"Science Stays True Here"
Journal of Mathematics and Statistical Science (ISSN 2411-2518), Vol.5, 263-273 | Science Signpost Publishing

Second Order Optimality Conditions for a Semilinear Elliptic Control Problem of Infinite Order with Pointwise Control Constraints

S. A. El-Zahaby and Samira El-Tamimy

Department of Mathematics, Faculty of Science,
Al-Azhar University [For Girls],Nasr City, Cairo, Egypt.
E-mail: elzahaby_somiah@hotmail.com, al.tamemy505@yahoo.com

Abstract

An optimal control problem for a semilinear elliptic equation with infinite order is investigated, where pointwise constraints are given on the control. First order necessary optimality conditions are derived, second-order sufficient optimality condition is established that consider strongly active constraints.

Keywords: Distributed control, semilinear elliptic equation, infinite order operator, pointwise control constraint, necessary optimality conditions, second order sufficient optimality conditions.

1 Introduction

Optimal control problems governed by a semilinear elliptic partial differential equations have already been considered by many authors. We refer only to the papers of Casas [1], to the book of Tröltzch [19], and the reference there in. Meanwhile, the existence of an optimal control and the first order necessary optimality conditions are well investigated. It is known that in the case of nonlinear equations the first order conditions are not in general sufficient for optimality so that we are going to derive a second order conditions. In this paper, we study an optimal control problem for a class of semilinear elliptic distributed control problem governed by elliptic operator of infinite order with pointwise control constraint. The aim is to derive the first-order necessary and the second order sufficient optimality conditions by using [4, 5, 19]. For the elliptic distributed control problems of infinite order the second-order sufficient conditions were estiblished in the paper by El-zahaby [9]. In the study of the Caushy Dirichlet problem by Dubinskii [6, 7]

$$\begin{aligned} L(u) &= \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma} u) = h(x), \qquad x \in \Omega \\ D^{|\omega|} u(x)|_{\partial\Omega} &= 0, \qquad |\omega| = 0, 1, 2, \cdots \end{aligned}$$

The Sobolev space of infinite order which defined by

$$w^{\infty}\{a_{\alpha}, p_{\alpha}\}(\Omega) = \{u(x) \in C_0^{\infty}(\Omega) : p(u) \equiv \sum_{|\alpha|=0}^{\infty} \|D^{\alpha} u\|_{p_{\alpha}}^{p_{\alpha}} < \infty\}$$

where $a_{\alpha} \geq 0$ and $p_{\alpha} \geq 1$ are numerical sequences and $\|.\|_p$ is the canonical norm in the space $L_p(G)$.

Gali et al. [14] presented a set of inequalities defining on an optimal control of a system governed by self-adjoint elliptic operators with an infinite number of variables.

Subsequently Lions suggested a problem related to this result but in different direction by taking the case of operators of an infinite order with finite dimensions.

Gali has solved this problem, the result has been published in [12].

Moreover, I. M. Gali et. al. [12–17] presented some control problems generated by both elliptic and hyperbolic linear operator of an infinite order with finite number of variables.

Necessary conditions for control problems governed by elliptic variational inequalities with an infinite number of variables, obtained by El-Zahaby [8]

El-Zahaby et al [11] obtained the optimal control of problems governed by variational inequalities of an infinite order with finite domain.

We refers for instance, to Cases [1] for the first-order necessary optimality conditions, Casas, Tröltzsch and Unger [5] for the second-order sufficient conditions.

For the elliptic case with quadratic objective and linear equation of an infinite order, this obtained by El-Zahaby et. al. [10], and a semilinear problem of an infinite order with finite dimension, this obtained by El-Zahaby [9].

For the papers which a close connection to our work, we refer to [2–5, 18] and reference given there in.

This paper is structed as follows:

In section two, introduce for functional space of an infinite order with finite dimension.

In section three, a semilinear elliptic control problem for infinite order operator with finite dimension is considered.

In section four, we derive the first-order necessary condition.

In section five, introduce the formal Lagrange method to determin the actual form of the adjoint equation.

In section six, second-order sufficient optimality conditions are obtained.

2 Some Function Spaces

The embedding problems for non-trivial Sobolev spaces of infinite order are investigated in [6, 7].

An embedding criterian established in terms of the characteristic functions of these space.

