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Characterizations of Generalized Convex Functions in Terms of Coderivative

Phan Nhat Tinh

Department of Mathematics, Faculty of sciences, University of Hue, Vietnam

Maria Do Rosário de Pinho

Faculty de Engenharia, Universidade do Porto, Porto, Portugal

Abstract

In this paper we characterize the convexity and the natural quasiconvexity of locally Lipschitz vector functions via the monotonicity and the quasimonotonicity of their Mordukhovich coderivatives.

Keywords: convexity, quasiconvexity, monotonicity, quasimonotonicity, coderivative, Clark-Rockafellar subgradient.

1. Introduction

In the last decades generalized convexity and generalized monotonicity have been widely and intensively study [8] since they have vast applications in several fields of sciences such as mathematical optimization, economics, finance etc.

The generalized convexity of functions are usually characterized by their generalized derivatives, directional derivatives or subdifferential [7] such as the Clarke generalized gradient and Jacobian [2], the generalized Dini directional derivative [2] and the Clarke-Rockafellar generalized subdifferential [13, 1, 18].

Recently, the concept of coderivative introduced by Mordukhovich has proved to be an efficient tool to treat problems in variational analysis, optimization and control optimization [14]. A natural question arises: can we use Mordukhovich coderivative to investigate characterizations of generalized convexity of

Corresponding author: Phan Nhat Tinh, Department of Mathematics, Faculty of sciences, University of Hue, Vietnam.

functions, particularly for vector-valued functions? The aim of this paper is to answer this question. Beside convex vector functions, the naturally quasiconvex vector functions introduced by Tanaka [17] are considered. As shown in [17, 11], this class of functions lies at the center of several kinds of generalized convex vector functions and it plays an important role in the proof of several basic theorems of vector optimization such as the sadle point theorem, the minimax theorem and the solvalidity theorem.

The paper is organized as follows. In the next section we introduce some preliminaries. Section 3 study characterizations of convex vector functions. The last section is devoted to characterizations of naturally quasiconvex vector functions.

2. Preliminaries

We denote the convex hull and the interior of a set $A \subset \mathbb{R}^n$ by co A and int A. Let $\phi : \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, +\infty]$ be lower semicontinuous and finite at $x \in \mathbb{R}^n$. The Dini upper directional derivative of ϕ at x in direction $u \in \mathbb{R}^n$, which is denoted $\phi'_+(x;u)$, is defined by

$$\phi'_+(x;u) := \limsup_{t \downarrow 0} \frac{\phi(x+tu) - \phi(x)}{t}.$$

The Clarke-Rockafellar subgradient [16] of ϕ at x is given by

$$\partial_{CR}\phi(x) := \{x^* \in \mathbb{R}^n \mid \langle x^*, u \rangle \le \phi^{\uparrow}(x;u), \forall u \in \mathbb{R}^n\},$$

where

$$\phi^{\uparrow}(x;u) := \sup_{\varepsilon > 0} \limsup_{\substack{x' \to_{\phi} x \\ t \downarrow 0}} \inf_{u' \in B(u,\varepsilon)} \frac{\phi(x' + tu') - \phi(x')}{t}$$

is the Clarke-Rockafellar directional derivative of ϕ at x in the direction u.

Now assume that ϕ is locally Lipschitz at x. Then the Clarke directional derivative [4] of ϕ at x in the direction u is defined as the following limit

$$\phi^{\circ}(x;u) := \limsup_{x' \to x;\, t \downarrow 0} \frac{\phi(x' + tu) - \phi(x')}{t}.$$

Obviously, $\phi'_+(x;u) \le \phi^\circ(x;u)$. The Clarke generalized gradient of ϕ at x, which is denoted $\partial_C \phi(x)$, is defined by

$$\partial_C \phi(x) := \{x^* \in \mathbb{R}^n \mid \langle x^*, u \rangle \le \phi^\circ(x;u), \forall u \in \mathbb{R}^n\}.$$

