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Fuzzy Almost p-Continuous and Almost p^*-Continuous Functions

Anjana Bhattacharyya

Department of Mathematics, Victoria Institution (College), 78 B, A.P.C. Road, Kolkata - 700009, India.

Abstract

This paper deals with two different types of fuzzy continuous functions, viz., fuzzy almost p-continuous and fuzzy almost p^*-continuous functions. It is shown that fuzzy almost p^*-continuous function is fuzzy almost p-continuous but the converse is true under certain condition only. Again it is proved that the composition of two fuzzy almost p^*-continuous functions is also so but this is not true for fuzzy almost p-continuous function. In the last section a new type of fuzzy regularity is introduced under which fuzzy almost p-continuous function is fuzzy almost p^*-continuous.

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1. Introduction and Preliminaries

Fuzzy preopen set is introduced by S. Nanda in [6]. Using this concept as a basic tool here we have first introduced an idempotent operator and using this operator two different types of fuzzy continuous-like functions are introduced and studied. Afterwards, a new type of fuzzy regularity is introduced in which these two functions coincide.

Throughout this paper, (X,τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [3]. In 1965, L.A. Zadeh introduced fuzzy set [8] A which is a function from a non-empty set X into the closed interval $I=[0,1]$, i.e., $A \in I^X$. The support [8] of a fuzzy set

Corresponding author: Anjana Bhattacharyya, Department of Mathematics, Victoria Institution (College), 78 B, A.P.C. Road, Kolkata - 700009, India. E-mail: anjanabhattacharyya@hotmail.com.

A, denoted by $suppA$ or A_0 and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [8] of a fuzzy set A in an fts X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [8] while AqB means A is quasi-coincident (q-coincident, for short) [7] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A\not{q}B$ respectively. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [3] and fuzzy interior [3] respectively.

A fuzzy set A in an fts (X, τ) is called fuzzy regular open [1] (resp., fuzzy preopen [6]) if $A = intclA$ (resp., $A \leq intclA$). The complement of a fuzzy preopen set is called fuzzy preclosed [6]. The union (intersection) of all fuzzy preopen (resp., fuzzy preclosed) sets contained in (resp., containing) a fuzzy set A is called fuzzy preinterior [6] (resp., fuzzy preclosure [6]) of A, denoted by $pintA$ (resp., $pclA$). A fuzzy set A in X is called a fuzzy neighbourhood (nbd, for short) [7] of a fuzzy point x_t if there exists a fuzzy open set G in X such that $x_t \in G \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open nbd of x_t. A fuzzy set A is said to be a fuzzy q-nbd of a fuzzy point x_t in an fts X if there is a fuzzy open set U in X such that $x_t qU \leq A$. If, in addition, A is fuzzy open (resp., fuzzy preopen), then A is called a fuzzy open [7] (resp., fuzzy preopen [6]) q-nbd of x_t. The collection of all fuzzy preopen (resp., fuzzy preclosed) sets in X is denoted by $FPO(X)$ (resp., $FPC(X)$).

2. Fuzzy p^*-Open and p^*-Closed Sets: Some Properties

In this section we first introduce fuzzy p^*-open and fuzzy p^*-closed sets. Some important results are established here. Afterwards it is shown that p^*-closure operator is an idempotent operator. Again it is established that the collection of all fuzzy p^*-open sets in an fts (X,τ) is strictly weaker than that of fuzzy preopen sets.

Definition 2.1. A fuzzy set A in an fts (X,τ) is called fuzzy p^*-open if $A \leq int(pclA)$. The complement of this set is called fuzzy p^*-closed set.

The collection of fuzzy p^*-open (resp., fuzzy p^*-closed) sets in (X,τ) is denoted by $FP^*O(X)$ (resp., $FP^*C(X)$).

The union (resp., intersection) of all fuzzy p^*-open (resp., fuzzy p^*-closed) sets contained in (containing) a fuzzy set A is called fuzzy p^*-interior (resp., fuzzy p^*-closure) of A, denoted by p^*intA (resp., p^*clA).

Definition 2.2. A fuzzy set A in an fts (X,τ) is called fuzzy p^*-nbd of a fuzzy point x_α if there exists a fuzzy p^*-open set U in X such that $x_\alpha \leq U \leq A$. If, in addition, A is fuzzy p^*-open, then A is called fuzzy p^*-open nbd of x_α.

Definition 2.3. A fuzzy set A in an fts (X,τ) is called fuzzy p^*-q-nbd of a fuzzy point x_α if there exists a fuzzy p^*-open set U in X such that $x_\alpha qU \leq A$. If, in addition, A is fuzzy p^*-open, then A is called fuzzy p^*-open q-nbd of x_α.

Result 2.4. Union (resp., intersection) of any two fuzzy p^*-open (resp., fuzzy p^*-closed) sets is also so.

Proof. Let A, B be two fuzzy p^*-open (resp., fuzzy p^*-closed) sets in an fts X. Then

$A \leq int(pclA), B \leq int(pclB)$ (resp., $cl(pintA) \leq A, cl(pintB) \leq B$). Now $int(pcl(A \vee B)) = int(pclA \vee pclB) \geq int(pclA) \vee int(pclB) \geq A \vee B$ (resp., $cl(pint(A \wedge B)) = cl(pintA \wedge pintB) \leq cl(pintA) \wedge cl(pintB) \leq A \wedge B$).

Remark 2.5. Intersection (resp., union) of two fuzzy p^*-open (resp., fuzzy p^*-closed) sets may not be so as it seen from the following example.