In this case

$$W^{\infty}\{a_\alpha, 2\} \subseteq L_2(R^n) \subseteq W^{-\infty}\{a_\alpha, 2\}$$

where,

$$W^{\infty}\{a_\alpha, 2\} = \{\phi \in C^{\infty}(R^n) : \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha \phi\|_2^2 < \infty\}$$

be Sobolev space of infinite order of periodic function defined on all of R^n and $W^{-\infty}\{a_\alpha, 2\}$ denotes their topological dual with respect to $L_2(R^n)$, we recall that $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index for differentiation and $|\alpha| = \alpha_1 + \cdots + \alpha_n$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots (\partial x_n^{\alpha_n})}, \qquad a_\alpha > 0,$$

is a numerical sequence, and $\|.\|_2$ is the canonical norm in the space $L_2(R^n)$, (all functions are assumed to be real value).

Analogous to the above chain we have

$$W_0^{\infty}\{a_\alpha, 2\} \subseteq L^2(R^n) \subseteq W_0^{-\infty}\{a_\alpha, 2\}$$

Let us consider the elliptic operator of an infinite order with finite dimension [12]

$$Ay = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y \quad a_\alpha > 0. \tag{2.1}$$

This operator is bounded self-adjoint elliptic operator mapping $W_0^{\infty}\{a_\alpha, 2\}$ onto $W_0^{-\infty}\{a_\alpha, 2\}$.

We introduce a continuous bilinear form on $W_0^{\infty}\{a_\alpha, 2\}$

$$\begin{aligned}
\pi(y, \phi) &= (Ay, \phi) \\
&= \sum_{|\alpha|=0}^{\infty} \Big((-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x), \phi(x)\Big)_{L^2(R^n)}, \ a_\alpha \geq 0 \\
&= \sum_{|\alpha|=1}^{\infty} \Big((-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x), \phi(x)\Big)_{L^2(R^n)} + q(x)(y(x), \phi(x))_{L^2(R^n)}
\end{aligned}$$

where $q(x)$ is a real valued function from $L_2(R^n)$ such that $q(x) \geq \nu$, $1 \geq \nu > 0$.

The ellipticity of A is sufficient from the coerciveness of $\pi(u, v)$ on $W^{\infty}\{a_\alpha, 2\}$, see [12] In fact

$$\begin{aligned}
\pi(u,u) &= (Au, u) \\
&= \sum_{|\alpha|=1}^{\infty} (a_\alpha D^\alpha u(x), D^\alpha u(x))_{L^2(R^n)} + q(x)(u(x), u(x))_{L^2(R^n)} \\
&\geq \sum_{|\alpha|=1}^{\infty} (a_\alpha D^\alpha u(x), D^\alpha u(x))_{L^2(R^n)} + \nu(u(x), u(x))_{L^2(R^n)} \\
&= \sum_{|\alpha|=1}^{\infty} \left(a_\alpha \|D^\alpha u\|^2_{L^2(R^n)} \right) + \nu \|u(x)\|^2_{L^2(R^n)} \\
&= \sum_{|\alpha|=1}^{\infty} \left(a_\alpha \|D^\alpha u\|^2_{L^2(R^n)} \right) + \nu \sum_{|\alpha|=1}^{\infty} \left(a_\alpha \|D^\alpha u\|^2_{L^2(R^n)} \right) \\
&- \nu \sum_{|\alpha|=1}^{\infty} \left(a_\alpha \|D^\alpha u\|^2_{L^2(R^n)} \right) + \nu \|u(x)\|^2_{L^2(R^n)} \\
&= \nu \|u\|^2_{W^\infty\{a_\alpha,2\}} + (1-\nu) \sum_{|\alpha|=1}^{\infty} \left(a_\alpha \|D^\alpha u\|^2_{L^2(R^n)} \right)
\end{aligned}$$

Then

$$\pi(u,u) \geq \nu \|u\|^2_{W^\infty\{a_\alpha,2\}}. \tag{2.2}$$

3 Formulation of the optimal control problem

In this paper, we consider the optimal control problem to minimize

$$(P) \quad \begin{cases}
\min J(y,u) := \dfrac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \dfrac{k}{2}\|u\|^2_{L^2(\Omega)}, & (3.1) \\
\text{subject to} & \\
Ay + d(x,y) = u \qquad \text{in} \quad \Omega & (3.2) \\
y^{|w|}|_\Gamma = 0 \qquad |w| = 0, 1, 2, \cdots & \\
\text{and to the constraints on the control} & \\
u_a(x) \leq u(x) \leq u_b(x) \qquad \text{a.e in} \quad \Omega & (3.3)
\end{cases}$$

where A is the elliptic operator of infinite order $A \in L(W^\infty\{a_\alpha, 2\}, W^{-\infty}\{a_\alpha, 2\})$, and having the form (2.1) and $d : \Omega \times R \to R$ is a function. The function u denotes the control in the space of control $\mathcal{U}$ and $y(u)$ is the solution (state of the function) associated to the control u, $u_a, u_b \in L^\infty(\Omega)$ with $u_a(x) \leq u_b(x)$.