For locally Lipschitz functions, the Clarke-Rockafellar subgradient coincides with the Clarke generalized gradient. See Clarke [4] and Rockafellar [15, 16] for further properties.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz at $\overline{x} \in \mathbb{R}^n$. The Clarke generalized Jacobian of f at $\overline{x} \in \mathbb{R}^n$ is defined as

$$\partial_C f(\overline{x}) := \mathrm{co}\{\lim_{k\to\infty} Jf(x_k) : x_k \to \overline{x}, Jf(x_k) \text{ exists}\},$$

where $Jf(x_k)$ denotes the Jacobian of f at x_k. When $m = 1$ the Clarke generalized Jacobian coincides with the Clarke generalized gradient.

For $\xi \in \mathbb{R}^m$, we define the function $\xi f : \mathbb{R}^n \to \mathbb{R}$ as follows.

$$\xi f(x) := \langle \xi, f(x) \rangle, \forall x \in \mathbb{R}^n.$$

The link between the Clarke generalized Jacobian of the vector function f and the Clarke directional derivative of the real function ξf at $\overline{x}$ in the direction $u \in \mathbb{R}^n$ is given by

$$(\xi f)^\circ(\overline{x};u) = \max_{M \in \partial_C f(\overline{x})} \langle \xi, Mu \rangle.$$

Next we recall some notions from [14]. Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map. The sequential Painlevé-Kuratowski upper limit of F as $x \to \overline{x}$ is defined by

$$\operatorname*{Lim\,sup}_{x\to\overline{x}} F(x) := \{y \in \mathbb{R}^m \mid \exists\, x_k \to \overline{x}, y_k \in F(x_k) \text{ s.t. } y_k \to y\}.$$

Definition 2.1 Let $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $\overline{x} \in \mathbb{R}^n$ and let $\varepsilon \ge 0$. The ε*-subdifferential* of ϕ

at $\bar{x}$ is the set $\hat{\partial}_\varepsilon \phi(\bar{x})$ defined by

$$\hat{\partial}_\varepsilon \varphi(\bar{x}) := \left\{ x^* \in \mathbb{R}^n \mid \liminf_{x \to \bar{x}} \frac{\phi(x) - \phi(\bar{x}) - \left\langle x^*, x - \bar{x} \right\rangle}{\| x - \bar{x} \|} \geq -\varepsilon \right\}.$$

We put $\hat{\partial}_\varepsilon \phi(\bar{x}) = \varnothing$ if $|\phi(\bar{x})| = \infty$. When $\varepsilon = 0$ the set $\hat{\partial}_\varepsilon \phi(\bar{x})$, denoted by $\hat{\partial}\phi(\bar{x})$, is called the *Fréchet subdifferential* of ϕ at $\bar{x}$. The *limiting subdifferential* (or *Mordukhovich subdifferential*) of ϕ at $\bar{x}$ is given by

$$\partial \phi(\bar{x}) := \operatorname*{Lim\,sup}_{x \to \bar{x};\, \varepsilon \downarrow 0} \hat{\partial}_\varepsilon \phi(x).$$

Definition 2.2 The *Mordukhovich normal cone* to $A \subset \mathbb{R}^n$ at $x \in A$ is defined by

$$N(x; A) := \partial \delta(x; A),$$

where $\delta(.; A)$ is the indicator function of A.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a vector function and denote its graph by gph f.

Definition 2.3 The *Mordukhovich coderivative* of f at $\bar{x} \in \mathbb{R}^n$ is the set-valued map $D^M f(\bar{x})$: $\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by

$$D^M f(\bar{x})(v) := \{u \in \mathbb{R}^n \mid (u, -v) \in N((\bar{x}, f(\bar{x})); \mathrm{gph} f)\}.$$

For locally Lipschitz functions, we have the link between the Modurkhovich coderivative and the Clarke generalized Jacobian as follows.

