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A Heuristic Derivation of a Summation Formula Involving Euler Numbers

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Introduction

In the paper [2], the Gregory-Leibniz series $\frac{\pi}{2}=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$ was shown to exhibit a remarkable feature. The sum to 500,000 terms exhibits an error as early as the sixth decimal place as expected by the alternating series test. However, the next ten decimal places are correct! And there is more of the same astonishing behavior as the following shows.

$$\frac{\pi}{2}\approx 2\sum_{n=1}^{500{,}000}\frac{(-1)^{n-1}}{2n-1}$$

(1) $$=1.570795\ 326794\ 896620\ 231321\ 691634\ 751442\ 098645\ 6996876$$

$$\underset{1}{\underline{5}} \qquad \underset{-1}{\underline{20}} \qquad \underset{5}{\underline{4}} \qquad \underset{-61}{\underline{645}}$$

Here every digit is correct expect those underlined. Below these underlined digits is the amount that must be added to make the number correct. (These numbers, 1, -1, 5, -61, …, are a special sequence known as Euler numbers as will be described below.)

It is the purpose of this short note to give a heuristic derivation of the Boole summation formula

$$\sum_{n=0}^{\infty}(-1)^n f\left(x+n+M+\frac{1}{2}\right)\sim\sum_{n=0}^{\infty}\frac{E_{2n}}{2^{2n+1}(2n)!}f^{(2n)}(x+M)\ ,$$

We use this formula to explain the remarkable features just described. The series on the right is called *asymptotic*, such series diverge in most interesting cases. Notice the sign " ~ " which means "is asymptotic to". By definition we say that $g(x)\sim h(x)$ if $\lim_{x\to\infty}\frac{h(x)}{g(x)}=1$.

Heuristic Derivation of the Boole Summation Formula

The Euler numbers E_n are defined by the generating function

$$\operatorname{sech}(s)=\sum_{n=0}^{\infty}\frac{E_n}{n!}\ s^n.$$

The first few Euler numbers are listed in the table:

E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9
1	0	-1	0	5	0	-61	0	1365	0

Because the hyperbolic secant is even, all the odd subscripted Euler numbers vanish and we can write

$$\operatorname{sech}(s) = \frac{2}{e^s + e^{-s}} = \frac{2e^s}{e^{2s}+1} = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!}\, s^{2n}.$$

Let $s = t/2$ and get

$$\frac{2e^{t/2}}{e^t+1} = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!}\, \frac{t^{2n}}{2^{2n}}$$

Multiplying by e^{Mt} we get

$$\frac{2e^{\left(M+\frac{1}{2}\right)t}}{e^t+1} = e^{Mt}\sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!}\, \frac{t^{2n}}{2^{2n}}\ .$$

We now begin our heuristic derivation of the summation formula. We begin by replacing t by the derivative operator D and get

(2) $$\frac{2e^{\left(M+\frac{1}{2}\right)D}}{e^D+1} = e^{MD}\sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!}\, \frac{D}{2^{2n}}^{2n}\ .$$

Of course this formal manipulation is not rigorous mathematics and perhaps shocking to the modern reader. But it has a long tradition of yielding difficult formulas in much fewer steps than rigorous approaches. See the book by Davis [3] for many examples. Without apology we continue in this spirit.

Using the geometric series $\frac{1}{e^D+1} = \sum_{n=0}^{\infty}(-1)^n e^{nD}$ we rewrite the left side of (2) as

$$\sum_{n=0}^{\infty}(-1)^n e^{\left(n+M+\frac{1}{2}\right)D} = e^{MD}\sum_{n=0}^{\infty} \frac{E_{2n}}{2(2n)!}\, \frac{D}{2^{2n}}^{2n}\ .$$

Now consider how the operator e^{aD} should act on the function $f(x)$

We replace e^{aD} with it's Taylor's series and get

$$e^{aD} f(x) = \sum_{n=0}^{\infty} \frac{a^n D^n}{n!} f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x) a^n}{n!} = f(x+a)\ .$$

Thus the operator e^{aD} transforms the function $f(x)$ to $f(x+a)$.