Let us consider the set of admissible control by

$$U_{ad} = \{u \in L^\infty(\Omega) : u_a(x) \leq u(x) \leq u_b(x) \quad f.a.a \quad x \in \Omega\}$$

The norm functional above is a convex with respect to y and the set U_{ad} is a convex , closed and bounded in $L^2(\Omega)$. Nevertheless, the functional J is in general nonconvex, because the equation (3.2) is nonlinear equation . Therefore, the discussion of the second order condition is reasonable. We should mention a theory of the optimality conditions for convex problems with a semilinear equations, of. [9, 19].

The partial differential equation is considered in the following sense.

Definition 3.1. A function $y \in W^{\infty}\{a_\alpha, 2\}$ is said to be a weak solution of the partial differential equation (3.2) if for all $v \in W^{\infty}\{a_\alpha, 2\}$ the equation

$$\sum_{|\alpha|=0}^{\infty} \int_\Omega a_\alpha (D^\alpha y)(x)(D^\alpha v)(x)dx + \int_\Omega d(x,y)vdx = \int_\Omega uvdx$$

is vaild

To make this well defined, we impose the following assumptions:

Assumption A1:

The set $\Omega \subset R^N, N \geq 2$ is assumed to be a bounded Lipschitz domain with sufficiently smooth boundary Γ and A is an elliptic operator and the coerciveness condition (2.2) of $\pi(y,y)$ on $W^{\infty}\{a_\alpha, 2\}$ is satisfied if $q(x) \geq \nu$, $1 \geq \nu > 0$.

The function $d : \Omega \times R \to R$ is bounded and measurable with respect to $x \in \Omega$ for any fixed $y \in R$ and is continuous. Furthermore, the function d is monotone increasing with respect to y for almost all x. This property of the function d guarantees the existence of unique weak solution of partial differential equation (3.2) for each fixed $u \in U_{ad}$ see [19]

Assumption A2:

The function d is twice differentiable with respect to y for almost every $x \in \Omega$. Morever, it satisfy the boundedness and a local lipschitz conditions of order $k = 2$. The bounds $u_a, u_b : E \to R$ belong to $L^\infty(E)$ for $E = \Omega$ and satisfy the conditions $u_a(x) \leq u_b(x)$ for almost every $x \in E$.

Additionally $d_y(x,y) \geq 0$ for almost every $x \in \Omega$ and all $y \in R$. Moreover, there is a set $E_d \subset \Omega$ of positive measure and a constant λ_d such that

$$d_y(x,y) \geq \lambda_d \qquad \forall \quad x \in E_d \; \forall \; y \in R.$$

The proof of the following theorem can be found in [9, 19]

Theorem 3.1. *(Well-Posedess of the state equation) With Assumption A1 holding, then the semilinear elliptic control problem* (3.2) *admits for every* $u \in L^2(\Omega)$ *a unique solution* $y \in W^{\infty}\{a_\alpha, 2\} \cap C(\bar{\Omega})$ *with*

$$\|y\|_{W^{\infty}\{a_\alpha,2\}} + \|y\|_{C(\bar{\Omega})} \leq C\|u\|_{L^2(\Omega)} \tag{3.4}$$

.

4 First-Order necessary Conditions

We introduce the control-to-state operator

$$G : L^2(\Omega) \to W^{\infty}\{a_\alpha, 2\}(\Omega) \cap C(\bar{\Omega}), \qquad u \mapsto y.$$