Lemma 2.4 ([10]) If *f* is locally Lipschitz at $\bar{x} \in \mathbb{R}^n$, then $D^M f(\bar{x})$ consists of $n \times m$-matrices and satisfies the following set equality

$$\left[\partial_C f(\bar{x})\right]^\top (v) = \left[\operatorname{co}\left(D^M f(\bar{x})\right)\right](v), \forall v \in \mathbb{R}^m.$$

(Where $\left[\partial_C f(\overline{x})\right]^\top$ is the set of transposed matrices of $\partial_C f(\overline{x})$.)

3. Characterizations of Convex Vector-Valued Functions

From now on we assume that $\mathbb{R}^m$ is ordered by a closed and convex cone $K \subset \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^m$ is a vector function and $E \subset \mathbb{R}^n$ is a nonempty open convex set. Each vector of $\mathbb{R}^n$ or $\mathbb{R}^m$ is also considered as a $n \times 1$ or $m \times 1$ matrix. By K' we denote the polar cone of K, i.e.,

$$K' := \{\xi \in \mathbb{R}^m \mid \langle \xi, v \rangle \geq 0, \forall v \in K\}.$$

We recall that f is called convex with respect to K on E if for every $x, y \in E, \lambda \in (0,1)$,

$$\lambda f(x) + (1-\lambda) f(y) \in f(\lambda x + (1-\lambda) y) + K.$$

Supposing $\operatorname{int} K \neq \emptyset$, f is called strictly convex with respect to K on E if for every $x, y \in E, x \neq y, \lambda \in (0,1)$,

$$\lambda f(x) + (1-\lambda) f(y) \in f(\lambda x + (1-\lambda) y) + \operatorname{int} K.$$

Denote the space of $m \times n$-matrices by $L(\mathbb{R}^n, \mathbb{R}^m)$. Let $\mathcal{F} : E \rightrightarrows L(\mathbb{R}^n, \mathbb{R}^m)$ be a set-valued map with nonempty values. For any $\xi \in \mathbb{R}^m$, we define the set-valued map $\xi\mathcal{F} : E \rightrightarrows L(\mathbb{R}^n, \mathbb{R})$ by

$$\xi\mathcal{F}(x) := \{\xi^\top M : M \in \mathcal{F}(x)\},$$

where $\xi^\top$ is the transpose of ξ.

Definition 3.1 We say that

i) $\mathcal{F}$ is monotone with respect to K on E if

$$\mathcal{F}(x)(y-x) + \mathcal{F}(y)(x-y) \subset -K, \forall x, y \in E.$$

ii) Supposing int $K \neq \emptyset$, $\mathcal{F}$ is strictly monotone with respect to K on E if

$$\mathcal{F}(x)(y-x)+\mathcal{F}(y)(x-y)\subset -\mathrm{int}K, \forall x, y\in E, x\neq y.$$

(Where $\mathcal{F}(x)(v):=\{Mv : M\in\mathcal{F}(x)\}$). Observe that when $m=1$ and $K=\mathbb{R}_+$, Definition 3.1 collapses to the classical concept of monotonicity, i.e., $\mathcal{F}$ is monotone (resp., strictly monotone) if

$$\forall x, y\in E, x\neq y, \xi\in\mathcal{F}(x), \eta\in\mathcal{F}(y), \langle\xi, y-x\rangle+\langle\eta, x-y\rangle\leq 0.$$

$$(\text{resp., } \langle\xi, y-x\rangle+\langle\eta, x-y\rangle< 0.)$$

The following results ([9]) will be needed.

Lemma 3.2 For any $c\in\mathbb{R}^m$, we have

i) $c\in K$ if and only if $\langle\xi, c\rangle\geq 0, \forall\xi\in K'\backslash\{0\}.$

ii) Supposing $\mathrm{int}\,K\neq\varnothing, c\in\mathrm{int}\,K$ if and only if $\langle\xi, c\rangle> 0, \forall\xi\in K'\backslash\{0\}.$

Proposition 3.3 i) $\mathcal{F}$ is monotone with respect to K if and only if $\xi\mathcal{F}$ is monotone in the classical sense for every $\xi\in K'\backslash\{0\}$.

ii) Supposing $\mathrm{int}\,K\neq\varnothing$, $\mathcal{F}$ is strictly monotone with respect to K if and only if $\xi\mathcal{F}$ is strictly monotone in the classical sense for every $\xi\in K'\backslash\{0\}$.