Example 2.6. Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4; B(a) = 0.5, B(b) = 0.55$. Then (X,τ) is an fts. Now $FPO(X,\tau) = \{0_X, 1_X, U, V, W\}$ where $1_X \setminus B < U \leq B$, $V > 1_X \setminus A, W \leq A$ and $FPC(X) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V, 1_X \setminus W\}$ where $1_X \setminus B \leq 1_X \setminus U < B$, $1_X \setminus V < A, 1_X \setminus W \geq 1_X \setminus A$. Consider two fuzzy sets C, D in X defined by $C(a) = 0.4$, $C(b) = 0.55, D(a) = 0.6, D(b) = 0.5$. Now $int(pclC) = int(1_X \setminus A) = B \geq C, int(pclD) = int(1_X) = 1_X > D \Rightarrow C, D$ are fuzzy p^*-open sets in (X,τ). Let $E = C \wedge D$. Then $E(a) = 0.4, E(b) = 0.5$. Now $int(pclE) = int(E) = 0_X \not\geq E \Rightarrow E$ is not fuzzy p^*-open in X.

Here also $1_X \setminus C, 1_X \setminus D$ are fuzzy p^*-closed sets in (X,τ). Now $F = (1_X \setminus C) \vee (1_X \setminus D)$ is defined by $F(a) = 0.6, F(b) = 0.5$ and $cl(pintF) = clU$ where $U(a) = U(b) = 0.5$. Then $clU = 1_X \setminus A \not\leq F \Rightarrow F$ is not fuzzy p^*-closed in (X,τ).

Result 2.7. $x_\alpha \in p^*clA$ iff every fuzzy p^*-open q-nbd U of x_α, UqA.

Proof. Let $x_\alpha \in p^*clA$ for any fuzzy set A in an fts (X,τ). Let $U \in FP^*O(X)$ with $x_\alpha qU$. Then $U(x) + \alpha > 1 \Rightarrow x_\alpha \not\leq 1_X \setminus U \in FP^*C(X)$. By Definition 2.1, $A \not\leq 1_X \setminus U \Rightarrow$ there exists $y \in X$ such that $A(y) > 1 - U(y) \Rightarrow A(y) + U(y) > 1 \Rightarrow UqA$.

Conversely, let the given condition hold. Let $U \in FP^*C(X)$ with $A \leq U$ ... (1). We have to show that $x_\alpha \in U$, i.e., $U(x) \geq \alpha$. If possible, let $U(x) < \alpha$. Then $1 - U(x) > 1 - \alpha \Rightarrow x_\alpha q(1_X \setminus U)$ where

$1_X \setminus U \in FP^*O(X)$. By hypothesis, $(1_X \setminus U)qA \Rightarrow$ there exists $y \in X$ such that $1 - U(y) + A(y) > 1 \Rightarrow A(y) > U(y)$, contradicts (1).

Result 2.8. $p^*cl(p^*clA) = p^*clA$ for any fuzzy set A in an fts (X, τ).

Proof. Let $A \in I^X$. Then $A \le p^*clA \Rightarrow p^*clA \le p^*cl(p^*clA)$ ... (1).

Conversely, let $x_\alpha \in p^*cl(p^*clA)$. If possible, let $x_\alpha \notin p^*clA$. Then there exists $U \in FP^*O(X)$, $x_\alpha qU, U\not{q}A$ ... (2). But as $x_\alpha \in p^*cl(p^*clA)$, $Uq(p^*clA) \Rightarrow$ there exists $y \in X$ such that $U(y) + (p^*clA)(y) > 1 \Rightarrow U(y) + t > 1$ where $t = (p^*clA)(y)$. Then $y_t \in p^*clA$ and $y_t qU$ where $U \in FP^*O(X)$. Then by definition, UqA, contradicts (2). So $p^*cl(p^*clA) \le p^*clA$ ... (3). Combining (1) and (3), we get the result.

Note 2.9. It is clear from above discussion that fuzzy p^*-open set is fuzzy preopen as $pclA \le clA$ for any fuzzy set A in an fts. But not conversely, as it seen from the following example.

Example 2.10. Consider Example 2.6. Here E is not fuzzy p^*-open. But $int(clE) = int(1_X \setminus A) = B > E$ and so E is fuzzy preopen.

Remark 2.11. It is clear from definition that every fuzzy open set is fuzzy p^*-open. But the converse may not be true follows from Example 2.6. Here C is fuzzy p^*-open, but C is not fuzzy open. So we have the following relation.

fuzzy open $\Rightarrow$ fuzzy p^*-open $\Rightarrow$ fuzzy preopen

3. Fuzzy Almost *p*-Continuous Function: Some Characterizations

In this section we first introduce a new type of fuzzy continuous-like function which implies fuzzy almost continuity but the converse need not be true.

Definition 3.1. A function $f : X \to Y$ is said to be fuzzy almost p-continuous if for each fuzzy point x_α in X and every fuzzy nbd V of $f(x_\alpha)$ in Y, $pcl(f^{-1}(V))$ is a fuzzy nbd of x_α in X.

Theorem 3.2. For a function $f: X \to Y$, the following statements are equivalent :

(a) f is fuzzy almost p-continuous,

(b) $f^{-1}(B) \le int(pcl(f^{-1}(B)))$, for all fuzzy open set B of Y,

(c) $f(clA) \le cl(f(A))$, for all $A \in FPO(X)$.

Proof (a) $\Rightarrow$ (b). Let B be any fuzzy open set in Y and $x_\alpha \in f^{-1}(B)$. Then $f(x_\alpha) \le B \Rightarrow B$ is a fuzzy nbd of $f(x_\alpha)$. By (a), $pcl(f^{-1}(B))$ is a fuzzy nbd of x_α in $X \Rightarrow x_\alpha \le int(pcl(f^{-1}(B)))$. Hence $f^{-1}(B) \le int(pcl(f^{-1}(B)))$.