We transformed the control problem (P) by using the solution operator G into the reduced quadratic optimization problems in term of u, namely

$$\begin{aligned} f(u) &:= J(y,u) \\ &= J(G(u),u) \\ &= \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{k}{2}\|u\|^2_{L^2(\Omega)} \\ &= \frac{1}{2}\int_\Omega (G(u) - y_d)^2 \; dx + \frac{k}{2}\int_\Omega u^2 \; dx \end{aligned}$$

the problem is to find $\min f(u)$. where

$$U_{ad} = \{u \in L^\infty(\Omega) : u_a(x) \leq u(x) \leq u_b(x)\}$$

Definition 4.1. A control $\bar{u} \in U_{ad}$ is said to be an optimal if it satisfies, together with the associated optimal state $\bar{y} = y(\bar{u})$, the inequality

$$J(y(\bar{u}), \bar{u}) \le J(y(u), u) \qquad \forall \quad u \in U_{ad}.$$

A control is said to be a locally optimal in the sense of $L^2(\Omega)$ if there exists some $\epsilon > 0$ such that the above inequality holds for all $u \in U_{ad}$ such that $\|u - \bar{u}\|_{L^2(\Omega)} < \epsilon$.

The next theorem states the existence of an optimal solution for (P).

Theorem 4.1. *Suppose that Assumptions A1, A2 are holds. If the admissible set is not empty, then Problem (P) admits at least one local solution in the sense of Definition 4.1.*

Proof. The proof is noreor less standard. In all what follows, we denote the optimal solution by $(\bar{y}, \bar{u})$ where $\bar{y} = G(\bar{u})$ and $\bar{u}$ is said to be an optimal control. By $\lambda > 0$, and Assumptions A1 and A2, we find a bounded minimizing sequence $\{u_n\} \subset L^2(\Omega)$, $y_n = G(u_n)$ and we can assume without loss of generality $u_n \to \bar{u}$, $n \to \infty$. By Theorem 3.1, the associated sequence $\{y_n\}$ is bounded in $W^\infty\{a_\alpha, 2\}$. Hence in [6, 7] $W^\infty\{a_\alpha, 2\}$ is completely imbedded in $L^2(\Omega)$, we can assume

$$y_n \to \bar{y} \qquad \text{in} \quad L^2(\Omega)$$

Together with boundedness in $C(\Omega)$ that follows from (3.4), this yields

$$d(x, y_n) \to d(x, y) \qquad \text{in} \quad L^2(\Omega)$$

$$\bar{y} = G(\bar{u})$$

The optimality of $\bar{u}$ is a standard conclusion, therefore

$$j = \lim_{n \to \infty} J(y_n, u_n) = J(\bar{y}, \bar{u})$$

□

Remark 4.1. Obviously, all admissible controls must be bounded and measurable, since $u_a(x), u_b(x) \in L^\infty(\Omega)$ imply $u \in L^\infty(\Omega)$ because of the constraint $u_a(x) \le u(x) \le u_b(x)$.

Theorem 4.2. *Suppose that $\bar{u}$ be a local solution to (P) and U_{ad} convex then the variational inequality*

$$f'(\bar{u})(u - \bar{u}) \ge 0 \qquad \textit{for all} \quad u \in U_{ad} \tag{4.1}$$

holds.

The proof follows from a more general result see [19].

Theorem 4.3. *Let $\bar{u} \in U_{ad}$ be a local solution to (P). Then there exist a unique adjoint state $p \in W^\infty\{a_\alpha, 2\}$ defined by*

$$\begin{aligned} Ap + d_y(x, y)p &= y - y_d && \textit{in} \quad \Omega \\ p^{|w|}|_\Gamma &= 0 && |w| = 0, 1, 2, \cdots \end{aligned} \tag{4.2}$$

where p is the adjoint solution and A is the adjoint operator which take the same form in (2.1).

such that the variational inequality is given by

$$\begin{aligned} f'(\bar{u})h &= f'(\bar{u})(u - \bar{u}) \quad \forall \; u \in U_{ad} \\ &= \int_\Omega (p(x) + k\bar{u}(x))h(x) \; dx \ge 0 \end{aligned} \tag{4.3}$$

where p solves the adjoint equation (4.2).

From (4.3) several important conclusions can be drawn. First, a standard pointwise discussion reveals the following implications $for \quad a.a \quad x \in \Omega$. If $k > 0$

$$\bar{u}(x) = \begin{cases} u_a(x), & \text{if} \quad p(x) + k\bar{u}(x) > 0; \\ \epsilon\left[u_a(x), u_b(x)\right], & \text{if} \quad p(x) + k\bar{u}(x) = 0; \\ u_b(x) & \text{if} \quad p(x) + k\bar{u}(x) < 0. \end{cases} \tag{4.4}$$

If $k > 0$, then the second implication in (4.4) can be resolved for u . This somehow explains the following important projection formula:

$$\bar{u}(x) = P_{[u_a(x), u_b(x)]}\left(\frac{-1}{k}p(x)\right)$$

where $P_{[a,b]}$, $a < b$, is the projection of R on $[a, b]$ given by

$$P_{[a,b]}u := \min(b, \max(a, u))$$

This formula follows from (4.3) because it implies that $\bar{u}$ solves the optimization problem

$$\min_{v\in[u_a(x),u_b(x)]}(p(x)v + \frac{k}{2}v^2)$$

The projection formula permits to deduce higher regularity of any local optimal control, if $k > 0$.