Proof. i) For the ’only if’ part, let $\xi\in K'\backslash\{0\}$ be arbitrary. For every $x, y\in E$, since $\mathcal{F}$ is monotone, $\mathcal{F}(x)(y-x)+\mathcal{F}(y)(x-y)\subset -K$. Then by Lemma 2.4, we have $(\xi\mathcal{F})(x)(y-x)+(\xi\mathcal{F})(y)(x-y)=\xi^{\top}\left[\mathcal{F}(x)(y-x)+\mathcal{F}(y)(x-y)\right]\subset -\mathbb{R}_+$ which implies the monotonicity of $\xi\mathcal{F}$.

For the ’if’ part, suppose in the contrary that $\mathcal{F}$ is not monotone. Then there exist $x, y\in E$ such that

$$\mathcal{F}(x)(y-x)+\mathcal{F}(y)(x-y)-K.$$

Let $z \in \mathcal{F}(x)(y-x)+\mathcal{F}(y)(x-y)$ and $z \notin -K$. By using the strong separation theorem, one can find $\xi \in L(\mathbb{R}^m,\mathbb{R})\backslash\{0\}$ so that $\langle \xi, z\rangle > v \in -K \sup\langle \xi, v\rangle$ which implies $\xi \in K'\backslash\{0\}$ and $\xi(z)>0$. Therefore $\xi\mathcal{F}$ is not monotone. We get a contradiction.

ii) The proof is quite similar to the one of i).

For any $x, y \in \mathbb{R}^n$ we set

$$\begin{aligned}
[x,y] &:= \{\lambda x + (1-\lambda y) \mid 0 \le \lambda \le 1\} \\
(x,y) &:= \{\lambda x + (1-\lambda y) \mid 0 < \lambda < 1\} \\
[x,y) &:= \{\lambda x + (1-\lambda y) \mid 0 < \lambda \le 1\}.
\end{aligned}$$

Lemma 3.4 (Diewert's mean value theorem [5]) Let $D \subset \mathbb{R}^n$ be a nonempty convex set and $\phi: D \to \mathbb{R}$ be a lower semicontinuous function. Then, for every $a,b \in D, a \neq b,$ there exists $c \in [a,b)$ satisfying

$$\phi'_{+}(c; b-a) \ge \phi(b)-\phi(a).$$

Now assume that f is locally Lipschitz on E. We define the set-valued map $[\mathrm{co}D^M f]^\top : E \rightrightarrows L(\mathbb{R}^n,\mathbb{R}^m)$ by

$$[\mathrm{co}D^M f]^\top(x) := [\mathrm{co}(D^M f(x))]^\top, \forall x \in E,$$

where $[\mathrm{co}(D^M f(x))]^\top$ is the set of transpose matrices of $\mathrm{co}(D^M f(x))$.

Theorem 3.5 Assume that f is locally Lipschitz on E. Then f is convex with respect to K on E if and only if $[\mathrm{co}D^M f]^\top$ is monotone with respect to K on E.

