"Science Stays True Here"
Journal of Mathematics and Statistical Science, 304-310 | Science Signpost Publishing

Fan-Gottesman Compactification and Completeness

Ceren Sultan ELMALI

Department of Mathematics, Faculty of Science, Erzurum Technical University, 25100 Erzurum-Turkey.

ceren.elmali@erzurum.edu.tr

Tamer UĞUR

Department of Mathematics, Faculty of Science, Atatürk University, 25240 Erzurum-Turkey.

tugur@atauni.edu.tr

Abstract

It is investigated that Fan-Gottesman compactification of (X,d) is homeomorphic to the completion of (X,d^{ϕ}) totally bounded metric space. As application, we construct Fan-Gottesman compactification of $(\mathbb{R}^n,d_E)$, where d_E is the euclidean metric and $n\geq 2$ and show that Fan-Gottesman compactification of $(\mathbb{R}^n,d_E)$ is homeomorphic to its completion.

Keywords: Fan-Gottesman compactification, completion.

1. The Metric $d^{\phi,m}$

Let (X,d) be a metric space and $m\in X$. Suppose that $\phi: X\times X\to\mathbb{R}$ is a nonnegative symmetric function. As a usual, two metrics d_1 and d_2 on a set X are called equivalent if (X,d_1) and (X,d_2) are homeomorfic.It is defined a metric $d^{\phi,m}$ on X which is equivalent to d in [1]. For each $x,y\in X$, let

---

Corresponding author: Ceren Sultan ELMALI, Department of Mathematics, Faculty of Science, Erzurum Technical University, 25100 Erzurum-Turkey. E-mail: ceren.elmali@erzurum.edu.tr.

$$\delta^{\phi,m}(x,y)=\min\left\{d(x,y),\frac{1}{1+d(m,x)}+\phi(x,y)+\frac{1}{1+d(m,y)}\right\}$$

And for each $x,y\in X$ and $n\in\mathbb{N}$, let

$$\Gamma_{x,y}^{n}=\left\{(x_0,x_1,\dots,x_n):x_0=x,x_n=y\text{ and }x_i\in X\text{ for all }i\right\}$$

and

$$\Gamma_{x,y}=_{n\in\mathbb{N}}\Gamma_{x,y}^{n}$$

Notice that $\Gamma_{x,y}\neq$ for all $x,y\in X$. In the following definition, the infimum runs over all elements of $\Gamma_{x,y}$.

1.1. Definition:

Suppose that $x,y\in X$. Let

$$d^{\phi,m}(x,y)=\inf_{\Gamma_{x,y}}{}^{n}_{i=1}\ \delta^{\phi,m}(x_{i-1},x_i)$$

For the sake of simplicity, It is simply writen d^{ϕ}, δ^{ϕ} to denote $d^{\phi,m}$, $\delta^{\phi,m}$ respectively. In particular, it is writen as

$$d^{\phi}(x,y)=\inf_{\Gamma_{x,y}}{}^{n}_{i=1}\ \delta^{\phi}(x_{i-1},x_i)$$

Young Deuk Kim showed that d^{ϕ} is a metric on X and (X,d^{ϕ}) is a homeomorphic to (X,d). Also he investigated that (X,d^{ϕ}) is totally bounded. He construct the completion $(\overline{X},\rho)$ of totally bounded metric space (X,d^{ϕ}). He applied to Euclidean metric space $\mathbb{R}^n$ with $n\geq 2$ in [1]. Therefore Young Deuk Kim showed that $(\mathbb{R}^n,d_E)$ is totally bounded and construct its completion.

$\overline{X}$ can be desired as the set of equivalence classes of all Cauchy sequences in (X,d^{ϕ}) with the equivalence relation

$$x_i \sim y_i \text{ if and only if } \lim_{i\to\infty} d^{\phi}(x_i, y_i) = 0$$

where a point x in X is identified to the equivalence class of constant Couchy sequence $\{x\}$.

Suppose that $\{x_i\},\{y_i\} \in \overline{X}$. The metric ρ is given by

$$\rho(\{x_i\},\{y_i\}) = \lim_{i\to\infty} d^{\phi}(x_i, y_i)$$

In particular, we have

$$\rho(\{x\},\{y\}) = d^{\phi}(x, y) \text{ for all } x, y \in X$$

Notice that X is dense subset of $\overline{X}$ and $(\overline{X}, \rho)$ is a compact metric space.