If $k = 0$, then we cannot apply the projection formula but only (4.4). Then the optimal control admits the value u_a or u_b, where $p(x) \neq 0$. The control of this type is called bang-bang control.

We recall some results about the differentiability of the functionals involved in the control problem. For the detailed proofs, we refer to Cases and Mateos [2].

Lemma 4.4. *(First-and second order derivative of G). Under the Assumption A1 and A2 on d. Then the mapping $G : L^2(\Omega) \to W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega})$ is twice continuously Frechet differentiable. Moreover, its first derivative, denoted by $w = G'(u)h$, $h \in L^2(\Omega)$ is given by the solution of the linearlized equation*

$$\begin{aligned} Aw + d_y(x, y)w = h \qquad & in \quad \Omega \\ w^{|\alpha|}|_\Gamma = 0 \qquad & |\alpha| = 0, 1, 2, \cdots \end{aligned} \tag{4.5}$$

Furthermore, for every $z \in L^2(\Omega)$ the second derivative $z = G''(u)[u_1, u_2]$ is the solution of

$$\begin{aligned} Az + d_y(x, y)z = -d_{yy}(x, y)y_1y_2 \qquad & in \quad \Omega \\ z^{|w|}|_\Gamma = 0 \qquad & |w| = 0, 1, 2, \cdots \end{aligned} \tag{4.6}$$

where $y = G(u)$ and $y_i = G'(u)u_i \in W^\infty\{a_\alpha, 2\}$ for $i = 0, 1, 2$.

The existence of the first-and second order derivatives G' and G'' is proved by the implict function theorem [19].

Lemma 4.5. *Let Assumptions A1 and A2 are fulfilled. Then f is twice continuously Frechet differentiable from $L^2(\Omega)$ to R. Its first derivative is given by* (4.3)

For the second derivative, we obtain

$$f''(u)[u_1, u_2] = (y_1, y_2)_{L^2(\Omega)} + k(u_1, u_2)_{L^2(\Omega)} - \int_\Omega d_{yy}(x, y)y_1y_2p dx \tag{4.7}$$

Proof. From definition of the reduced cost functional

$$f(u) := J(G(u), u) = \frac{1}{2}\int_\Omega (G(u) - y_d(x))^2 + \frac{k}{2}\int_\Omega u^2(x)dx$$

We get

$$f'(u)h = (y - y_d, w)_{L^2(\Omega)} + k(u, h)_{L^2(\Omega)},$$

where $y = G(u)$ and $w = G'(u)h$ denotes the weak solution of the linearized equation (4.5) with the right hand side h.

Now, choosing p as a test function in the weak formulation of (4.5) and inserting w in the weak formulation of equation (4.2), we obtain

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} \int_\Omega a_\alpha D^{2\alpha} wpdx + \int_\Omega d_y(x, y)wpdx = \int_\Omega hpdx$$

$$\sum_{|\alpha|=0}^{\infty} \int_\Omega a_\alpha (D^\alpha w)(x)(D^\alpha p)(x)dx + \int_\Omega d_y(x, y)wpdx = \int_\Omega hpdx$$

and

$$\sum_{|\alpha|=0}^{\infty} \int_\Omega a_\alpha (D^\alpha p)(x)(D^\alpha w)(x)dx + \int_\Omega d_y(x, y)pwdx = \int_\Omega (y - y_d)wdx$$

Substracting one equation from the other finally yields

$$(y - y_d, w)_{L^2(\Omega)} = (h, p)_{L^2(\Omega)}$$

As a simple conclusion, the following expression for the directional derivative of the reduced functional f at $\bar{u}$ in the direction $h \in L^2(\Omega)$ yields

$$f'(\bar{u})h = \int_\Omega (p(x) + k\bar{u}(x))h(x)dx$$

We obtain the desired necessary optimality condition.