Proof. For the 'if' part, we assume that $[\mathrm{co}D^M f]^\top$ is monotone with respect to K on E. Suppose that f is not convex on E. Then there exist $\xi \in K'\backslash\{0\}$, $x, y \in E, x \neq y, \lambda \in (0,1)$ such that

$$(\xi f)(\lambda x+(1-\lambda)y) > \lambda(\xi f)(x)+(1-\lambda)(\xi f)(y). \tag{1}$$

Set $z := \lambda x+(1-\lambda)y$. By Diewert's mean value theorem, there exist $c_1 \in [x,z), c_2 \in [y,z)$ such that

$$(\xi f)'_+(c_1; z-x) \geq (\xi f)(z)-(\xi f)(x), (\xi f)'_+(c_2; z-y) \geq (\xi f)(z)-(\xi f)(y).$$

Since $z-x=(1-\lambda)(y-x), z-y=\lambda(x-y)$ and the Dini upper direction derivative is positive homogenous , we have

$$\begin{aligned}\lambda(1-\lambda)(\xi f)'_+(c_1; y-x) &\geq \lambda[(\xi f)(z)-(\xi f)(x)]\\ \lambda(1-\lambda)(\xi f)'_+(c_2; y-x) &\geq (1-\lambda)[(\xi f)(z)-(\xi f)(y)].\end{aligned} \tag{2}$$

By (1),

$$\begin{aligned}&\lambda[(\xi f)(z)-(\xi f)(x)] > \lambda(1-\lambda)[(\xi f)(y)-(\xi f)(x)],\\ &(1-\lambda)[(\xi f)(z)-(\xi f)(y)] > \lambda(1-\lambda)[(\xi f)(x)-(\xi f)(y)]\end{aligned}$$

which together with (2) imply

$$(\xi f)'_+(c_1; y-x)+(\xi f)'_+(c_2; x-y) > 0. \tag{3}$$

Observe that we can find a number $\alpha > 0$ such that

$$y-x=\alpha(c_2-c_1). \tag{4}$$

By positive homogeneity of Dini upper directional derivatives, (3) and (4) give us

$$(\xi f)'_+(c_1; c_2-c_1)+(\xi f)'_+(c_2; c_1-c_2) > 0.$$

Taking in account Lemma 4 and the properties of Clarke subdifferential, we have

$$
\begin{aligned}
&\sup_{M\in[\mathrm{co}D^M f(c_1)]^\top} \xi^\top M(c_2-c_1)+\sup_{N\in[\mathrm{co}D^M f(c_2)]^\top} \xi^\top N(c_1-c_2)\\
&=\max_{M\in\partial_C f(c_1)} \xi^\top M(c_2-c_1)+\max_{N\in\partial_C f(c_2)} \xi^\top N(c_1-c_2)\\
&=(\xi f)^\circ(c_1;c_2-c_1)+(\xi f)^\circ(c_2;c_1-c_2)\\
&\geq(\xi f)'_+(c_1;c_2-c_1)+(\xi f)'_+(c_2;c_1-c_2)\\
&>0.
\end{aligned}
$$

Therefore there exist $M\in\left[\mathrm{co}(D^M f(c_1))\right]^\top, N\in\left[\mathrm{co}(D^M f(c_2))\right]^\top$ so that

$$\xi^\top M(c_2-c_1)+\xi^\top N(c_1-c_2)>0$$

which implies $\xi\left[\mathrm{co}D^M f\right]^\top$ is not monotone on E by definitions. Using Proposition 2, we see that $[\mathrm{co}D^M f]^\top$ is not monotone (with respect to K on E) contradicting the assumption. Thus f is convex with respect to K on E.

For the 'only if' part, let $\xi\in K'\backslash\{0\}$, $x,y\in E$ and $M\in[\mathrm{co}(D^M f(x))]^\top, N\in[\mathrm{co}(D^M f(x))]^\top$ be arbitrary. By Lemma 2.4, there exist $M_1\in\partial_C f(x), M_1\in\partial_C f(x)$ such that

$$\xi^\top M(y-x)+\xi^\top N(x-y)=\xi^\top M_1(y-x)+\xi^\top N_1(x-y). \tag{5}$$

By convexity of the scalar function ξf, we have

$$\xi^\top M_1(y-x)+\xi^\top N_1(x-y)\leq(\xi f)'_+(x,y-x)+(\xi f)'_+(y,x-y)\leq 0. \tag{6}$$

The relation (5) and (6) imply the monotonicity of $\xi[\mathrm{co}(D^M f)]^\top$. Since this is true for any $\xi\in K'\backslash\{0\}$, $[\mathrm{co}(D^M f)]^\top$ is monotone with respect to K on E by Proposition 2.

