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Raw and Central Moments of a Variety of Generalized Beta-Binomial Distributions via Stirling Numbers

Rachel Ogal

School of Mathematics, CBPS College, University of Nairobi, P. O. Box 30197-00100, Nairobi, Kenya.

Abstract

In this article, we consider the construction of various compound Binomials by mixing the success parameter with generalized Beta distributions within the unit interval. We then proceed to calculate the moments of the derived mixtures using Stirling numbers of the second kind.

Keywords: Binomial mixtures, Central moments, Factorial moments, Generalized Beta distributions, Raw moments, Stirling numbers.

1. Introduction

If we let a random variable X to represent the total number of successes in n Bernoulli experiments, then X is said to be Binomially distributed and has parameters n and ϑ.

ϑ denotes the probability of success hence lies within the unit interval, $0 < \vartheta < 1$, and can be taken to be a continuous random variable with density $g(\vartheta)$. This formulates the density $f(x)$ referred to as a Binomial mixture.

$$f(x) = \int_0^1 \binom{n}{x} \vartheta^x (1-\vartheta)^{n-x} g(\vartheta) d\vartheta \qquad (1)$$

The Beta distribution offers a tractable prior to the Binomial, and has been used in this sense (see, for instance, Skellam (1948); Ishii and Hayakawa (1960); Chatfield and Goodhardt (1970)).

Various generalizations of the Beta distribution have also been developed and used as priors to the Binomial (see, for instance, Gerstenkorn (2004); Rodriguez-Avi et al. (2007)).

We aim to construct Binomial mixtures whose priors are generalizations of the Beta, and derive some of their properties using stirling numbers of the second kind.

2. Stirling Numbers of the Second Kind

Stirling numbers are close relatives of the Binomial coefficients and are found in two kinds: stirling numbers of the first kind, stirling numbers of the second kind. They are named after James Stirling (1692-1770).

Consider the descending factorial $x_{(k)} = x(x-1)(x-2)\ldots(x-k+1)$, which can be identified as a polynomial of x of degree k. Multiplying it out, and rearranging the terms gives us the following formula (as appearing in Gould (1978))

$$x_{(k)} = \sum_{j=0}^{k} s(k,j)x^{j}$$

where $k = 0,1,\ldots$ and $s(k,j)$ denotes the stirling numbers of the first kind.

Inversely, the k^{th} power of x may be written in terms of a polynomial of factorials of x of degree j and stirling numbers of the second kind, $S(k,j)$, (see Joarder and Mahmood (1997)).

$$x^{k} = \sum_{j=0}^{k} S(k,j)x_{(j)} \tag{2}$$

where $k = 0,1,2,\ldots$.

An inductive proof for 2 is given in Gould (1978). We concentrate on stirling numbers of the second kind, $S(k,j)$ and use them to calculate raw moments, hence central moments of derived Binomial mixtures.

k\j	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0
3	0	1	3	1	0	0	0	0	0	0
4	0	1	7	6	1	0	0	0	0	0
5	0	1	15	25	10	1	0	0	0	0
6	0	1	31	90	65	15	1	0	0	0
7	0	1	63	301	350	140	21	1	0	0
8	0	1	127	966	1701	1050	266	28	1	0
9	0	1	255	3025	7770	6951	2646	462	36	1

Table 1. Stirling numbers of the second kind, $S(k,j)$

Table 1 illustrates the $S(k,j)$, up to $k=9$. It can be extended using simple algebraic manipulation (as given by Joarder and Mahmood (1997)) where for instance, to calculate $S(4,j)$ for varying values of j we expand 2 for $k=4$ getting

$$\begin{aligned} x^4 &= S(4,0)x_{(0)} + S(4,1)x_{(1)} + S(4,2)x_{(2)} + S(4,3)x_{(3)} + S(4,4)x_{(4)} \\ &= S(4,1)x + S(4,2)\left[x^2 - x\right] + S(4,3)\left[x^3 - 3x^2 + 2x\right] + \\ &\quad S(4,4)\left[x^4 - 6x^3 + 11x^2 - 6x\right] \end{aligned}$$

We then equate the coefficients of x, x^2, x^3 and x^4 found on the RHS, to corresponding values on the LHS, giving us the following sets of equations

$$\begin{aligned} S(4,4) &= 1 \\ S(4,3) - 6S(4,4) &= 0 \\ S(4,2) - 3S(4,3) + 11S(4,4) &= 0 \\ S(4,1) - S(4,2) + 2S(4,3) - 6S(4,4) &= 0 \end{aligned}$$

Upon solving these we get $S(4,4)=1, S(4,3)=6, S(4,2)=7$ and $S(4,1)=1$.