Applying again the Chain rule, we can calculate the second derivative as follow. First, we obtain

$$f'(u)u_1 = D_y J(G(u), u)G'(u)u_1 + D_u J(G(u), u)u_1$$

Next, we calculate the direction derivative of $f'(u)u_1$ in direction u_2. Invoking the product and chain rules, we find that

$$\begin{aligned} f''(u)[u_1, u_2] = & \\ &= D_y^2 J(G(u), u)[G'(u)u_1, G'(u)u_2] + D_u D_y J(G(u), u)[G'(u)u_1, u_2] \\ &\quad + D_y J(G(u), u)G''(u)[u_1, u_2] + D_y D_u J(G(u), u)[u_1, G'(u)u_2] \\ &\quad + D_u^2 J(G(u), u)[u_1, u_2] \\ &= J''(y, u)[(y_1, u_1), (y_2, u_2)] + D_y J(y, u)G''(u)[u_1, u_2] \end{aligned} \tag{4.8}$$

A similar discussion as above, where the abbreviations $z := G''(u)[u_1, u_2]$ denotes the weak solution of (4.6) with this, we obtain the expression

$$D_y J(y, u)z = \int_\Omega (y - y_d)z(x)dx$$

which can be transformed by using the adjoint state p, which is the weak solution to (4.2) hence, we have $(y - y_d, z)_{L^2(\Omega)} = -(d_{yy}(x, y)y_1 y_2, p)_{L^2(\Omega)}$. Using this in (4.8) finally yields to equation (4.7) □

5 Formal Lagrange method

The Lagrangian function associated with the problem (3.1)-(3.3) is defined by

$$\mathcal{L}(y,u,p) := J(y,u) - (Ay + d(x,y) - u, p)_{L^2(\Omega)} \tag{5.1}$$

we expect the existence of a function $p \in W^{\infty}\{a_\alpha, 2\} \cap C(\bar{\Omega})$

$$\frac{\partial \mathcal{L}}{\partial y}(\bar{y},\bar{u},p) = 0 \quad for \quad all \quad y \in W^{\infty}\{a_\alpha,2\}, \quad \frac{\partial \mathcal{L}}{\partial u}(\bar{y},\bar{u},p) \geq 0 \quad for \quad all \quad u \in U_{ad}$$

This leads to the weak formulation for the solution to the problem (4.2) and the variational inequalities

$$\int_\Omega (p(x) + k\bar{u}(x))h(x)\ dx \geq 0$$

The second derivative of $\mathcal{L}$ with respect to (y,u) is given by

$$\mathcal{L}''(\bar{y},\bar{u},p)[(y_1,u_1),(y_2,u_2)] = D^2_{(y,u)}\mathcal{L}(\bar{y},\bar{u},p)[(y_1,u_1),(y_2,u_2)]$$

It follow from (4.7) that

$$f''(u)[u_1,u_2] = \int_\Omega (y_1y_2 + ku_1u_2 - pd_{yy}(x,y)y_1y_2)dx = \mathcal{L}''(\bar{y},\bar{u},p)[(y_1,u_1),(y_2,u_2)]$$

Theorem 5.1. *Suppose that Assumption A1, A2 holds. Then the reduced functional $f : L^\infty(\Omega) \to R$,*

$$f(u) = J(y,u) = J(G(u),u),$$

is twice continuously Frechet differentiable. The second derivative of f can be expressed in the form

$$f''(u)[u_1,u_2] = \mathcal{L}''(\bar{y},\bar{u},p)[(y_1,u_1),(y_2,u_2)] \tag{5.2}$$

6 Second-order Sufficient Conditions

The requirement of coercivity or non-negativity of quadratic form $J''(\bar{u})v^2$ would be a too strong requirement. Therefore, we introduce the cone of critical directions as follows:

Definition 6.1. Let $\tau > 0$ be real number. Then the set

$$A_\tau(\bar{u}) = \{x \in \Omega : |(p(x) + k\bar{u}(x))| \geq \tau\}$$

is called a strongly active set.

We define the cone of ciritical direction by

$$\widehat{C}_\tau := \left\{ h \in L^2(\Omega) \;\middle|\; \begin{array}{lll} h(x) = 0 & a.e. \quad in\ A_\tau & \\ h(x) \geq 0 & where \quad \bar{u}(x) = u_a(x)\ and & x \notin A_\tau \\ h(x) \leq 0 & where \quad \bar{u}(x) = u_b(x)\ and & x \notin A_\tau \end{array} \right. \tag{6.1}$$

With these definition at hand, one can prove by standard arguments the following theorem covering the local optimality of $\bar{u}$ [19].

