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Note on the Relation Between Cesaro Averages for Discrete Weighted Shift Operator and the Behavior of the Resolvent

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Abstract

The behavior of the limit of the Cesaro averages was studied in different ergodic theorems. In this note we study these averages for discrete weighted shift operators. The relation between the coefficients of the mentioned operators, the behavior of the Cesaro averages and the behavior of the resolvent is given.

Keywords: *Cesaro averages, power-bounded operator, Kriess resolvent condition, Ritt resolvent condition, weighted shift operator.*

1. Introduction

Let T be a linear bounded operator on a Banach space X. By $R(T)$ we mean *the spectral radius* of T. Without restriction of generality we will consider a case where $R(T)=1$, then the spectrum $\sigma(T)$ belongs to the open unit disk $\mathbb{D}=\{\lambda:|\lambda|<1\}$ and the resolvent can be represented by power series

$$(T-\lambda I)^{-1}=-\sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}T^{n}(|\lambda|>1).$$

It follow that

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$$\| (T-\lambda I)^{-1} \| \le \varphi_T(\frac{1}{|\lambda|})(|\lambda|>1), \tag{1}$$

where

$$\varphi_T(z)=\sum_0^\infty \| T^{n-1} \| z^n,$$

is an analytic function on $\mathbb{D}$.

We remind that T is called *power-bounded* operator if there exist a constant *C* such that

$$\sup_{n\ge 0} \| T^n \| = C < \infty.$$

Well-known that if T is power-bounded and the spectrum $\sigma(T)\in\mathbb{D}$, then the resolvent holds the so-called *Kriess resolvent condition* (see [2])

$$\| (T-\lambda I)^{-1} \| \le \frac{C}{|\lambda|-1}(|\lambda|>1). \tag{2}$$

If the spectrum is the smallest possible such that $\sigma(T)=\{1\}$, then the resolvent holds the so-called *Ritt resolvent condition* (see [3])

$$\| (T-\lambda I)^{-1} \| \le \frac{C}{|\lambda-1|}(|\lambda|>1). \tag{3}$$

However, if X is a finite dimensional space and the operator T satisfies the condition (2), then the operator T is power-bounded. But if X is infinite dimensional space, then the condition (2) only holds

$$\| T^n \| = O(n)(n\to\infty)$$

Moreover, in ([7], [13]) its shown that if the operator T satisfies the condition (3), then the operator T is power-bounded even if the space X is infinite dimensional.

If the the spectrum $\sigma(T)\in\overline{D}:=\{\lambda:|\lambda|\le 1\}$, in [3] obtained that the resolvent holds the (γ,ρ)

Kriess condition

$$\| (T-\lambda I)^{-1} \| \leq C \exp[\frac{\rho}{(|\lambda|-1)^{\gamma}}](|\lambda|>1) \tag{4}$$

If and only if

$$\| T^n \| \leq C \exp[\omega n^{\beta}] \quad \text{as} \quad n \to \infty,$$

where C is a constant and the relations between the orders and types of powers of operator and the resolvent are respectively given by:

$$\beta = \frac{\gamma}{\gamma+1}; \omega = \frac{(\rho\gamma)^{\frac{1}{\gamma+1}}}{\beta}.$$

One of the most well-known facts is that the Cesaro averages

$$\Psi_n(T) = \frac{1}{n}\sum_{k=0}^{n-1} T^k$$

which is studied in different ergodic theorems relate to powers of operator. This relation has been studied by many authors, such as J. Strikwerda and B. Wade in [15] there were shown that the boundedness of the second Cesaro averages of the power $e^{it}T$ is equivalent with the condition (2). If the operator T satisfies the condition (3), then the norm of the Cesaro averages is behave like $O(1)$ as $n \to \infty$, this equivalence was given by B. Negy and J. Zemanek in [13]. For background we refer to [5], [6], [9], [11], [12] and [16].

In this note we study the behavior of the Cesaro averages $\Psi_n(T)$ where the operator T belongs to concretely class of operators so-called *a discrete weighted shift operators*. And the main aim is to give the relation between the coefficients of the mentioned operator, behavior of $\Psi_n(T)$ as well as the behavior of the resolvent.