Theorem 3.6 Assume that $\mathrm{int}\,K\neq\varnothing$ and f is locally Lipschitz on E. Then f is strictly convex with respect to K on E if and only if $[\mathrm{co}D^M f]^\top$ is strictly monotone with respect to K on E.

Proof. It is quite similar the proof of Theorem 3.5.

4. Characterizations of quasiconvex vector-valued functions

Let E, K, f be as in Section 3.

Definition 4.1 [17] i) The function f is said to be naturally quasiconvex with respect to K on E if for every $x, y \in E, z \in [x, y]$, we have

$$f(z) \in [f(x), f(y)] - K.$$

ii) Supposing $\operatorname{int} K \neq \varnothing$, f is said to be strictly naturally quasiconvex with respect to K on E if for every $x, y \in E, x \neq y, z \in (x, y)$,

$$f(z) \in [f(x), f(y)] - \operatorname{int} K.$$

Obviously, when $m = 1$ and $K = \mathbb{R}_+$ the above definition collapses to the classical concept of quasiconvexity of scalar functions, i.e., f is said to be quasiconvex (resp., strictly quasiconvex) if for every $x, y \in E, x \neq y, z \in (x, y)$,

$$f(z) \leq \max\{f(x), f(y)\} (\text{resp., } f(z) < \max\{f(x), f(y)\}).$$

Proposition 4.2 i) The function f is naturally quasiconvex with respect to K on E if and only if ξf is quasiconvex on E for every $\xi \in K' \setminus \{0\}$.

ii) Supposing $\operatorname{int} K \neq \varnothing$, f is strictly naturally quasiconvex with respect to K on E if and only if ξf is strictly quasiconvex on E for every $\xi \in K' \setminus \{0\}$.

Proof. i) For the 'only if' part, let $\xi \in K' \setminus \{0\}$ be arbitrary. For every $x, y \in E, z \in [x, y]$, we have $f(z) \in [f(x), f(y)] - K$. Then $(\xi f)(z) \in [(\xi f)(x), (\xi f)(y)] - \mathbb{R}_+$ which implies

$(\xi f)(z) \le \max\{(\xi f)(x), (\xi f)(y)\}$. Hence ξf is quasiconvex. Conversely, seeking a contradiction suppose that f is not naturally quasiconvex. Then there exist $x, y \in D, z \in [x, y]$ such that

$$f(z) \notin [f(x), f(y)] - K.$$

The set $[f(x), f(y)] - K$ is closed, since K is closed, and $[f(x), f(y)]$ is compact. Then, by the strong separation theorem, there exists $\xi \in \mathbb{R}^m \setminus \{0\}$ such that

$$(\xi f)(z) > \sup \xi^\top ([f(x), f(y)] - C). \tag{7}$$

Next we show that $\xi \in K'$. Indeed, suppose that $\xi \notin K'$. Then there is $c \in K$ so that $\langle \xi, c \rangle < 0$. Then there exists $t > 0$ such that $t\langle \xi, c \rangle < (\xi f)(x) - (\xi f)(z)$, i.e., $(\xi f)(z) < (\xi f)(x) - \langle \xi, tc \rangle$. But this contradicts (7).

Since

$$\begin{aligned} \sup \xi^\top ([f(x), f(y)] - C) &= \sup \xi^\top [f(x), f(y)] \\ &= \max\{(\xi f)(x), (\xi f)(y)\}, \end{aligned} \tag{8}$$

(7) and (8)imply that ξf is not quasiconvex which contradicts assumptions. Hence f is naturally quasiconvex.

ii) Analogously.

Remark 4.3 *Proposition 4.2 i) differ from [6, Proposition 3.9] since the ordering cone is not required to have a nonempty interior.*

Let $\mathcal{F} : E \rightrightarrows L(\mathbb{R}^n, \mathbb{R}^m)$ be a set valued map with nonempty values.