Lemma 6.1. *Suppose that Assumption A1, A2 holds. And let the functional*

$$f : L^\infty(\Omega) \to R.$$

be given by

$$f(u) := J(y,u) = J(G(u),u)$$

Then for each $M > 0$ there exist a constant $L(M) > 0$ such that

$$|f''(u+h)[u_1,u_2] - f''(u)[u_1,u_2]| \leq L(M)\|h\|_{L^\infty(\Omega)}\ \|u_1\|_{L^2(\Omega)}\|u_2\|_{L^2(\Omega)} \forall\ u_1,u_2,h \in L^\infty(\Omega). \tag{6.2}$$

Under to the given consideration, the theorem of Tröltzch [19] can be formulated to

Theorem 6.2. *(Second order necessary condition). If $\bar{u}$ is a local solution to (P) , then there holds*

$$f''(\bar{u})u^2 \geq 0 \qquad \forall\ u \in \widehat{C}_\tau$$

The following condition is a second-order sufficient condition. There exists some $\delta > 0$ such that

$$f''(\bar{u})u^2 \geq \hat{\delta}\|u\|^2_{L^2(\Omega)} \qquad \forall\ u \in \widehat{C}_\tau$$

By (5.2), this is equivalent to the condition

$$\mathcal{L}''(\bar{y},\bar{u},p)(y,u)^2 \geq \hat{\delta}\|u\|^2_{L^2(\Omega)} \qquad \forall\ u \in \widehat{C}_\tau$$

The following theorem on a second order optimality conditions deals with the two norm-discrepancy, i.e, the functional f is twice differentiable in $L^\infty(\Omega)$, but the inequality $f''(\bar{u})u^2 \geq \hat{\delta}\|u\|^2_{L^2(\Omega)}$ holds in $L^2(\Omega)$, for instance, [18].

Theorem 6.3. *Under Assumptions A1, A2. Let the control $\bar{u} \in U_{ad}$, together with the associated state $\bar{y} = G(\bar{u})$ and the adjoint state p, satisfy the first-order necessary optimality condition stated in* (4.3). *If, in addition $(\bar{y},\bar{u})$ satisfies the second-order sufficient condition*

$$f''(\bar{u})u^2 \geq \hat{\delta}\|u\|^2_{L^2(\Omega)} \qquad \forall\ u \in \widehat{C}_\tau \tag{6.3}$$

Then there exist a constant $\epsilon > 0$ and $\hat{\delta} > 0$ such that we have the quadratic growth condition holds:

$$\frac{\delta}{4}\|u-\bar{u}\|^2_{L^2(\Omega)} + J(\bar{y},\bar{u}) \leq J(y,u)\ if\ \|u-\bar{u}\|_{L^\infty(\Omega)} < \epsilon\ \forall u \in U_{ad} \tag{6.4}$$

In particular, $\bar{u}$ is a localy optimal control with respect to the norm $\|.\|_\infty$

Proof. Using the known series expansion of Taylor at $u(x)$. We obtain that

$$J(y,u) := f(u) = f(\bar{u}) + f'(\bar{u})(u-\bar{u}) + \frac{1}{2}f''(\bar{u}+\theta(u-\bar{u}))(u-\bar{u})^2$$

with $\theta \in (0,1)$.

In view of the first-order necessary condition, the first-order term is nonnegative. Indeed, it follows from the variationall inequality that

$$f'(u)(u-\bar{u})\ = \int_\Omega (p(x) + k\bar{u}(x))h(x)\ dx \geq 0 \tag{6.5}$$

Next, note that $(u-\bar{u}) \in \widehat{C}_\tau$. We estimate the second-order term from below, we have

$$\begin{aligned} &f''(\bar{u}+\theta(u-\bar{u}))(u-\bar{u})^2 = \left(f''(\bar{u})(u-\bar{u})^2 + [f''(\bar{u}+\theta(u-\bar{u})-f''(\bar{u}))]\,(u-\bar{u})^2\right) \\ &\geq \delta\|u-\bar{u}\|^2_{L^2(\Omega)} - L(M)\|(u-\bar{u})\|_{L^\infty(\Omega)}\ \|(u-\bar{u})\|^2_{L^2(\Omega)} \\ &\geq \frac{\delta}{2}\|u-\bar{u}\|^2_{L^2(\Omega)} \end{aligned} \tag{6.6}$$

Provided that $\|u - \bar{u}\|_{L^\infty(\Omega)} \le \epsilon$ for some sufficiently small $\epsilon > 0$.
In summary, we have

$$\frac{\delta}{4}\|u - \bar{u}\|^2_{L^2(\Omega)} + J(\bar{y}, \bar{u}) \le J(y, u)$$

□

References

[1] E. Casas, *Pontyagins principle for optimal control problems governed by semilinear elliptic equations*, SIAM J.International series of numerical mathematics **118** (1994), 97–114.