(b) $\Rightarrow$ (a). Let x_α be a fuzzy point in X and B be a fuzzy nbd of $f(x_\alpha)$ in Y. Then $x_\alpha \le f^{-1}(B) \le int(pcl(f^{-1}(B)))$ (by (b)) $\le pcl(f^{-1}(B)) \Rightarrow pcl(f^{-1}(B))$ is a fuzzy nbd of x_α in X.

(b) $\Rightarrow$ (c). Let $A \in FPO(X)$. Then $1_Y \setminus cl(f(A))$ is a fuzzy open set in Y. By (b), $f^{-1}(1_Y \setminus cl(f(A))) \le int(pcl(f^{-1}(1_Y \setminus cl(f(A))))) = int(pcl(1_X \setminus f^{-1}(cl(f(A))))) \le$ $int(pcl(1_X \setminus f^{-1}(f(A)))) \le int(pcl(1_X \setminus A)) = 1_X \setminus cl(pintA) = 1_X \setminus clA$. Then $1_X \setminus f^{-1}(cl(f(A))) \le 1_X \setminus clA \Rightarrow clA \le f^{-1}(cl(f(A))) \Rightarrow f(clA) \le cl(f(A))$.

(c) $\Rightarrow$ (b). Let B be any fuzzy open set in Y. Then $pint(f^{-1}(1_Y \setminus B)) \in FPO(X)$. By (c), $f(cl(pint(f^{-1}(1_Y \setminus B)))) \le cl(f(pint(f^{-1}(1_Y \setminus B)))) \le cl(f(f^{-1}(1_Y \setminus B))) \le cl(1_Y \setminus B) =$ $1_Y \setminus B \Rightarrow f^{-1}(B) = 1_X \setminus f^{-1}(1_Y \setminus B) \le 1_X \setminus cl(pint(f^{-1}(1_Y \setminus B))) = 1_X \setminus cl(pint(1_X \setminus f^{-1}(B))) = int(pcl(f^{-1}(B)))$.

Note 3.3. It is clear from Theorem 3.2 that the inverse image under fuzzy almost p-continuous function of any fuzzy open set is fuzzy p^*-open and hence fuzzy preopen.

Theorem 3.4. For a function $f : X \to Y$, the following statements are equivalent :

(a) f is fuzzy almost p-continuous,

(b) $f^{-1}(B) \le int(pcl(f^{-1}(B)))$, for all fuzzy open set B of Y,

(c) for each fuzzy point x_α in X and each fuzzy open nbd V of $f(x_\alpha)$, there exists $U \in FP^*O(X)$ containing x_α such that $f(U) \le V$,

(d) $f^{-1}(F) \in FP^*C(X)$, for all fuzzy closed sets F in Y,

(e) for each fuzzy point x_α in X, the inverse image under f of every fuzzy nbd of $f(x_\alpha)$ is a fuzzy p^*-nbd of x_α in X,

(f) $f(p^*clA) \le cl(f(A))$, for all $A \in I^X$,

(g) $p^*cl(f^{-1}(B)) \le f^{-1}(clB)$, for all $B \in I^Y$,

(h) $f^{-1}(intB) \le p^*int(f^{-1}(B))$, for all $B \in I^Y$,

(i) for every basic open fuzzy set V in Y, $f^{-1}(V) \in FP^*O(X)$.

Proof (a) $\Leftrightarrow$ (b). Follows from Theorem 3.2 (a) $\Leftrightarrow$ (b).

(b) $\Rightarrow$ (c). Let x_α be a fuzzy point in X and V be a fuzzy open nbd of $f(x_\alpha)$. By (b), $f^{-1}(V) \le int(pcl(f^{-1}(V)))$ ... (1) . Now $f(x_\alpha) \le V \Rightarrow x_\alpha \in f^{-1}(V)$ $(= U$, say). Then $x_\alpha \in U$ and by (1), $U(= f^{-1}(V)) \in FP^*O(X)$ and $f(U) = f(f^{-1}(V)) \le V$.

(c) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and let $x_\alpha \le f^{-1}(V)$. Then $f(x_\alpha) \le V \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$. By (c), there exists $U \in FP^*O(X)$ containing x_α such that $f(U) \le V$. Then $x_\alpha \le U \le f^{-1}(V)$. Now $U \le int(pclU)$. Then $U \le int(pclU) \le int(pcl(f^{-1}(V))) \Rightarrow x_\alpha \le U \le int(pcl(f^{-1}(V))) \Rightarrow f^{-1}(V) \le int(pcl(f^{-1}(V)))$.

(b) $\Leftrightarrow$ (d). Obvious.

(b) $\Rightarrow$ (e). Let W be a fuzzy nbd of $f(x_\alpha)$. Then there exists a fuzzy open set V in Y such that $f(x_\alpha) \le V \le W \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$. Then by (b), $f^{-1}(V) \in FP^*O(X)$ and $x_\alpha \le f^{-1}(V) \le f^{-1}(W) \Rightarrow f^{-1}(W)$ is a fuzzy p^*-nbd of x_α.

(e) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and $x_\alpha \le f^{-1}(V)$. Then $f(x_\alpha) \le V \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$. By (e), $f^{-1}(V)$ is a fuzzy p^*-nbd of x_α. Then there exists $U \in FP^*O(X)$ containing x_α such that $U \le f^{-1}(V) \Rightarrow x_\alpha \le U \le int(pclU) \le int(pcl(f^{-1}(V))) \Rightarrow$ $f^{-1}(V) \le int(pcl(f^{-1}(V)))$.