Using this we let the last operator expression act on the function $f(x)$ and get

$$\sum_{n=0}^{\infty}(-1)^n e^{\left(n+M+\frac{1}{2}\right)D} f(x)=\sum_{n=0}^{\infty}\frac{E_{2n}}{2^{2n+1}(2n)!}e^{MD}D^{2n}f(x)\ ,$$

and so

(3) $$\sum_{n=0}^{\infty}(-1)^n f\left(x+n+M+\frac{1}{2}\right)\sim\sum_{n=0}^{\infty}\frac{E_{2n}}{2^{2n+1}(2n)!}f^{(2n)}(x+M)\ .$$

This is the Boole summation formula found on page 684 of [2] where a more careful, but more difficult, mathematical analysis is suggested. From the left side of (3) we see that for convergence we need $f(x)\to 0$ as $x\to\infty$. (But with so much heuristic thinking, we might be inclined to ignore this.) The series on the right of (3) is asymptotic, and diverges in most interesting cases. We note that the Euler numbers grow very rapidly. In fact

$$\left|E_{2n}\right|\sim\frac{4^{n+1}(2n)!}{\pi^{2n+1}}\ ,$$

so $f(x)$ should be infinitely differentiable and these derivatives should also approach zero as $x\to\infty$. Perhaps more is needed, but our heuristic reasoning obscures these conditions. This completes our heuristic derivation of the needed summation formula (3).

Application of the Summation Formula

For completeness, we give our version of the application found in [2]. Let $f(x)=\dfrac{1}{2x}$ and observe that $f^{(k)}(x)=\dfrac{(-1)^k k!}{2x^{k+1}}$ so we have from (3)

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2(x+n+M)+1}\sim\sum_{n=0}^{\infty}\frac{E_{2n}}{2^{2n+2}(x+M)^{2n+1}}\ .$$

Next we set $x=0$ and take M to be a multiple of 4 and write $M=N/2$ (so that N is even) to get

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+N+1}\sim\frac{1}{2}\sum_{n=0}^{\infty}\frac{E_{2n}}{N^{2n+1}}$$

Simply changing the index on LHS we get

$$\sum_{n=N/2+1}^{\infty}\frac{(-1)^{n-1+N/2}}{2n-1}=\sum_{n=N/2+1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\sim\frac{1}{2}\sum_{n=0}^{\infty}\frac{E_{2n}}{N^{2n+1}}\ .$$

Since $\dfrac{\pi}{4}=\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{2n-1}=\sum_{n=1}^{N/2}\dfrac{(-1)^{n-1}}{2n-1}+\sum_{n=N/2+1}^{\infty}\dfrac{(-1)^{n-1}}{2n-1}$ we can now write

(4) $$\frac{\pi}{4}-\sum_{n=1}^{N/2}\frac{(-1)^{n-1}}{2n-1}\sim\frac{1}{2}\sum_{n=0}^{\infty}\frac{E_{2n}}{N^{2n+1}}.$$

(This is equation (a) of Theorem 1 on page 683 of [2].)

We can now write (4) as

$$\frac{\pi}{2}\sim 2\sum_{n=1}^{N/2}\frac{(-1)^{n-1}}{2n-1}+\left(\frac{1}{N}-\frac{1}{N^3}+\frac{5}{N^5}-\frac{61}{N^7}+\frac{1365}{N^9}-\cdots\right)$$

Now set $N=1,000,000$ and get

$$\frac{\pi}{2}\sim 2\sum_{n=1}^{500,000}\frac{(-1)^{n-1}}{2n-1}+\left(\frac{1}{10^6}-\frac{1}{10^{18}}+\frac{5}{10^{30}}-\frac{61}{10^{42}}+\frac{1365}{10^{54}}-\cdots\right) .$$

Carefully comparing this with (1), reveals the secret behind the remarkable result displayed in the introduction.

Final Remarks

The Euler-Maclaurin summation formula, is the best known asymptotic formula. An assessable rigorous proof is found in [1], while several interesting heuristic proofs in the spirit of this paper are in [3]. However, the Boole summation formula, and the special case examined here, is much less familiar.

Rigorous derivations of summation formulas are much more involved than the simple reasoning presented here. We hope that the heuristic reasoning will make this interesting phenomenon assessable to a wider audience. Hardy's book [4] is the classic on asymptotic and other divergent series. A simpler examination of an asymptotic series is found in [5].

References

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