2. Weighted Shift Operators and Ergodic Theorems

An operator T on a Banach space $F(X)$ of functions on a set X is called *a weighted shift operator* (in short, WSO), if it can be represented by

$$aT_{\alpha} = a(x)u(\alpha(x)), x \in X, \tag{5}$$

where $\alpha : X \to X$ is a given map and $a(x)$ is a scalar function on X.

The operators of the form (5), as well as the operator algebras generated by them and related functional equations in different function spaces were studied by a number of authors and have various applications to the theory of dynamical systems, integro-functional, functional-differential, functional and difference equations, nonlocal boundary value problems and in other areas (see [1]).

Below we consider a case where X be a compact space, $\alpha : X \to X -$ is an invertible continuous map, $F(X) = C(X)$ and $a(x) \in C(X)$.

Firstly, we need to review the theorem on the spectral radius of WSO.

Theorem 1. [1] Let X be a compact space, $\alpha : X \to X -$ is an invertible continuous map and coefficient $a(x) \in C(X)$. Then the spectral radius of weighted shift operators on $C(X)$ can be given by:

$$R(aT_{\alpha}) = \max_{\nu \in M_{\alpha}(X)} \exp[\int_X \ln |a(x)| d\nu],$$

where $M_{\alpha}(X)$ is a set of all probability measures on X, invariant with respect to α.

The proof of the above theorem is based on the following ergodic theorem which is obtained by A. Antonevich and A. Lebedev in ([4], Theorem 2.1).

Theorem 2. Let X be a compact space, $\alpha : X \to X$ be a continuous map and $f \in C(X)$. Then

$$\lim_{n \to \infty} \max_{x} \frac{1}{n} \sum_{k=0}^{n-1} f(\alpha^k(x)) = \max_{\nu \in M_{\alpha}(X)} [\int_X f(x) d\nu].$$

Let $f(x) = \ln |a(x)|$, then

$$\ln \| [aT_{\alpha}]^n \| = \max_{x \in X} \prod_{k=0}^{n-1} | a(\alpha^k(x)) | = \max_{x \in X} \sum_{k=0}^{n-1} f(\alpha^k(x)).$$

Denoted by

$$\Psi_n(f) = \max_{x \in X} \frac{1}{n} \sum_{k=0}^{n-1} f(\alpha^k(x)).$$

In particular, if $R(aT_\alpha)=1$ then $\frac{\Psi_n}{n}\to 0$ and

$$\|[aT_\alpha]^n\|=\exp[n\Psi_n(f)](n>0), \tag{6}$$

from this follows that the rate of growth of $\|[aT_\alpha]^n\|$ depends on the rate of convergence of $\Psi_n(f)$ to zero.

3. The Main Result

Let's note here that the studying of weighted shift operators (5) is reduced to investigate the class of operators in which we are interesting, i.e. discrete weighted shift operators.

Let $l_p(\mathbb{N})$ be the space of one-sided sequence of numbers

$$u=(u(x)), x\in \mathbf{N}, u(x)\in\mathbb{C},$$

with the finite norm

$$\|u\|_p=(\sum_x |u(x)|^p)^{1/p}.$$

Denote by T the shift operator:

$$Tu(x)=u(x+1).$$

For any given bounded sequence $a(x)$, the operator acting on the space $l_p(\mathbb{N})$ by the formula

$$aT(u)(x)=a(x)u(x+1) \tag{7}$$

is said to be *a discrete weighted shift operator*, i.e $aT:=B$.

If $a(x)\to 1$ as $x\to+\infty$, then $\sigma(B)=\mathbb{D}$ and $R(B)=1$, in this case, the sequence $\Psi_n(f)$ can be rewritten in the form

$$\max_x \frac{1}{n}\sum_{j=0}^{n} f(x+j) \tag{8}$$

where $f(x)=\ln|a(x)|$.

Showed in [2] that if the sequence $\Psi_n(f)$ has arbitrarily slowly decreasing behavior then the growth of the resolvent of operator (7) can be arbitrary.