(d) $\Rightarrow$ (f). Let $A \in I^X$. Then $cl(f(A))$ is a fuzzy closed set in Y. By (d), $f^{-1}(cl(f(A))) \in FP^*C(X)$ containing A. Therefore, $p^*clA \le p^*cl(f^{-1}(cl(f(A)))) =$ $f^{-1}(cl(f(A))) \Rightarrow f(p^*clA) \le cl(f(A))$.

(f) $\Rightarrow$ (d). Let B be a fuzzy closed set in Y. Then $f^{-1}(B) \in I^X$. By (f), $f(p^*cl(f^{-1}(B))) \le cl(f(f^{-1}(B))) \le clB = B \Rightarrow p^*cl(f^{-1}(B)) \le f^{-1}(B) \Rightarrow f^{-1}(B) \in FP^*C(X)$.

(f) $\Rightarrow$ (g). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (f), $f(p^*cl(f^{-1}(B))) \le cl(f(f^{-1}(B))) \le clB$ $\Rightarrow p^*cl(f^{-1}(B)) \le f^{-1}(clB)$.

(g) $\Rightarrow$ (f). Let $A \in I^X$. Let $B = f(A)$. Then $B \in I^Y$. By (g), $p^*clA = p^*cl(f^{-1}(B))$ $\le f^{-1}(clB) = f^{-1}(cl(f(A))) \Rightarrow f(p^*clA) \le cl(f(A))$.

(b) $\Rightarrow$ (h). Let $B \in I^Y$. Then $intB$ is a fuzzy open set in Y. By (b), $f^{-1}(intB) \le$ $int(pcl(f^{-1}(intB))) \Rightarrow f^{-1}(intB) \in FP^*O(X) \Rightarrow f^{-1}(intB) = p^*int(f^{-1}(intB)) \le p^*int(f^{-1}(B))$.

(h) $\Rightarrow$ (b). Let A be any fuzzy open set in Y. Then $f^{-1}(A) = f^{-1}(intA) \le p^*int(f^{-1}(A))$ (by (h)) $\Rightarrow f^{-1}(A) \in FP^*O(X)$.

(b) $\Rightarrow$ (i). Obvious.

(i) $\Rightarrow$ (b). Let W be any fuzzy open set in Y. Then there exists a collection $\{W_\alpha : \alpha \in \Lambda\}$ of fuzzy basic open sets in Y such that $W = \bigvee_{\alpha\in\Lambda} W_\alpha$. Now $f^{-1}(W) = f^{-1}(\bigvee_{\alpha\in\Lambda} W_\alpha) = \bigvee_{\alpha\in\Lambda} f^{-1}(W_\alpha) \in FP^*O(X)$ (by (i) and by Result 2.4). Hence (b) follows.

Theorem 3.5. A function $f : X \to Y$ is fuzzy almost p-continuous iff for each fuzzy point x_α in X and each fuzzy open q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy p^*-open set W in X with $x_\alpha qW$ such that $f(W) \le V$.

Proof. Let f be fuzzy almost p-continuous function and x_α be a fuzzy point in X and V be a fuzzy open set in Y with $f(x_\alpha)qV$. Let $f(x) = y$. Then $V(y) + \alpha > 1 \Rightarrow V(y) > 1-\alpha \Rightarrow V(y) > \beta > 1-\alpha$, for some real number β. Then V is a fuzzy open nbd of y_β. By Theorem 3.4 (a) $\Rightarrow$ (c), there exists $W \in FP^*O(X)$ containing x_β, i.e., $W(x) \ge \beta$ such that $f(W) \le V$. Then $W(x) \ge \beta > 1-\alpha \Rightarrow x_\alpha qW$ and $f(W) \le V$.

Conversely, let the given condition hold and let V be a fuzzy open set in Y. Put $W = f^{-1}(V)$. If $W = 0_X$, then we are done. Suppose $W \ne 0_X$. Then for any $x \in W_0$, let $y = f(x)$. Then $W(x) = [f^{-1}(V)](x) = V(f(x)) = V(y)$. Let us choose $m \in \mathcal{N}$ where $\mathcal{N}$ is the set of all natural numbers such that $1/m \le W(x)$. Put $\alpha_n = 1 + 1/n - W(x)$, for all $n \in \mathcal{N}$. Then for $n \in \mathcal{N}$ and $n \ge m$, $1/n \le 1/m \Rightarrow 1 + 1/n \le 1 + 1/m \Rightarrow \alpha_n = 1 + 1/n - W(x) \le 1 + 1/m - W(x) \le 1$. Again $\alpha_n > 0$, for all $n \in \mathcal{N} \Rightarrow 0 < \alpha_n \le 1$ so that $V(y) + \alpha_n > 1 \Rightarrow y_{\alpha_n} qV \Rightarrow V$ is a fuzzy open q-nbd of y_{α_n}. By the given condition, there exists $U_n^x \in FP^*O(X)$ such that $x_{\alpha_n} qU_n^x$ and $f(U_n^x) \le V$, for all $n \ge m$. Let $U^x = \vee\{U_n^x : n \in \mathcal{N}, n \ge m\}$. Then $U^x \in FP^*O(X)$ (by Result 2.4) and $f(U^x) \le V$. Again

$n \geq m \Rightarrow U_n^x(x) + \alpha_n > 1 \Rightarrow U_n^x(x) + 1 + 1/n - W(x) > 1 \Rightarrow U_n^x(x) > W(x) - 1/n \Rightarrow U_n^x(x) \geq W(x)$, for each $x \in W_0$. Then $W \leq U_n^x$, for all $n \geq m$ and for all $x \in W_0 \Rightarrow W \leq U^x$, for all $x \in W_0 \Rightarrow W \leq \bigvee_{x \in W_0} U^x = U$ (say) ... (1) and $f(U^x) \leq V$, for all $x \in W_0 \Rightarrow f(U) \leq V = U \leq f^{-1}(f(U)) \leq f^{-1}(V) = W$ ... (2). By (1) and (2), $U = W = f^{-1}(V) \Rightarrow f^{-1}(V) \in FP^*O(X)$. Hence by Theorem 3.2, f is fuzzy almost p-continuous.

