*-$compressible ring with proper involution.

Lemma 2.7. *Let R be a $*-$compressible ring and $*$ is proper involution then the set of $*-$reversible elements is closed under multiplication.*

Proof. Let a, b are $*-$reversible elements in R then there exists $y \neq 0, x \neq 0$, such that

$$ax = 0 \Rightarrow xa^* = 0.$$
$$yb = 0 \Rightarrow b^*y = 0.$$

Hence, $xa^*b^* = 0 \Rightarrow b(xa^*) = 0 \Rightarrow (ab)x = 0$, thus (ab) is a left zero divisor and since $b^*y = 0 \Rightarrow a^*b^*y = 0 \Rightarrow b^*ya = 0 \Rightarrow y(ab) = 0$. Therefore (ab) is a zero divisor. Now, let $(ab)h = 0$, then $[h(ab)]^2 = 0$ and by proposition 2.3, $h(ab) = -(ab)^*h^*$, so $h(ab)(ab)^*h^* = 0$, and $h(ab) = 0$ since $*$ is proper involution. Thus $a^*b^*h = a^*b^*hb^*a^* = b^*hb^*a^*a = 0$ and $hb^*a^*ab = 0$ since a, b are $*$-reversible elements. So, $h(ab)^*(ab)h^* = 0$. Again $*$ is proper involution then $h(ab)^* = 0$ and (ab) is a right $*-$reversible element.

To prove that (ab) is a left $*-$reversible element, let $h(ab) = 0$. By the same argument above it follows, $(ab)hh^*(ab)^* = 0$ and $(ab)h = 0$, hence $ha^*b^* = 0$ and $b^*a^*ha^*b^* = bb^*a^*ha^* = h^*(ab)(ab)^*h = 0$, thus $h^*(ab) = (ab)^*h = 0$. So, (ab) is a left $*-$reversible element and the set of $*-$reversible elements are closed under multiplication. □

Lemma 2.8. *Let a be a $*-$reversible element in a $*-$compressible ring R, with proper involution $*$, then:*

1. *If* $(ar)h = 0$ *then* $h(ar)^* = 0$ *for* $r, h \in R$.
2. *If* $h(ar) = 0$ *then* $(ar)^*h = 0$ *for* $r, h \in R$.
3. *If* $h(ra) = 0$ *then* $(ra)^*h = 0$ *for* $r, h \in R$.
4. *If* $(ra)h = 0$ *then* $h(ra)^* = 0$ *for* $r, h \in R$.

Proof. (1) Let $(ar)h = 0$, so $arhr^* = rhr^*a^*a = 0$ since a is a $*-$reversible element. Hence $[h(r^*a^*)(ar)]^2 = 0$. By proposition 2.3, $h(r^*a^*)(ar) = -(r^*a^*)(ar)h^*$ and we have, $h(r^*a^*)(ar)(r^*a^*)(ar)h^* = 0$. Since $*$ is proper involution, $h(r^*a^*)(ar) = 0$, it follows $h(ar)^*(ar)h = 0$ and again $*$ is proper involution, $h(ar)^* = 0$.

(2) Let $h(ra) = 0$, so $r^*h(ra) = aa^*r^*hr = 0$ and $[(ra)(a^*r^*)h]^2 = 0$. By the same argument above we have $(ra)(ra)^*h = 0$, thus $h^*(ra)(ra)^*h = 0$. Since $*$ is proper involution, $h^*(ra) = (ra)^*h = 0$.

(3) If $h(ar) = 0$ then $(rha)^2 = 0$ and $rha = -a^*h^*r^* = a^*h^*r^* = 0$ since R is $*$-compressible. Thus $0 = r^*rha = a^*r^*rha^*$, it follows $ha^*(a^*r^*r)^* = ha^*r^*rah^* = 0$ by (1). So $h(ra)^*(ra)h^* = 0$ and since $*$ is proper involution, $(ra)h^* = h(ra)^* = 0$, so $((ra)^*h)^2 = 0$ and $(ra)^*h = -h^*(ra)$, thus $((ra)^*h)(-h^*(ra)) = ((ra)^*h)(h^*(ra)) = 0$ and we get $(ra)^*h = 0$.

(4) Let $(ra)h = 0$, it follows, $(ahr)^2 = 0$ and $ahr = -r^*h^*a^*$, then $(ahr)(r^*h^*a^*) = 0$. Since $*$ is proper involution, $a^*r^*h^*a^* = 0$ and $aa^*r^*h^* = 0$, by (1) $h^*(aa^*r^*)^* = 0$ and we have, $h^*(ra)(a^*r^*)h = 0$. Thus $h^*(ra) = 0$ and $[(ra)h^*]^2 = 0$, hence $(ra)h^*h(ra)^* = 0$, so $(ra)h^* = h(ra)^* = 0$.

□

We conclude this note by the following theorem which is analogous to theorem 4 and remark 4 of [1].

Theorem 2.9. *Let R be a $*-$compressible and $*$ is proper involution. Then*

1. *If $s_1s_2s_3 = 0$ where s_i are $*-$reversible elements, then the product of the s_i's is zero in any order.*

2. *If $sxdyt = 0$, where s,d,t are $*-$reversible elements and $x,y \in R$, then :*

$$dytsx = tsxdy = xdyts = ytsxd = 0$$
$$sxytd = dysxt = sytxd = dsxyt = 0$$

Proof. (1) By lemma 2.7, s_is_j is $*-$reversible element for $i,j = 1,2,3$. Since $s_1s_2s_3 = 0$, it follows, $s_3^*s_1s_2 = s_2^*s_3^*s_1 = s_1^*s_2^*s_3^* = s_3s_2s_1 = 0$. Also, $s_3^*(s_1s_2) = (s_1s_2)^*s_3^* = s_2^*s_1^*s_3^* = s_3s_1s_2 = 0$, by similar way we can deduce that $s_2s_3s_1 = s_1s_3s_2 = s_2s_1s_3 = 0$.

(2) Since s,t,d are $*-$reversible elements and by lemma 2.8, we can get the following:

$$dytsx = tsx(dy)^* = (dy)^*(tsx)^* = tsxdy = 0$$
$$tsxdy = dy(tsx)^* = (tsx)^*(dy)^* = dytsx = 0$$
$$xdyts = yts(xd)^* = (xd)^*(yts)^* = ytsxd = 0$$

also, we can have,

$$xdyts = yts(xd)^* = s(xd)^*(yt)^* = (xd)^*(yt)^*s^* = sytxd = 0$$
$$tsxdy = (dy)^*tsx = (sx)^*(dy)^*t = t^*(sx)^*(dy)^* = dysxt = 0$$
$$ytsxd = sxd(yt)^* = d(yt)^*(sx)^* = (yt)^*(sx)^*d^* = dsxyt = 0$$

□

Conclusion

In this note we proved that semiprime ring R is a subdirect product of rings without $*$-reversible elements if and only if R is $*$-compressible. We proved some results related to $*$-compressible ring.

Data Availability
No data were used to support this study.

Conflicts of Interest
The author declares that there is no conflict of interest regarding the publication of this paper.

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