Definition 4.4 We say that

i) $\mathcal{F}$ is quasimonotone with respect to K on E if for every $x, y \in E, M \in F(x), N \in F(y)$

$$[M(y - x), N(x - y)] \cap -K \neq \varnothing.$$

ii) Supposing $\operatorname{int} K \neq \emptyset$, F is strictly quasimonotone with respect to K on E if for every $x, y \in E, x \neq y, M \in \mathcal{F}(x), N \in \mathcal{F}(y)$

$$[M(y-x), N(x-y)] \cap -\operatorname{int} K \neq \emptyset.$$

We observe that when $m=1$ and $K=\mathbb{R}_+$, Definition 4.4 collapses to the classical concept of quasimonotonicity.

Proposition 4.5 i) $\mathcal{F}$ is quasimonotone with respect to K on E if and only if $\xi\mathcal{F}$ is quasimonotone (in the classical sense) on E for every $\xi \in K' \backslash \{0\}$.

ii) Supposing $\operatorname{int} K \neq \emptyset$, $\mathcal{F}$ is strictly quasimonotone with respect to K on E if and only if $\xi\mathcal{F}$ is strictly quasimonotone (in the classical sense) on E for every $\xi \in K' \backslash \{0\}$.

Proof. i) For the ’only if’ part , let $\xi \in K' \backslash \{0\}, x, y \in E$ and $M \in \mathcal{F}(x), N \in \mathcal{F}(y)$ be arbitrary. Since $\mathcal{F}$ is quasimonotone,

$$[M(y-x), N(x-y)] \cap -K \neq \emptyset.$$

Hence,

$$[\xi^\top M(y-x), \xi^\top N(x-y)] \cap -\mathbb{R}_+ \neq \emptyset.$$

Equivalently,

$$\min\{\xi^\top M(y-x), \xi^\top N(x-y)\} \leq 0,$$

which implies the monotonicity of $\xi\mathcal{F}$.

Conversely, suppose in the contrary that $\mathcal{F}$ is not quasimonotone. Then there exists $x, y \in E, M \in \mathcal{F}(x), y \in \mathcal{F}(y)$ such that

$$[M(y-x), N(x-y)] \cap -K = \emptyset. \tag{9}$$

Since K is closed, using the strong separation theorem, one can find $\xi \in \mathbb{R}^m \setminus \{0\}$ so that

$$\sup \xi^\top(-K) < \min \xi^\top [M(y-x), N(x-y)].$$

This inequality implies that $\xi \in K' \setminus \{0\}$ and

$$\min\{\xi^\top M(y-x), \xi^\top N(x-y)\} > 0.$$

Therefore $\xi\mathcal{F}$ is not quasimonotone. We get a contradiction.

ii) Analogously.

Lemma 4.6 [1, Proposition 2.2] Let X be a Banach space. Then, the Clarke Rockerfellar subdifferential of any quasiconvex function $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is quasimonotone.

Theorem 4.7 Assume that f is locally Lipschitz on E. Then f is naturally quasiconvex with respect to K on E if and only if $[\mathrm{co}D^M f]^\top$ is quasimonotone with respect to K on E.

Proof. For the 'only if' part, let $\xi \in K' \setminus \{0\}, x, y \in E$ and $M \in [\mathrm{co}D^M f]^\top(x), N \in [\mathrm{co}D^M f]^\top(y)$ be arbitrary. Then ξf is quasiconvex on E by Proposition 4.2 Since f is locally Lipschitz, the Clarke generalized gradient of ξf coincides with the Clarke-Rockafellar subgradient. Hence by Lemma 4.6, $\xi\partial_C f$ is quasimonotone on E, which together with Lemma 4.6 give us

$$\min\{\xi^\top M(y-x), \xi^\top N(x-y)\} \le 0.$$

Hence $\xi[\mathrm{co}D^M f]^\top$ is quasimonotone on E for any $\xi \in K' \setminus \{0\}$. By Proposition 3, $[\mathrm{co}D^M f]^\top$ is quasimonotone with respect to K on E.