Remark 3.6. If $f : X \to Y$ is fuzzy almost p-continuous, then by Note 3.3, inverse image of every fuzzy regular open set is fuzzy p^*-open. But the converse may not be true, as it seen from the following example.

Example 3.7. Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.4,$ $B(a) = 0.5, B(b) = 0.6$. Then (X,τ) and (X,τ_1) are fts's. Now $FPO(X,\tau) = \{0_X, 1_X, U, V\}$ where $U \leq A$, $V \not\leq 1_X \setminus A$ and $FPC(X,\tau) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\}$ where $1_X \setminus U \geq 1_X \setminus A$, $1_X \setminus V \not\geq A$. Consider the identity function $i : (X,\tau) \to (X,\tau_1)$. The fuzzy regular open sets in (X,τ_1) are only 0_X and 1_X. So obviously inverse image of every fuzzy regular open sets in (X,τ_1) is fuzzy p^*-open in (X,τ). But $B \in \tau_1$, $i^{-1}(B) = B \not\leq int_\tau(pcl_\tau(i^{-1}(B))) = int_\tau(pcl_\tau(B)) = int_\tau(1_X \setminus A) = A \Rightarrow i$ is not fuzzy almost p-continuous function.

Remark 3.8. The inverse image of a fuzzy preopen set under fuzzy almost p-continuous function may not be fuzzy p^*-open follows from the following example.

Example 3.9. Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B\}$, $\tau_1 = \{0_X, 1_X, C\}$ where $A(a) = 0.5,$ $A(b) = 0.3, B(a) = 0.5, B(b) = 0.4, C(a) = 0.5, C(b) = 0.4$. Then (X,τ) and (X,τ_1) are fts's. Consider the identity function $i : (X,\tau) \to (X,\tau_1)$. Clearly i is fuzzy almost p-continuous. Indeed, $FPO(X,\tau) = \{0_X, 1_X, U, V\}$ where $U \leq B, V \not\leq 1_X \setminus A$ and $FPC(X,\tau) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\}$

where $1_X \setminus U \geq 1_X \setminus B, 1_X \setminus V \ngeq A$. Then $i^{-1}(C) = C = int_\tau(pcl_\tau(i^{-1}(C))) = int_\tau(pcl_\tau(C)) = int_\tau(1_X \setminus B) = B = C$. Consider a fuzzy set D in X defined by $D(a) = 0.5, D(b) = 0.7$. Then $int_{\tau_1}(cl_{\tau_1}D) = int_{\tau_1}1_X = 1_X > D \Rightarrow D \in FPO(X, \tau_1)$. Now $i^{-1}(D) = D$. $int_\tau(pcl_\tau(i^{-1}(D))) = int_\tau(pcl_\tau D) = int_\tau D = B \ngeq D \Rightarrow D \notin FP^*O(X)$.

Let us now recall the following definition and theorem from [5] for ready references.

Definition 3.10 [5]. A function $f : X \to Y$ is said to be fuzzy almost continuous if for each fuzzy point x_α in X and each fuzzy nbd V of $f(x_\alpha)$ in Y, $cl(f^{-1}(V))$ is a fuzzy nbd of x_α in X.

Theorem 3.11 [5]. A function $f : X \to Y$ is fuzzy almost continuous iff $f^{-1}(B) \leq int(cl(f^{-1}(B)))$, for all fuzzy open set B in Y.

Remark 3.12. Since for any fuzzy set A in an fts (X, τ), $pclA \leq clA$, it is immediate that every fuzzy almost p-continuous function is fuzzy almost continuous. But the converse is not true, in general, follows from the following example.

Example 3.13. Let $X = \{a\}$, $\tau = \{0_X, 1_X, B\}$, $\tau_1 = \{0_X, 1_X, A\}$ where $A(a) = 1/3, B(a) = 0.4$. Then (X, τ) and (X, τ_1) are fts's. Now $FPO(X, \tau) = \{0_X, 1_X, U, V\}$ where $U \leq B, V > 1_X \setminus B$ and $FPC(X, \tau) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\}$ where $1_X \setminus U \geq 1_X \setminus B, 1_X \setminus V < B$. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Clearly i is fuzzy almost continuous. Indeed, other than 0_X and 1_X, A is the only fuzzy open set in (X, τ_1). Now $int_\tau(cl_\tau(i^{-1}(A))) = int_\tau(cl_\tau A) = int_\tau(1_X \setminus B) = B \geq A = i^{-1}(A)$. But i is not fuzzy almost p-continuous. Because, $int_\tau(pcl_\tau(i^{-1}(A))) = int_\tau(pcl_\tau A) = int_\tau A = 0_X \ngeq A$.

Lemma 3.14 [2]. Let Z, X, Y be fts's and $f_1 : Z \to X$ and $f_2 : Z \to Y$ be functions. Let $f : Z \to X \times Y$ be defined by $f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1, U_2 are fuzzy sets in Z, X, Y respectively such that $f(B) \leq U_1 \times U_2$, then $f_1(B) \leq U_1$ and $f_2(B) \leq U_2$.