Another method would be to use the formula

$$S(k,j)=\frac{1}{j!}\sum_{i=0}^{j}(-1)^{i}\binom{r}{i}\left(r-i\right)^{k} \tag{3}$$

for $0\le j\le k$ and $j>0$, as given by (Roberts 1984)

3. Factorial Moments

Let X be a Binomial mixture, then the jth factorial moment of X about the origin, $E\left[X_{(j)}\right]$ is given by

$$E\left[X_{(j)}\right]=\sum_{x=0}^{n}\int_{0}^{1}x_{(j)}\binom{n}{x}\vartheta^{x}\left(1-\vartheta\right)^{n-x}f(\vartheta)d\vartheta$$

$$=\int_{0}^{1}\left[\sum_{x=0}^{n}\frac{x!}{(x-j)!}\frac{n!}{n-x!x!}\vartheta^{x}\left(1-\vartheta\right)^{n-x}\right]f(\vartheta)d\vartheta$$

$$=\int_{0}^{1}\left[\sum_{x=0}^{n}\binom{n-j}{x-j}n_{(j)}\vartheta^{x}\left(1-\vartheta\right)^{n-x}\right]f(\vartheta)d\vartheta$$

$$=\int_{0}^{1}n_{(j)}\vartheta^{j}\left[\sum_{x=0}^{n}\binom{n-j}{x-j}\vartheta^{x-j}\left(1-\vartheta\right)^{n-x}\right]f(\vartheta)d\vartheta$$

$$=(n)_{j}E\left[\vartheta^{j}\right] \tag{4}$$

4. Raw and Central Moments

The raw moments of a discrete random variable X are defined by the formula

$$m_{k}=\sum_{i=1}^{n}x_{i}^{k}P(x=x_{i})=E\left[X^{k}\right]$$

where k refers to the order of the moment.

Consider 2, taking the expectation of both sides of the equation gives us

$$E\left[X^{k}\right]=\sum_{j=0}^{k}S(k,j)E\left[X_{(j)}\right] \tag{5}$$

hence a general formula for calculating the raw moments of the discrete generalized Beta-binomial distributions.

The moments about the mean, or central moments of univariate discrete distributions are defined by the formula

$$\mu_r = \sum_{i=1}^{n} (x - m_1)^r P(x = x_i) \tag{6}$$

(6) can be simplified using Binomial expansion to get the following relation

$$\begin{aligned}\mu_r &= \sum_{i=1}^{n} (x - m_1)^r P(x = x_i) \\ &= \sum_{j=0}^{r} \binom{r}{j} (x_i)^{r-j} (-1)^j (m_1)^j \sum_{i=1}^{n} P(X = X_i) \\ &= \sum_{j=0}^{r} \binom{r}{j} (-1)^j (m_1)^j m_{r-j}, (r \geq 1)\end{aligned}$$

The second order central moment μ_2, is called the variance, commonly denoted by $Var(X)$ or σ^2. In this paper, we only focus on the variances of generated Binomial mixtures.

5. Application

5.1. Beta-binomial distribution

From 1, the probability function of a Beta-binomial distribution is given by

$$\begin{aligned}f(x) &= \int_0^1 \binom{n}{x} \vartheta^x (1-\vartheta)^{n-x} \frac{\vartheta^{\alpha-1}(1-\vartheta)^{\beta-1}}{B(\alpha,\beta)} d\vartheta \\ &= \binom{n}{x} \frac{B(x+\alpha, n-x+\beta)}{B(\alpha,\beta)},\end{aligned}$$

$$(x = 0, 1, \ldots, n;\ \alpha, \beta > 0;\ B(\alpha,\beta) = \frac{\Gamma\alpha\Gamma\beta}{\Gamma\alpha+\beta}) \tag{7}$$

