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A New Type of Crisp Set via λ-Shading

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Abstract

In this paper a new class of crisp subsets has been introduced and studied which inherits λ-α-almost compactness of a space X, endowed with a fuzzy topology. Also a new type of continuous function between two fuzzy topological spaces is introduced under which λ-α-almost compactness for crisp subsets remains invariant.

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Introduction

From very beginning many mathematicians are engaged themselves for studying different types of compactness in fuzzy setting by using the definition of fuzzy cover initiated by Chang [4]. The concept of fuzzy cover was generalized by Gantner et al. [5] in 1978 by introducing a new concept of cover termed as α-shading ($0 < \alpha < 1$). In this paper we use this idea of α-shading but in a different terminology, viz., λ-shading ($0 < \lambda < 1$) for avoiding confusion of the term $'\alpha'$ used in α-open set and α-shading.

Preliminaries

A fuzzy set A in an fts X means a function from X to the closed interval $I = [0,1]$ of the real line, i.e., $A \in I^X$ [7]. By a crisp subset of an fts X, we mean an ordinary subset A of X, i.e., $A \subseteq X$, where the underlying structure on X is a fuzzy topology τ. For a fuzzy set A in an fts X, clA and $intA$ stand for fuzzy closure and fuzzy interior of A in X respectively [4]. The support of a fuzzy set A in X will be denoted by $suppA$ [7] and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A

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fuzzy point in X with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. For two fuzzy sets A and B in X, we write $A \leq B$ if $A(x) \leq B(x)$, for all $x \in X$, while we write AqB if A is quasi-coincident (q-coincident, for short) with B [6], i.e., $A(x) + B(x) > 1$, for some $x \in X$. The negation of these two statements are written as $A \not\leq B$ and $A\not{q}B$ respectively. A fuzzy set B is called a quasi neighbourhood (q-nbd, for short) of a fuzzy set A if there is a fuzzy open set U in X such that $AqU \leq B$ [6]. If, in addition, B is fuzzy open, then B is called fuzzy open q-nbd of A. A fuzzy neighbourhood (nbd, for short) [6] A of a fuzzy point x_t in an fts X is defined in the usual way, i.e., whenever for some fuzzy open set U in X, $x_t \leq U \leq A$; A is a fuzzy open nbd of x_t if A is fuzzy open, in addition.

1. Fuzzy α-Open and Fuzzy α-Closed Sets and λ-α-Almost Compact Space

Let us recall some definitions for ready references.

Definition 1.1 [3]. A fuzzy set A in an fts X is said to be fuzzy α-open if $A \leq intclintA$. The complement of a fuzzy α-open set is called fuzzy α-closed.

Definition 1.2 [3]. The smallest fuzzy α-closed set containing a fuzzy set A in X is called fuzzy α-closure of A and is denoted by αclA, i.e., $\alpha clA = \wedge\{U : A \leq U$ and U is fuzzy α-closed $\}$. A fuzzy set A in X is fuzzy α-closed if $A = \alpha clA$.

Definition 1.3 [3]. For a fuzzy set A in an fts X, the fuzzy α-interior of a fuzzy set A denoted by $\alpha intA$ is defined by the union of all fuzzy α-open sets contained in A, i.e., $\alpha intA = \vee\{V : V \leq A$ and V is fuzzy α-open in $X\}$. A fuzzy set A in X is fuzzy α-open if $A = \alpha intA$.

Definition 1.4 [1]. A fuzzy set A in an fts X is called a fuzzy α-open q-nbd of a fuzzy point x_t in X if there exists a fuzzy α-open set V in X such that $x_t qV \leq A$.

Definition 1.5 [5]. Let A be a crisp subset of an fts X. A collection $\mathcal{U}$ of fuzzy sets in X is called a λ-shading (where $0<\lambda<1$) (formerly known as α-shading (where $0<\alpha<1$)) of A if for each $x\in A$, there is some $U_x\in\mathcal{U}$ such that $U_x(x)>\lambda$. Taking $A=X$, we arrive at the definition of λ-shading of an fts X, as proposed by Gantner et. al [5].

If the members of a λ-shading $\mathcal{U}$ of A (or of X) are fuzzy α-open sets in X, then $\mathcal{U}$ is called a fuzzy α-open λ-shading of A (resp., of X).

Definition 1.6 [2]. Let X be an fts and A, a crisp subset of X. A is called λ-α-almost compact if each fuzzy α-open λ-shading of A has a finite α-proximate λ-subshading, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{\alpha clU : U\in U_0\}$ is again a λ-shading of A. If $A=X$, in addition, then X is called a λ-α-almost compact space.