Theorem 3.15. Let Z, X, Y be fts's. For any functions $f_1 : Z \to X, f_2 : Z \to Y$, if $f : Z \to X \times Y$, defined by $f(x) = (f_1(x), f_2(x))$, for all $x \in Z$, is fuzzy almost p-continuous, so are f_1 and f_2.

Proof. Let U_1 be any fuzzy open q-nbd of $f_1(x_\alpha)$ in X for any fuzzy point x_α in Z. Then $U_1 \times 1_Y$ is a fuzzy open q-nbd of $f(x_\alpha)$, i.e., $(f(x))_\alpha$ in $X \times Y$. Since f is fuzzy almost p-continuous, there exists $V \in FP^*O(Z)$ with $x_\alpha q V$ such that $f(V) \leq U_1 \times 1_Y$. By Lemma 3.14, $f_1(V) \leq U_1$, $f_2(V) \leq 1_Y$. Consequently, f_1 is fuzzy almost p-continuous.

Similarly, f_2 is fuzzy almost p-continuous.

Lemma 3.16 [1]. Let X, Y be fts's and let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. Then if A, B are fuzzy sets in X and Y respectively, $g^{-1}(A \times B) = A \bigcap f^{-1}(B)$.

Theorem 3.17. Let $f : X \to Y$ be a function from an fts X to an fts Y and $g : X \to X \times Y$ be the graph function of f. If g is fuzzy almost p-continuous, then f is so.

Proof. Let g be fuzzy almost p-continuous and B be a fuzzy set in Y. Then by Lemma 3.16, $f^{-1}(B) = 1_X \bigcap f^{-1}(B) = g^{-1}(1_Y \times B)$. Now if B is fuzzy open in Y, then $1_Y \times B$ is fuzzy open in $X \times Y$. Again, $g^{-1}(1_Y \times B) = f^{-1}(B) \in FP^*O(X)$ (by hypothesis) $\Rightarrow f$ is fuzzy almost p-continuous.

4. Fuzzy Almost p^*-Continuous Function: Some Characterizations

In this section a new type of function, viz., fuzzy almost p^*-continuous function is introduced and shown that inverse image under this function of any fuzzy p^*-open set is fuzzy p^*-open. It is shown that fuzzy almost p^*-continuous function is fuzzy almost p-continuous and the converse is true under certain condition.

Definition 4.1. A function $f : X \to Y$ is called fuzzy almost p^*-continuous if the inverse image of every fuzzy p^*-open set in Y is fuzzy p^*-open in X.

Theorem 4.2. For a function $f : X \to Y$, the following statements are equivalent :

(a) f is fuzzy almost p^*-continuous,

(b) for each fuzzy point x_α in X and each fuzzy p^*-open nbd V of $f(x_\alpha)$, there exists a fuzzy p^*-open nbd U of x_α in X and $f(U) \le V$,

(c) $f^{-1}(F) \in FP^*C(X)$, for all $F \in FP^*C(Y)$,

(d) for each fuzzy point x_α in X, the inverse image under f of every fuzzy p^*-open nbd of $f(x_\alpha)$ is a fuzzy p^*-open nbd of x_α in X,

(e) $f(p^*clA) \le p^*cl(f(A))$, for all $A \in I^X$,

(f) $p^*cl(f^{-1}(B)) \le f^{-1}(p^*clB)$, for all $B \in I^Y$,

(g) $f^{-1}(p^*intB) \le p^*int(f^{-1}(B))$, for all $B \in I^Y$.

Proof. The proof is similar to that of Theorem 3.4 and hence is omitted.

Theorem 4.3. A function $f : X \to Y$ is fuzzy almost p^*-continuous iff for each fuzzy point x_α in X and corresponding to any fuzzy p^*-open q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy p^*-open q-nbd W of x_α in X such that $f(W) \le V$.

Proof. The proof is similar to that of Theorem 3.5 and hence is omitted.

Remark 4.4. It is clear from definition that composition of two fuzzy almost p^*-continuous functions is fuzzy almost p^*-continuous.

Theorem 4.5. If $f: X \to Y$ is fuzzy almost p^*-continuous and $g: Y \to Z$ is fuzzy almost p-continuous, then $g \circ f: X \to Z$ is fuzzy almost p-continuous.

Proof. Obvious.

Remark 4.6. Every fuzzy almost p^*-continuous function is fuzzy almost p-continuous, but the converse is not true, in general, follows from the following example.

Example 4.7. Let $X = \{a\}$, $\tau = \{0_X, 1_X, A, B\}$, $\tau_1 = \{0_X, 1_X, C\}$ where $A(a) = 0.52$, $B(a) = 0.45, C(a) = 0.52$. Then (X, τ) and (X, τ_1) are fts's. Now $FPO(X, \tau) = \{0_X, 1_X, U, V, W\}$ where $1_X \setminus A < U \leq A, V > 1_X \setminus B, W \leq B$ and $FPC(X, \tau) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V, 1_X \setminus W\}$ where $1_X \setminus A \leq 1_X \setminus U < A, 1_X \setminus V < B, 1_X \setminus W \geq 1_X \setminus B$. $FPO(X, \tau_1) = \{0_X, 1_X, T\}$ where $T > 1_X \setminus C$ and $FPC(X, \tau_1) = \{0_X, 1_X, 1_X \setminus T\}$ where $1_X \setminus T < C$. Consider the identity function $i: (X, \tau) \to (X, \tau_1)$. Other than 0_X and 1_X, C is the only fuzzy open set in (X, τ_1). Now $i^{-1}(C) = C = int_\tau(pcl_\tau(C)) = int_\tau(1_X \setminus W)$ (where $(1_X \setminus W)(a) \geq 0.55) = A = C \Rightarrow i$ is fuzzy almost p-continuous. Let D be a fuzzy set in (X, τ_1), defined by $D(a) = 0.53$. Now $pcl_{\tau_1}(D) = 1_X$ and so $int_{\tau_1}(pcl_{\tau_1}(D)) = 1_X > D \Rightarrow D \in FP^*O(X, \tau_1)$. Then $i^{-1}(D) = D$. But $int_\tau(pcl_\tau(i^{-1}(D))) = int_\tau(pcl_\tau(D)) = int_\tau(1_X \setminus B) = A < D \Rightarrow i$ is not fuzzy almost p^*-continuous.