5.1.1. Factorial Moments

From 4, the jth factorial moments of the Beta-binomial are given by

$$E\left[X_{(j)}\right] = n_{(j)} E(\vartheta^j)$$

where $E(\vartheta^j)$ refers to the jth raw moment of the Beta prior calculated as

$$E(\vartheta^{j}) = \int_0^1 \frac{\vartheta^{j+\alpha-1}\left(1-\vartheta\right)^{\beta-1}}{B(\alpha,\beta)} d\vartheta$$
$$= \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)}$$

Giving

$$E\left[X_{(j)}\right] = n_{(j)} \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)} \tag{8}$$

5.1.2. Raw and Central Moments

To find the kth raw moment of the Beta-binomial distribution, we substitute 8 into 5, to give

$$E\left[X^{k}\right] = \sum_{j=0}^{k} S(k,j) E\left[X_{(j)}\right]$$

$$= \sum_{j=0}^{k} S(k,j) n_{(j)} \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)} \tag{9}$$

The mean is then calculated as

$$E\left[X\right] = \sum_{j=0}^{1} S(1,j) E\left[X_{(j)}\right]$$
$$= n \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

The 2nd moment m_2, is

$$E\left[X^{2}\right] = \sum_{j=0}^{2} S(2,j) n_{(j)} \frac{B(j+\alpha,\beta)}{B(\alpha,\beta)}$$
$$= \frac{nB(\alpha+1,\beta)}{B(\alpha,\beta)} \left[1 + (n-1)\frac{B(\alpha+2,\beta)}{B(\alpha+1,\beta)}\right]$$

The variance $m_2 - m_1^2$ is

$$Var(X) = \frac{nB(\alpha+1,\beta)}{B(\alpha,\beta)} \left[1 + (n-1)\frac{B(\alpha+2,\beta)}{B(\alpha+1,\beta)}\right] - \left[\frac{nB(\alpha+1,\beta)}{B(\alpha,\beta)}\right]^2$$
$$= \frac{nB(\alpha+1,\beta)}{B(\alpha,\beta)} \left[1 + (n-1)\frac{B(\alpha+2,\beta)}{B(\alpha+1,\beta)} - \frac{nB(\alpha+1,\beta)}{B(\alpha,\beta)}\right]$$

5.2. McDonald's Generalized Beta-binomial distribution

The probability function of a McDonald's Generalized Beta-binomial (MGBB) distribution is given by

$$f(x)=\int_0^1 \binom{n}{x} \vartheta^x (1-\vartheta)^{n-x} \frac{\lambda \vartheta^{\alpha\lambda-1}\left(1-\vartheta^{\lambda}\right)^{\beta-1}}{B(\alpha,\beta)} d\vartheta$$

Further evaluation gives us

$$f(x)=\frac{\binom{n}{x}\lambda}{B(\alpha,\beta)}\int_0^1 \vartheta^{x+\alpha\lambda-1}(1-\vartheta)^{n-x}\left(1-\vartheta^{\lambda}\right)^{\beta-1} d\vartheta$$

$$=\frac{\binom{n}{x}\lambda}{B(\alpha,\beta)}\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^k\int_0^1 \vartheta^{x+\alpha\lambda-1+k\lambda}(1-\vartheta)^{n-x} d\vartheta$$

$$=\binom{n}{x}\lambda\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^k \frac{B(x+\alpha\lambda+k\lambda, n-x+1)}{B(\alpha,\beta)},$$

$$(x=0,1,\ldots,n;\alpha,\beta,\lambda>0) \tag{10}$$

as found by Manoj et al. (2013).

We seek to re-write 10 in such a way as to eliminate the infinite series that occurs within it, hence easen computation. This leads to what we call the **Moments method** of deriving Binomial mixtures, as given by Sivaganesan and Berger (1993).

The **Moments method** further evaluates 1 by expanding the $(1-\vartheta)^{n-x}$, where $(n-x)$ is a positive integer, found within the integral to give

$$f(x)=\binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k\int_0^1 \vartheta^{x+k} g(\vartheta) d\vartheta$$

$$=\binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k E(\vartheta^{x+k}) \tag{11}$$

In this definition, $E(\vartheta^{x+k})$ is the $(x+k)$th moment of the mixing distribution $g(\vartheta)$.