Theorem 3. Let

$$\varphi(z)=\sum_{n=1}^{\infty}\varphi_n z^n$$

be analytic function $\mathbb{D}$. There exists a discrete weighted shift operator B such that $\sigma(B)=\mathbb{D}$ and

$$\|(B-\lambda I)^{-1}\|\geq\varphi_B(\frac{1}{|\lambda|})(|\lambda|>1).$$

Moreover, for special class of coefficients $a(x)$ we obtained there that the resolvent can be found in a clear form, it means that the estimate of the resolvent coincided with right side of (1).

Theorem 4. Let $a(x)\to 1$ as $x\to+\infty$ and $a(x)$ are monotonically increasing. Then

$$\|(B-\lambda I)^{-1}\|=\varphi_B(\frac{1}{|\lambda|})(|\lambda|>1),$$

where $\varphi_B(\frac{1}{|\lambda|})$ is given in (1).

Below the notation $\Psi_1\sim\Psi_2$ means that Ψ_1 and Ψ_2 has the same order and type.

Theorem 5. Let the coefficients of discrete weighted shift operator B hold

$$a(x)\sim 1+\frac{\xi}{x^{\upsilon}}(x\to+\infty), \tag{9}$$

where $0<\upsilon<1$ and $\xi>0$. Then satisfying the following conditions:

1. $\Psi_n(B)=\dfrac{\xi}{1-\upsilon}n^{-\upsilon}(n>0).$
2. The resolvent holds

$$\|(B-\lambda I)^{-1}\|\sim e^{\rho\frac{1}{(|\lambda|-1)^{\gamma}}}(|\lambda|>1),$$

where

$$\gamma=\frac{1-\upsilon}{\upsilon};$$

$$\rho=[\frac{1}{1-\upsilon}(\xi)^{\frac{2-\upsilon}{\upsilon}}].$$

Proof. The estimate of powers of the operator (7) is given by

$$\| B^n \| = \sup_x \prod_{j=1}^{n-1} |a(x+j)| \, (n>0). \tag{10}$$

The condition (9) is equivalent to

$$a(x) \sim \exp[\frac{\xi}{x^{\upsilon}}](x \to +\infty).$$

From (10) follows

$$\| B^n \| = \exp[\xi \sum_1^{n-1} \frac{1}{x^{\upsilon}}]. \tag{11}$$

Obviously,

$$\sum_1^{n-1} \frac{1}{x^{\upsilon}} \sim \int_1^n x^{-\upsilon} d\upsilon = \frac{n^{1-\upsilon}}{(1-\upsilon)} (0<\upsilon<1).$$

By theorem 5, we have that the sequence (8) tends to 0, only if

$$\Psi_n(B) = \frac{\xi}{(1-\upsilon)n^{\upsilon}}.$$

This proves (1).

By applying (Theorem 3, [3]), we have that if the norm of power of operator B holds (11), then the upper estimate of the resolvent near the unite disc is holding the condition (4).

To obtain the lower estimate we need the following lemma.

Lemma 1. If the coefficients $a(x)>0$ are monotonically decreasing and $a(x) \to 1$, then the resolvent of the operator (5) for $|\lambda|>1$ on $l_1(\mathbb{N})$ holds:

$$\| (B-\lambda I)^{-1} \| \geq \sum_{k=0}^{n} \frac{\prod_{x=1}^{n} |a(x)|}{|\lambda|^{k+1}}.$$

Proof. From (10) we have for monotonically decreasing coefficients that

$$\| B^n \| = \prod_{x=1}^{n} a(x).$$

Let

$$e_n(x) = \begin{cases} 1, & x=n; \\ 0, & x \neq n. \end{cases}$$

By computation the sequence, we have for $n = 0$;

$$\frac{1}{\lambda}e_0 = (\ldots, 0, 0, 0, \underbrace{\frac{1}{\lambda}}, \ldots),$$

$n = 1$;

$$\frac{1}{\lambda^2}Be_0 = (\ldots, 0, 0, \underbrace{\frac{1}{\lambda^2}a(1)}, \ldots),$$

$n = 2$;

$$\frac{1}{\lambda^3}B^2e_0 = (\ldots, 0, \underbrace{\frac{1}{\lambda^3}a(1)a(2)}, \ldots),$$