Note 4.8. From Remark 3.12 and Remark 4.6, we say that every fuzzy almost p^*-continuous function is fuzzy almost p-continuous and hence fuzzy almost continuous. But the converses are not true follow from Example 4.7. Here also $i^{-1}(C) = C = int_\tau(cl_\tau(C)) = int_\tau(1_X \setminus B) = A = C \Rightarrow i$ is fuzzy almost continuous.

To achieve the converse of Remark 4.6, we have to introduce some sort of fuzzy open-like function, as follows.

Definition 4.9. A function $f : X \to Y$ is said to be fuzzy preopen if $f(U)$ is fuzzy preopen in Y for every fuzzy preopen set U in X.

Lemma 4.10. If $f : X \to Y$ is fuzzy preopen function, then $f^{-1}(pclU) \leq pcl(f^{-1}(U))$, for any fuzzy set U in Y.

Proof. let $x_\alpha \not\leq pcl(f^{-1}(U))$ for some fuzzy set U in Y. Then there exists $W \in FPO(X)$ such that $x_\alpha q W$, $W \not{q} f^{-1}(U) \Rightarrow f(W) \not{q} U$. As f is fuzzy preopen function, $f(W) \in FPO(Y)$. Now $x_\alpha q W \Rightarrow f(x_\alpha) q f(W) \Rightarrow f(W)$ is a fuzzy preopen q-nbd of $f(x_\alpha)$ in Y, but $f(W) \not{q} U \Rightarrow f(x_\alpha) \not\leq pclU \Rightarrow x_\alpha \not\leq f^{-1}(pclU)$.

Theorem 4.11. If $f : X \to Y$ is fuzzy almost p-continuous and fuzzy preopen function, then f is fuzzy almost p^*-continuous function.

Proof. Let $V \in FP^*O(Y)$. Then $V \leq int(pclV)$. Since f is fuzzy almost p-continuous, $f^{-1}(V) \leq f^{-1}(int(pclV)) \leq int(pcl(f^{-1}(int(pclV))))$ (by Theorem 3.4 (a) $\Leftrightarrow$ (b)) $\leq int(pcl(f^{-1}(pclV)))$ $\leq int(pcl(pcl(f^{-1}(V))))$ (by Lemma 4.10) $= int(pcl(f^{-1}(V))) \Rightarrow f^{-1}(V) \in FP^*O(X) \Rightarrow f$ is fuzzy almost p^*-continuous.

5. Fuzzy p^*-Regular Space

In this section a new type of fuzzy regularity is introduced and shown that in this space fuzzy closed (resp., fuzzy open) set and fuzzy p^*-closed (resp., fuzzy p^*-open) set coincide.

Definition 5.1. An fts (X, τ) is said to be fuzzy p^*-regular if for each fuzzy p^*-closed set F in X and each fuzzy point x_α in X with $x_\alpha q(1_X \setminus F)$, there exist a fuzzy open set U in X and a fuzzy p^*-open set V in X such that $x_\alpha qU$, $F \le V$ and $U\not{q}V$.

Theorem 5.2. For an fts (X,τ), the following statements are equivalent:

(a) X is fuzzy p^*-regular,

(b) for each fuzzy point x_α in X and each fuzzy p^*-open set U in X with $x_\alpha qU$, there exists a fuzzy open set V in X such that $x_\alpha qV \le p^*clV \le U$,

(c) for each fuzzy p^*-closed set F in X, $\bigcap\{clV : F \le V, V \in FP^*O(X)\} = F$,

(d) for each fuzzy set G in X and each fuzzy p^*-open set U in X such that GqU, there exists a fuzzy open set V in X such that GqV and $p^*clV \le U$.

Proof (a)$\Rightarrow$(b). Let x_α be a fuzzy point in X and U, a fuzzy p^*-open set in X with $x_\alpha qU$. By (a), there exist a fuzzy open set V and a fuzzy p^*-open set W in X such that $x_\alpha qV$, $1_X \setminus U \le W$, $V\not{q}W$. Then $x_\alpha qV \le 1_X \setminus W \le U \Rightarrow x_\alpha qV$ and $p^*clV \le p^*cl(1_X \setminus W) = 1_X \setminus W \le U \Rightarrow x_\alpha qV \le p^*clV \le U$.

(b)$\Rightarrow$(a). Let F be a fuzzy p^*-closed set in X and x_α be a fuzzy point in X with $x_\alpha q(1_X \setminus F)$. Then $1_X \setminus F \in FP^*O(X)$. By (b), there exists a fuzzy open set V in X such that $x_\alpha qV \le p^*clV \le 1_X \setminus F$. Put $U = 1_X \setminus p^*clV$. Then $U \in FP^*O(X)$ and $x_\alpha qV$, $F \le U$ and $U\not{q}V$.