2. α^λ-Closed Sets: Some Properties

Let us now introduce a class of crisp sets in an fts X, as follows.

Definition 2.1. Let (X,τ) be an fts and $A\subseteq X$. A point $x\in X$ is said to be an α^λ-limit point of A if for every fuzzy α-open set U in X with $U(x)>\lambda$, there exists $y\in A\setminus\{x\}$ such that $(\alpha clU)(y)>\lambda$. The set of all α^λ-limit points of A will be denoted by $[A]^\lambda_\alpha$.

The α^λ-closure of A, to be denoted by $\alpha^\lambda-clA$, is defined by $\alpha^\lambda-clA=A\bigcup[A]^\lambda_\alpha$.

Definition 2.2. A crisp subset A of an fts X is said to be α^λ-closed if it contains all its α^λ-limit points. Any subset B of X is called α^λ-open if $X\setminus B$ is α^λ-closed.

Remark 2.3. It is clear from Definition 2.1 that for any set $A\subseteq X$, $A\subseteq\alpha^\lambda-clA$ and $\alpha^\lambda-clA=A$ if and only if $[A]^\lambda_\alpha\subseteq A$. Again it follows from Definition 2.1 that A is α^λ-closed if and only if $\alpha^\lambda-clA=A$. It is also clear that $A\subseteq B\subseteq X\Rightarrow[A]^\lambda_\alpha\subseteq[B]^\lambda_\alpha$.

Theorem 2.4. An α^λ-closed subset A of a λ-α-almost compact space X is λ-α-almost compact.

Proof. Let $A(\subseteq X)$ be α^λ-closed in a λ-α-almost compact space X. Then for any $x\notin A$, there is a fuzzy α-open set U_x in X such that $U_x(x) > \lambda$, and $(\alpha clU_x)(y) \le \lambda$, for every $y \in A$. Consider the collection $\mathcal{U} = \{U_x : x \notin A\}$. For proving A to be λ-α-almost compact, consider a fuzzy α-open λ-shading $\mathcal{V}$ of A. Clearly $\mathcal{U} \bigcup \mathcal{V}$ is a fuzzy α-open λ-shading of X. Since X is λ-α-almost compact, there exists a finite subcollection $\{V_1, V_2, ..., V_n\}$ of $\mathcal{U} \bigcup \mathcal{V}$ such that for every $t \in X$, there exists $V_i (1 \le i \le n)$ such that $(\alpha clV_i)(t) > \lambda$. For every member U_x of $\mathcal{U}$, $(\alpha clU_x)(y) \le \lambda$, for every $y \in A$. So if this subcollection contains any member of $\mathcal{U}$, we omit it and hence we get the result.

To achieve the converse of Theorem 2.4, we define the following.

Definition 2.5. An fts (X, τ) is said to be λ-α-Urysohn if for any two distinct points x, y of X, there exist a fuzzy open set U and a fuzzy α-open set V in X with $U(x) > \lambda$, $V(y) > \lambda$ and $min((\alpha clU)(z), (\alpha clV(z)) \le \lambda$, for each $z \in X$.

Theorem 2.6. A λ-α-almost compact set in a λ-α-Urysohn space X is α^{λ}-closed.

Proof. Let A be λ-α-almost compact set and $x \in X \setminus A$. Then for each $y \in A$, $x \ne y$. As X is λ-α-almost compact, there exist a fuzzy open set U_y and a fuzzy α-open set V_y in X such that $U_y(x) > \lambda, V_y(y) > \lambda$ and $min((\alpha clU_y)(z), (\alpha clV_y)(z)) \le \lambda$, for all $z \in X$ ... (1).

Then $U = \{V_y : y \in A\}$ is a fuzzy α-open λ-shading of A and so by λ-α-almost compactness of A, there exist finitely many points $y_1, y_2, ..., y_n$ of A such that $U_0 = \{\alpha clV_{y_1}, \alpha clV_{y_2}, ..., \alpha clV_{y_n}\}$ is again a λ-shading of A. Now $U = U_{y_1} \bigcap ... \bigcap U_{y_n}$ being a fuzzy open set is a fuzzy α-open set in X such that $U(x) > \lambda$. In order to show that A to be α^{λ} -closed, it now suffices to show that $(\alpha clU)(y) \le \lambda$, for each $y \in A$. In fact, if for some $z \in A$, we assume $(\alpha clU)(z) > \lambda$, then as $z \in A$, we have $(\alpha clV_{y_k})(z) > \lambda$, for some k $(1 \le k \le n)$. Also

$(\alpha clU_{y_k})(z) > \lambda$. Hence $min((\alpha clU_{y_k})(z), (\alpha clV_{y_k})(z)) > \lambda$, contradicting (1).