So we have

$$(B-\lambda I)^{-1}e_0 = (\ldots, 0, 0, 0, \underbrace{\frac{1}{\lambda}, \frac{1}{\lambda^2}a(1), \ldots, \frac{1}{\lambda^{k+1}}\prod_{x=1}^{n}|a(x)|}, \ldots),$$

it follows that

$$\|(B-\lambda I)^{-1}e_0\| = \sum_{k=0}^{n}\frac{\prod_{x=1}^{n}|a(x)|}{|\lambda|^{k+1}}.$$

Obviously, the resolvent on $l_1(\mathbb{N})$ space holds

$$\|(B-\lambda I)^{-1}\| \geq \|(B-\lambda I)^{-1}e_0\| = \sup_n \sum_{k=0}^{n}\frac{\|B^k\|}{|\lambda|^{k+1}} (|\lambda| > 1).$$

By the above lemma its clear now that if the norm of power of operator B holds (11), then the lower estimate of the resolvent holds the condition (4).

In particular, the resolvent grows asymptotically equal to (γ, ρ) Kriess condition such that

$$\|(B-\lambda I)^{-1}\| \sim e^{\rho\frac{1}{(|\lambda|-1)^{\gamma}}}, \tag{12}$$

where

$$\gamma = \frac{1-\upsilon}{\upsilon};$$

$$\rho = [\frac{\xi}{1-\upsilon}(\xi)^{\frac{1-\upsilon}{\upsilon}}].$$

This proves (2).

Corollary 1. Assume that υ in (9) equal to 1. Then satisfying the following conditions:

1. $\Psi_n(B)=\frac{\xi \ln n}{n}(n>1)$.
2. The resolvent holds

$$\|(B-\lambda I)^{-1}\| \sim \frac{C_1}{(|\lambda|-1)^{\xi+1}}(|\lambda|>1).$$

Proof. The assumption that υ in (9) equal to 1, means that the condition (9) is equivalent to

$$a(x)\sim \exp[\frac{\xi}{x}](x\to+\infty).$$

Then from (10) follows

$$\|B^n\|=\exp[\xi\sum_1^{n-1}\frac{1}{x}]. \tag{13}$$

Note that

$$\sum_1^{n-1}\frac{1}{x}\sim\int_1^n\frac{1}{x}dx=\ln n.$$

Now, we have

$$\|B^n\|=n^{\xi}(\xi,n>0). \tag{14}$$

By theorem 5, we have that the sequence (8) tends to 0 only if

$$\Psi_n(B)=\frac{\xi\ln n}{n}.$$

This proves (1).

Let us consider the following inequality

$$\varphi_B(\frac{1}{|\lambda|})\le\|(B-\lambda I)^{-1}\|\le\tilde{\varphi}_B(\frac{1}{|\lambda|}), \tag{15}$$

where

$$\varphi_B(\frac{1}{|\lambda|})=\sum_{n=0}^{+\infty}\frac{\prod_{j=0}^{n-1}|a(k+j)|}{|\lambda|^n},$$

and

$$\tilde{\varphi}_B(\frac{1}{|\lambda|})=\sum_{n=0}^{+\infty}\frac{\|B^n\|}{|\lambda|^{n+1}}.$$

In (8, [Example 1. P.124]) shown that for any analytic function $f(z)$ can be represented by power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = n^{\alpha}, \alpha > 0.$ The asymptotic behavior of $f(z)$ on the unit disc is:

$$|f(z)| = \frac{C_1}{||z|-1|^{\alpha+1}}.$$

By applying this result to the functions $\varphi_B(\frac{1}{|\lambda|})$ and $\widetilde{\varphi}_B(\frac{1}{|\lambda|})$, this give us that if the norm of powers of the operator (7) holds (13), then the grows of the resolvent asymptotically equal to

$$\|(B-\lambda I)^{-1}\| \sim \frac{C_1}{(|\lambda|-1)^{\xi+1}} (|\lambda|>1). \quad (16)$$

This proves (2).

In the results (Theorems 8 and Corollary 1) we obtained that such a small change in the coefficients of discrete weighted shift operators leads to a significant change in the behavior of the resolvent as well as the behavior of the sequence Ψ_n.

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