(b) $\Rightarrow$ (c). Let F be fuzzy p^*-closed set in X. It is clear that $F \le \bigcap\{clV : F \le V, V \in FP^*O(X)\}$.

Conversely, let $x_\alpha \not\le F$. Then $F(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus F)$ where $1_X \setminus F \in FP^*O(X)$. By (b), there exists a fuzzy open set U in X such that $x_\alpha qU \le p^*clU \le 1_X \setminus F$. Put $V = 1_X \setminus p^*clU$. Then

$F \le V$ and $U\not{q}V \Rightarrow x_\alpha \not\le clV \Rightarrow \bigcap\{clV : F \le V, V \in FP^*O(X)\} \le F$.

(c)$\Rightarrow$(b). Let V be any fuzzy p^*-open set in X and x_α be any fuzzy point in X with $x_\alpha qV$. Then $V(x)+\alpha>1 \Rightarrow x_\alpha \not\le (1_X \setminus V)$ where $1_X \setminus V \in FP^*C(X)$. By (c), there exists $G \in FP^*O(X)$ such that $1_X \setminus V \le G$ and $x_\alpha \not\le clG \Rightarrow$ there exists a fuzzy open set U in X with $x_\alpha qU$, $U\not{q}G \Rightarrow U \le 1_X \setminus G \le V$ $\Rightarrow x_\alpha qU \le p^*clU \le p^*cl(1_X \setminus G) = 1_X \setminus G \le V$.

(c)$\Rightarrow$(d). Let G be any fuzzy set in X and U be any fuzzy p^*-open set in X with GqU. Then there exists $x \in X$ such that $G(x)+U(x)>1$. Let $G(x)=\alpha$. Then $x_\alpha qU \Rightarrow x_\alpha \not\le 1_X \setminus U$ where $1_X \setminus U \in FP^*C(X)$. By (c), there exists $W \in FP^*O(X)$ such that $1_X \setminus U \le W$ and $x_\alpha \not\le clW \Rightarrow (clW)(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus clW)$. Let $V = 1_X \setminus clW$. Then V is fuzzy open in X and $V(x)+\alpha>1 \Rightarrow V(x)+G(x)>1 \Rightarrow VqG$ and $p^*clV = p^*cl(1_X \setminus clW) \le p^*cl(1_X \setminus W) = 1_X \setminus W \le U$.

(d)$\Rightarrow$(b). Obvious.

Note 5.3. It is clear from Theorem 5.2 that in a fuzzy p^*-regular space, every fuzzy p^*-closed set is fuzzy closed and hence every fuzzy p^*-open set is fuzzy open. As a result, in a fuzzy p^*-regular space, the collection of all fuzzy closed (resp., fuzzy open) sets and fuzzy p^*-closed (resp., fuzzy p^*-open) sets coincide.

Theorem 5.4. If $f : X \to Y$ is fuzzy almost p-continuous function and Y is fuzzy p^*-regular space, then f is fuzzy almost p^*-continuous.

Proof. Let x_α be a fuzzy point in X and V be any fuzzy p^*-open q-nbd of $f(x_\alpha)$ in Y where Y is fuzzy p^*-regular space. By Theorem 5.2 (a)$\Rightarrow$(b), there exists a fuzzy open set W in Y such that $f(x_\alpha)qW \le p^*clW \le V$. Since f is fuzzy almost p-continuous, by Theorem 3.5, there exists $U \in FP^*O(X)$ with $x_\alpha qU$ and $f(U) \le W \le V$. By Theorem 4.3, f is fuzzy almost p^*

-continuous function.

Let us now recall following definitions from [3,4] for ready references.

Definition 5.5 [3]. A collection $\mathcal{U}$ of fuzzy sets in an fts X is said to be a fuzzy cover of X if $\bigcup \mathcal{U} = 1_X$. If, in addition, every member of $\mathcal{U}$ is fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of X.

Definition 5.6 [3]. A fuzzy cover $\mathcal{U}$ of an fts X is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = 1_X$.

Definition 5.7 [4]. An fts (X, τ) is said to be fuzzy almost compact if every fuzzy open cover $\mathcal{U}$ of X has a finite proximate subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{clU : U \in \mathcal{U}_0\}$ is again a fuzzy cover of X.

Theorem 5.8. If $f : X \to Y$ is a fuzzy almost p-continuous surjective function and X is fuzzy p^*-regular and fuzzy almost compact space, then Y is fuzzy almost compact space.

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of Y. Then as f is fuzzy almost p-continuous, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy p^*-open cover and hence fuzzy open cover of X as X is fuzzy p^*-regular space. Since X is fuzzy almost compact, there are finitely many members $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $\bigcup_{i=1}^{n} cl(f^{-1}(U_i)) = 1_X$. Since X is fuzzy p^*-regular space, by Theorem 5.2, $clA = p^*clA$ and so $1_X = \bigcup_{i=1}^{n} p^*cl(f^{-1}(U_i)) \Rightarrow 1_Y = f(\bigcup_{i=1}^{n} p^*cl(f^{-1}(U_i))) =$ $\bigcup_{i=1}^{n} f(p^*cl(f^{-1}(U_i))) \leq \bigcup_{i=1}^{n} cl(f(f^{-1}(U_i)))$ (by Theorem 3.4 (a)$\Rightarrow$(f)) $\leq \bigcup_{i=1}^{n} cl(U_i) \Rightarrow \bigcup_{i=1}^{n} cl(U_i) = 1_Y$ $\Rightarrow Y$ is fuzzy almost compact space.

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