Corollary 2.7. In a λ-α-almost compact, λ-α-Urysohn space X, a subset A of X is λ-α-almost compact if and only if it is α^λ-closed.

Theorem 2.8. In a λ-α-almost compact space X, every cover of X by α^λ-open sets has a finite subcover.

Proof. Let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be a cover of X by α^λ-open sets. Then for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since $X \setminus U_x$ is α^λ-closed, there exists a fuzzy α-open set V_x in X such that $V_x(x) > \lambda$ and $(\alpha clV_x)(y) \le \lambda$, for each $y \in X \setminus U_x$ ... (1).

Then $\{V_x : x \in X\}$ forms a fuzzy α-open λ-shading of the λ-α-almost compact space X. Thus there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that $\{\alpha clV_{x_i} : i = 1, 2, ..., n\}$ is a λ-shading of X ... (2).

We claim that $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$ is a finite subcover of $\mathcal{U}$. If not, then there exists $y \in X \setminus \bigcup_{i=1}^{n} U_{x_i} = \bigcap_{i=1}^{n} (X \setminus U_{x_i})$. Then by (1), $(\alpha clV_{x_i})(y) \le \lambda$, for $i = 1, 2, ..., n$ and so $(\bigcup_{i=1}^{n} \alpha clV_{x_i})(y) \le \lambda$, contradicting (2).

Theorem 2.9. Let (X, τ) be an fts. If X is λ-α-almost compact, then every collection of α^λ-closed sets in X with finite intersection property has non-empty intersection.

Proof. Let $\mathcal{F} = \{F_i : i \in \Lambda\}$ be a collection of α^λ-closed sets in a λ-α-almost compact space X having finite intersection property. If possible, let $\bigcap_{i \in \Lambda} F_i = \phi$. Then $X \setminus \bigcap_{i \in \Lambda} F_i = \bigcup_{i \in \Lambda} (X \setminus F_i) = X \Rightarrow \mathcal{U} = \{X \setminus F_i : i \in \Lambda\}$ is an α^λ-open cover of X. Then by Theorem 2.8, there is a finite subset Λ_0 of Λ such that $\bigcup_{i \in \Lambda_0} (X \setminus F_i) = X \Rightarrow \bigcap_{i \in \Lambda_0} F_i = \phi$, a contradiction.

3. α^{λ}-Continuity: Some Applications

In this section, we now introduce a class of functions under which λ-α-almost compactness remains invariant.

Definition 3.1. Let X, Y be fts's. A function $f : X \to Y$ is said to be α^{λ}-continuous if for each point $x \in X$ and each fuzzy α-open set V in Y with $V(f(x)) > \lambda$, there exists a fuzzy α-open set U in X with $U(x) > \lambda$ such that $\alpha clU \leq f^{-1}(\alpha clV)$.

Theorem 3.2. If $f : X \to Y$ is α^{λ}-continuous (where X, Y are, as usual, fts's), then the following are true :

(a) $f([A]^{\lambda}_{\alpha}) \subseteq [f(A)]^{\lambda}_{\alpha}$, for every $A \subseteq X$.

(b) $[f^{-1}(A)]^{\lambda}_{\alpha} \subseteq f^{-1}([A]^{\lambda}_{\alpha})$, for every $A \subseteq Y$.

(c) For each α^{λ}-closed set A in Y, $f^{-1}(A)$ is α^{λ}-closed in X.

(d) For each α^{λ}-open set A in Y, $f^{-1}(A)$ is α^{λ}-open in X.

Proof (a). Let $x \in [A]^{\lambda}_{\alpha}$ and U be any fuzzy α-open set in Y with $U(f(x)) > \lambda$. Then there is a fuzzy α-open set V in X with $V(x) > \lambda$ and $\alpha clV \leq f^{-1}(\alpha clU)$. Now $x \in [A]^{\lambda}_{\alpha}$ and V is a fuzzy α-open set in X with $V(x) > \lambda \Rightarrow \alpha clV(x_0) > \lambda$, for some $x_0 \in A \setminus \{x\}$ $\Rightarrow \lambda < \alpha clV(x_0) \leq (f^{-1}(\alpha clU))(x_0) = (\alpha clU)(f(x_0))$ where $f(x_0) \in f(A) \setminus \{f(x)\} \Rightarrow f(x) \in [f(A)]^{\lambda}_{\alpha}$. Thus (a) follows.