Conversely, suppose in the contrary that f is not naturally quasiconvex on E. By Proposition 4.5, there is $\xi \in K' \setminus \{0\}$ so that ξf is not quasiconvex. Then there are $x, y \in E, x \neq y, z \in (x, y)$ satisfying

$$(\xi f)(z) > \max\{(\xi f)(x), (\xi f)(y)\}.$$

By Diewert's mean valued theorem, there exist $c_1 \in [x,z), c_2 \in [y,z)$ such that

$$(\xi f)'_+(c_1; z-x) \geq (\xi f)(z) - (\xi f)(x) > 0$$
$$(\xi f)'_+(c_2; z-y) \geq (\xi f)(z) - (\xi f)(y) > 0.$$

Observe that there are $\alpha, \beta > 0$ so that $z - x = \alpha(c_2 - c_1), z - y = \beta(c_1 - c_2)$, then by the positive homogeneity of the Dini upper directional derivative, we have

$$(\xi f)'_+(c_1; c_2 - c_1) > 0; (\xi f)'_+(c_2; c_1 - c_2) > 0.$$

Using properties of Clark generalized Jacobian, we can find $M \in \partial_C f(c_1), N \in \partial_C f(c_2)$ satisfying

$$\xi^\top M(c_2 - c_1) \geq (\xi f)'_+(c_1; c_2 - c_1) > 0$$
$$\xi^\top N(c_1 - c_2) \geq (\xi f)'_+(c_2; c_1 - c_2) > 0.$$

By Lemma 4, there exist $M_1 \in [\mathrm{co}D^M f]^\top(c_1), N_1 \in [\mathrm{co}D^M f]^\top(c_2)$ so that

$$\xi^\top M_1(c_2 - c_1) = \xi^\top M(c_2 - c_1) > 0$$
$$\xi^\top N_1(c_1 - c_2) = \xi^\top N(c_1 - c_2) > 0$$

which implies that $\xi[\mathrm{co}D^M f]^\top$ is not quasimonotone on E. Then by Proposition 4.5, $[\mathrm{co}D^M f]^\top$ is not quasimonotone. We get a contradiction.

Theorem 4.8 Assume that $\mathrm{int}K \neq \varnothing$ and f is locally Lipschitz on E. If $[\mathrm{co}D^M f]^\top$ is strictly quasimonotone with respect to K on E then, f is strictly naturally quasiconvex with respect to K on E.

Proof. It is similar the proof of the 'if' part of Theorem 4.7.

We should note that in general the converse statement of Theorem 4.8 is not true. For instant, consider the function $\phi : (-1, 3) \to \mathbb{R}$ defined as follows.

$$\phi(x) := \begin{cases} -x^2, & -1 < x < 0 \\ x^2, & 0 \le x < 1 \\ 1-(x-2)^2, & 1 \le x < 2 \\ 2+(x-2)^2, & 2 \le x < 3. \end{cases}$$

Then

$$\phi'(x) = \begin{cases} -2x, & -1 < x < 0 \\ 2x, & 0 \le x < 1 \\ -2(x-2), & 1 \le x < 2 \\ 2(x-2), & 2 \le x < 3 \end{cases}$$

and

$$\left[\mathrm{co}(D^M f(x))\right]^\top = \{\phi'(x)\}, \ \forall x \in (-1,3).$$

We see that ϕ is strictly quasiconvex but $\left[\mathrm{co}D^M f\right]^\top$ is not strictly quasimonotone since there are $x = 0, y = 2, M = 0 \in \left[\mathrm{co}D^M f(x)\right]^\top$, $N = 0 \in \left[\mathrm{co}D^M f(y)\right]^\top$ with $\min\{M(y-x), N(x-y)\} = 0$.

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