2. Fan-Gottesman Compactification

A compactification of a topological space X is a compact hausdorff space Y containing X as a subspace such that $\overline{X} = Y$. It is known that every metric space has a compactification [3]. In addition there are a lot of compactification methods appling different topological space such as Aleksandrov (one-point), Wallman, Stone-Cech. But we study with Fan-Gottesman compactification.

Fan-Gottesman compactification of a regular space is introduced and studied by Fan Ky and Noel Gottesman. Let β is a class of open sets in X. β contains $\varnothing$ and satisfies following three conditions,

(1) If $B_1, B_2 \in \beta$, then $B_1 \cap B_2 \in \beta$

(2) If $B \in \beta$, then $X - cl_X B \in \beta$, where closure of B in X will be denoted $cl_X B$.

(3) For every open set U in X and every $B \in \beta$ such that $cl_X B \subset U$, there exist a set $D \in \beta$ such that $cl_X B \subset D \subset cl_X D \subset U$.

Then β is called normal base.

It is considered that is a regular space having a normal base for open set which satisfies above three properties of normal base. A chain family on β is a non-empty family of sets of β such that

$$cl_X B_1 \cap cl_X B_2 \cap \ldots \cap cl_X B_n \neq \emptyset$$

for any finite number of sets B_i of the family. Every chain family on β is contained in at least one maximal chain family on β from Zorn lemma. Maximal chain families on β will be denoted by letters as $a^*, b^*, \ldots$ and also the set of all maximal chain families on β will be denoted by X^*. X^* is a compact hausdorff spaces and compactification of regular spaces. Afterwards this compactification is called Fan-Gottesman compactification [2].

We apply Fan-Gottesman compactification to metric space (X,d). Firstly, we take a base β for open set in metric space (X,d). We check whether β satisfies normal base conditions or not. β consist of $S(p,\delta)=\{x : d(p,x) \prec \delta, \delta \in \mathbb{R}\}$ for $\forall p \in X$.

(1) If $S(p_1,\delta)$, $S(p_2,\delta) \in \beta$, then we must show $S(p_1,\delta) \cap S(p_2,\delta) \in \beta$.

i) If $S(p_1,\delta) \cap S(p_2,\delta) = \emptyset$, it is trivial.

ii) If $S(p_1,\delta) \cap S(p_2,\delta) \neq \emptyset$, $S(p_1,\delta) \cap S(p_2,\delta)$ is a open set in (X,d). Since intersection of two open set is open in metric space.

(2) If $S(p,\delta) \in \beta$, then we must show $X - cl(S(p,\delta)) \in \beta$, where closure of B in X will be denoted clB. If we show that every point of $X - cl(S(p,\delta))$ is a interior point of it, the proof is completed. Let x be a arbitrary point in $X - cl(S(p,\delta))$. Then $x \in X$ and $x \notin cl(S(p,\delta))$. Since (X,d) is a metric space, there exist $\exists S(x,\varepsilon)$ such that $x \in S(x,\varepsilon) \subset X - cl(S(p,\delta))$. Thus $X - cl(S(p,\delta))$ is open and $X - cl(S(p,\delta)) \in \beta$.

(3) We take $U = S(p,\delta)$ and $B = S(p,\frac{\delta}{3})$ such that $clB \subset U$. Therefore there exists a set

$D = S\left(p, \frac{\delta}{2}\right) \in \beta$ for every $\delta \in R$ such that $clB \subset D \subset clD \subset U$. It is clearly that the process is correct for every $p \in X$ and every $\delta \succ 0$. Thus β is a normal base for (X, d).

A binding family x^* is defined as a family of element in β such that $\bigcap_{i=1}^{m} clB_i \neq \varnothing$ for every finite family $B_1, B_2, ..., B_m$ in x^*. By Zorn's lemma, every binding family on β is contained in at least one maximal binding family on β. For every $B \in \beta$, B^* is defined as the set of all maximal binding families x^* such that $D \in x^*$ for some $D \in \beta$ where $clD \subset B$. The set X^* of all maximal binding families is equipped with a topology having as a base for its open sets the class β^* of all sets $B^*, B \in \beta$. This space X^* is a Fan-Gottesman compactification of X.