(b) By (a), $f([f^{-1}(A)]^{\lambda}_{\alpha}) \subseteq [ff^{-1}(A)]^{\lambda}_{\alpha} \subseteq [A]^{\lambda}_{\alpha} \Rightarrow [f^{-1}(A)]^{\lambda}_{\alpha} \subseteq f^{-1}([A]^{\lambda}_{\alpha})$.

(c) We have $[A]^{\lambda}_{\alpha} = A$. By (b), $[f^{-1}(A)]^{\lambda}_{\alpha} \subseteq f^{-1}([A]^{\lambda}_{\alpha}) = f^{-1}(A) \Rightarrow [f^{-1}(A)]^{\lambda}_{\alpha} = f^{-1}(A) \Rightarrow$ $f^{-1}(A)$ is α^{λ}-closed set in X.

(d) Follows from (c).

Theorem 3.3. Let X, Y be fts's and $f: X \to Y$ be fuzzy α^{λ}-continuous function. If $A(\subseteq X)$ is λ-α-almost compact, then so is $f(A)$ in Y.

Proof. Let $\mathcal{V} = \{V_i : i \in \Lambda\}$ be a fuzzy α-open λ-shading of $f(A)$, where A is λ-α-almost compact set in X. For each $x \in A$, $f(x) \in f(A)$ and so there exists $V_x \in V$ such that $V_x(f(x)) > \lambda$. As f is fuzzy α^{λ}-continuous, there exists a fuzzy α-open set U_x in X such that $U_x(x) > \lambda$ and $f(\alpha cl U_x) \le \alpha cl V_x$. Then $\{U_x : x \in X\}$ is a fuzzy α-open λ-shading of A. By λ-α-almost compactness of A, there are finitely many points $a_1, a_2, ..., a_n$ in A such that $\{\alpha cl U_{a_i} : i = 1, 2, ..., n\}$ is again a λ-shading of A.

We claim that $\{\alpha cl V_{a_i} : i = 1, 2, ..., n\}$ is a λ-shading of $f(A)$. In fact, $y \in f(A) \Rightarrow$ there exists $x \in A$ such that $y = f(x)$. Now there is an U_{a_j} (for some $j, 1 \le j \le n$) such that $(\alpha cl U_{a_j})(x) > \lambda$ and hence $(\alpha cl V_{a_j})(y) \ge f(\alpha cl U_{a_j})(y) \ge \alpha cl U_{a_j}(x) > \lambda$.

Let us now give a definition of a function under which α^{λ}-closedness of a set remains invariant.

Definition 3.4. Let X, Y be fts's. A function $f: X \to Y$ is said to be fuzzy α-open if $f(A)$ is fuzzy α-open in Y whenever A is fuzzy α-open in X.

Remark 3.5. For a fuzzy α-open function $f: X \to Y$, for every fuzzy α-closed set A in X, $f(A)$ is fuzzy α-closed in Y.

Theorem 3.6. If $f: (X, \tau) \to (Y, \tau_1)$ is a bijective fuzzy α-open function, then the image of an α^{λ}-closed set in (X, τ) is α^{λ}-closed in (Y, τ_1).

Proof. Let A be a α^{λ}-closed set in (X, τ) and let $y \in Y \setminus f(A)$. Then there exists a unique $z \in X$ such that $f(z) = y$. As $y \notin f(A)$, $z \notin A$. Now, A being α^{λ}-closed in X, there exists a fuzzy α-open set V in X such that $V(z) > \lambda$ and $\alpha cl V(p) \le \lambda$, for each $p \in A$ ... (1).

As f is fuzzy α-open, $f(V)$ is a fuzzy α-open set in Y, and also $(f(V))(y) = V(z) > \lambda$ as f is bijective. Let $t \in f(A)$. Then there is a unique $t_0 \in A$ such that $f(t_0) = t$. As f is bijective and fuzzy α-open, by Remark 3.5, $\alpha clf(V) \leq f(\alpha clV)$. Then $(\alpha clf(V))(t) \leq f(\alpha clV)(t) = \alpha clV(t_0) \leq \lambda$ as f id bijective, by (1). Thus y is not an α^λ-limit point of $f(A)$. Hence the proof.

From Theorem 3.2 (c) and Theorem 3.6, it follows that

Corollary 3.7. Let $f : X \to Y$ be a fuzzy α^λ-continuous, bijective and fuzzy α-open function. Then A is α^λ-closed in Y if and only if $f^{-1}(A)$ is α^λ-closed in X.

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