[2] E. Casas and M. Mateos, *second order optimality conditions for semilinear elliptic control problems with finitely many state constraints*, SIAMJ. Control. Optim. **40** (2002), 1431–1454.

[3] E. Casas, J. C. D. L. Reyes, and F. Tröltzsch, *Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints*, SIAMJ. Optim. **19** (2008), 616–643.

[4] E. Casas and F. Tröltzsch, *Second order necessary and sufficient optimality conditions for optimization problems and applications to control theory*, SIAMJ. Control. Optim. **13** (2002), 406–431.

[5] E. Casas, F. Tröltzsch, and A. Unger, *Second order sufficient optimality conditions for some state-constrained control problems of semilinear equations*, SIAM J. Control and Optimization **38** (2000), no. 5, 1369–1391.

[6] A. Ju. Dubinskii, *Sobolev space of infinite order and the behavior of solutions of some boundeary value problems with unbounded increase of the order of the equation*, Math. USSR Sbornik **27** (1975), no. 2.

[7] ———, *Some imbeding theorem for sobolev spaces of infinite order*, Soviet Math. Dokl. **19** (1979), 1271–1274.

[8] S. A. El-Zahaby, *Necessary conditions for control problem governed by elliptic variational inequalities with an infinite number of variables*, Information Optimization Sciences **15** (1994), 171–178.

[9] ———, *Second order sufficient optimality conditions for semilinear elliptic control problem of infinite order*, International Journal of Pure and Applied Mathematical **100** (2015), 169–192.

[10] S. A. El-Zahaby and Samira El-Tamimy, *Optimal control problems of systems governed by elliptic operator of infinite order with pointwise control constraints*, Journal of Mathematics and Statistical Science (ISSN2411-2518, USA) **December 25** (2015).

[11] S. A. El-Zahaby and GH. H. Mostafa, *Optimal control of variational inequalities of infinite order*, J. of Math. Egyptian Soc. **13** (2005), 99–106.

[12] I. M. Gali, *Optimal control of a system governed by elliptic operator of inifinte order ordinary and partial differential-equations*, Proceedings Dunder Scolled, Springer-Verlag series, Lecture Notes in Mathematics **964** (1984), 263–272.

[13] I. M. Gali and H. A. El-Saify, *Distributed control of systems covherned by dirichlet and neumann problems for a self-adjoint elliptic operator with an infinite number of variables*, J. of Optimization Theory and Applications **39** (1983), no. February, 293–298.

[14] ———, *Optimal control of a system governed by a self adjoint elliptic operator with infinite number of variable*, Proceeding of international conference on "Functional Differential Systems and Related Topics" **Poland** (May, 1991).

[15] I. M. Gali, H. A. El-Saify, and El-Zahaby, *Optimal control of a system governed by hyperbolic operator*, Springer-Verlg Series, Lecture Notes in Math. Proceedings at the Eighth Conference on Ordinary and Partial Differential Equations, Dundee, Scotland, Apper in Lecture Notes in Math Springer-Verlg Series **June** (1984), 157–163.

[16] ______, *Distributed control of a system governed by dirichlet and neumann problem for elliptic equations of inifite order*, The international conference in Functional-Differential Systems and Related Topics **III** (May 1983).

[17] I. M. Gali and El-Zahaby, *Optimal control of a system governed by variational inequalities for infinte order elliptuc operator*, Kyngpook Mathematical Journal **31** (December 1991), no. 2, 193–200.

[18] A.D. Ioffe, *Necessary and sufficient conditions for a local minimum 3: Second order conditions and augmented duality.*, SIAM J. Control and optimization (1979), 266–288.

[19] F. Tröltzsch, *Optimal control of partial differential equation: Theory, methods and applications*, American Mathematical Society Provndence, Rhode, Island (2010).

Published: Volume 2019, Issue 10 / October 25, 2019