Fan-Gottesman compactification of (X, d) metric space is homeomorphic to Aleksandrov compactification αX. Really it suffices to show that $X^* - X$ has exactly one point in order that it is shown $X^* \cong \alpha X$. Because studied spaces (X, d) is local compact $X - B$ or $cl_X B$ is compact for all B open subsets of X. Let $b^* = \{B \in \beta : X - B \text{ is compact}\}$. As X is not compact, we get $\bigcup_{i=1}^{n}(X - B_i) \neq X$ for any finite number of sets $B_i \in b^*$. This implies that b^* is binding family on β. Similarly we take $A \in a^*$ such that $cl_X A$ is not compact. Hence $X - A$ is compact for every $A \in a^*$. Then $a^* \subset b^*$ and thus $a^* = b^*$. This shows that b^* is the only point in $X^* - X$. Therefore it is gotten $X^* \cong \alpha X$.

Notice that X is a dense subset of $\overline{X}$ and $(\overline{X}, \rho)$ is a compact metric space [3]. Then our compactification Fan-Gottesman (X^*, d^*) of (X, d) is the completion $(\overline{X}, \rho)$ of the totally bounded metric space (X, d^{ϕ}).

3. An Application to $\left(\mathbb{R}^n, d_E\right)$

Now we look at application to $\left(\mathbb{R}^n, d_E\right)$ of above statement. We must show the completion of $\left(\mathbb{R}^n, d^{\phi}\right)$ is a homeomorphic to Fan-Gottesman compactification of $\left(\mathbb{R}^n, d_E\right)$. Firsly, we construct Fan-Gottesman compactification of $\left(\mathbb{R}^n, d_E\right)$. We consider that β consist of $S\left(p,1\right)=\left\{x : d_E\left(p,x\right)\prec 1\right\}$ for $\forall\ \ p\in\mathbb{R}^n$ as similar to above. It is clearly that β is a normal base in $\left(\mathbb{R}^n, d_E\right)$. A binding family x^* is family of element in β such that $\bigcap_{i=1}^{m} cl\left(S\left(p_i,1\right)\right)\neq\varnothing$ for every finite family $S\left(p_1,1\right), S\left(p_2,1\right),\ldots,S\left(p_m,1\right)$ in x^* and every binding family on β is contained in at least one maximal binding family on β. For every $S\left(p,1\right)\in\beta$, B^* is defined as the set of all maximal binding families x^* such that $D\in x^*$ for some $D\in\beta$ where $clD\subset S\left(p,1\right)$. The set $\left(\mathbb{R}^n\right)^*$ of all maximal binding families is equipped with a topology having as a base for its open sets the class β^* of all sets $B^*, B\in\beta$. This space $\left(\mathbb{R}^n\right)^*$ is a Fan-Gottesman compactification of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a local compact hausdorff space, we get $\left(\mathbb{R}^n\right)^*=\alpha\mathbb{R}^n=\left\{x\in\mathbb{R}^n : d_E\left(0,x\right)\leq 1\right\}$.

We consider the function defined by Young Deuk Kim. He defined

$$h:\left(\alpha\mathbb{R}^n, d_E\right)\rightarrow\left(\overline{\mathbb{R}^n},\rho\right)$$

as follows

$$h\left(x\right)=\begin{cases}\dfrac{1}{1-d_E\left(0,x\right)}x \text{ (the constant Cauchy sequence) if } d_E\left(0,x\right)\prec 1\\ \left\{a_i x_i\right\} \qquad\qquad\qquad\qquad \text{if } d_E\left(0,x\right)=1\end{cases}$$

where $\left\{a_i x_i\right\}$ is a Cauchy sequence not equivalent to any constant Cauchy sequence and $a_i=\sum_{i\in I}\frac{1}{i}$. And also he showed that h is a homeomorphism [1].

We know that $\left(\mathbb{R}^n\right)^* = \alpha\mathbb{R}^n$ and if we use this function, we'll get Fan-Gottesman compactification of $\left(\mathbb{R}^n, d_E\right)$, $\left(\mathbb{R}^n\right)^*$ is homeomorphic to the completion of $\left(\mathbb{R}^n, d^{\phi}\right)$, $\left(\overline{\mathbb{R}^n}, \rho\right)$.

References

[1]. K. Fan and N. Gottesman, On compactification of Freduental and Wallman, Indag. Math. 13 (1952) 184-192.

[2]. Y.D. Kim, Equivalent metrics and compactification, arXiv: 0710. 0080v1 [math.GT] 2007.

[3]. J.R. Munkres, Topology at firs course, Prentice-Hall, New Jersey, 1975.

Published: Volume 2017, Issue 10 / October 25, 2017