For the McDonald's Generalized Beta distribution, the $(x+k)$th moment is

$$E(\vartheta^{j}) = \frac{\lambda}{B(\alpha,\beta)}\int_0^1 \vartheta^{x+k+\alpha\lambda-1}\left(1-\vartheta^{\lambda}\right)^{\beta-1} d\vartheta$$
$$= \frac{B(\frac{x+k}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)} \quad (12)$$

Substituting (12) into (11) gives us

$$f(x) = \binom{n}{x}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k \frac{B(\frac{x+k}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)} \quad (13)$$

(13) is a second way of writing the MGBB distribution.

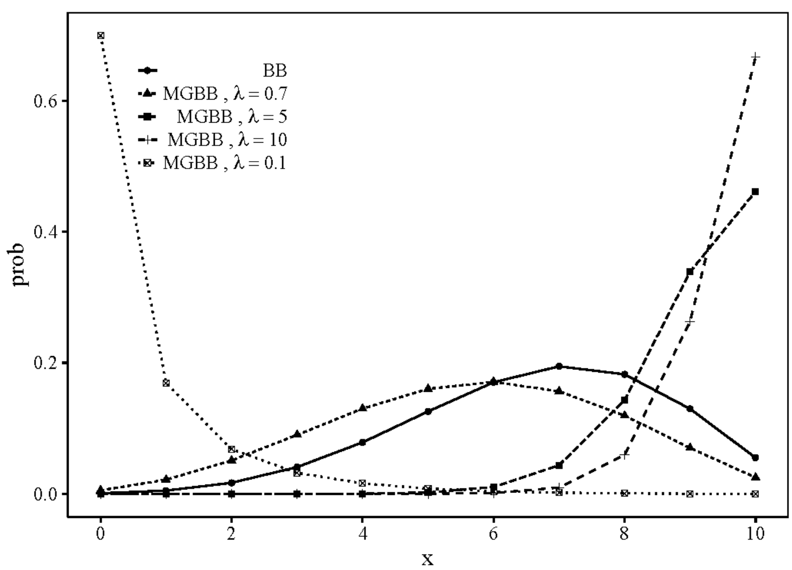

Figure 1. Probability function plots of the Beta-binomial (BB) for $\alpha = 8, \beta = 4, n = 10$ and the MGBB for $\alpha = 8, \beta = 4$ and varying values of λ

Figure 1 displays a graphical comparison of the BB with the MGBB distribution for varying values of λ. The MGBB distribution becomes more negatively skewed as λ increases in value. For $\lambda = 1$ the MGBB reduces to the BB distribution.

5.2.1. Factorial Moments

The jth raw moment of the McDonald's Generalized Beta distribution from (12) is

$$\frac{B(\frac{j}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)}$$

So that the jth factorial moments of the MGBB distribution are

$$E\left[X_{(j)}\right] = n_{(j)}\frac{B(\frac{j}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)} \tag{14}$$

5.2.2. Raw and Central Moments

Using 14, the kth raw moment of 10 can be easily derived as

$$E\left[X^k\right] = \sum_{j=0}^{k} S(k,j)n_{(j)}\frac{B(\frac{j}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)} \tag{15}$$

So that m_1 is $n\frac{B(\frac{1}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)}$ and m_2 is given by

$$E\left[X^2\right] = S(2,1)n_{(1)}\frac{B(\frac{1}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)} + S(2,2)n_{(2)}\frac{B(\frac{2}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)}$$
$$= \frac{nB(\frac{1}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)}\left[1+(n-1)\frac{B(\frac{2}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)}\right]$$

The variance, $m_2 - m_1^2$, is

$$Var(X)=\frac{nB(\frac{1}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)}\left[1+(n-1)\frac{B(\frac{2}{\lambda}+\alpha,\beta)}{B(\frac{1}{\lambda}+\alpha,\beta)}\right]-n^2\left[\frac{B(\frac{1}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)}\right]^2$$

$$=\frac{nB(\frac{1}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)}\left[1+(n-1)\frac{B(\frac{2}{\lambda}+\alpha,\beta)}{B(\frac{1}{\lambda}+\alpha,\beta)}-n\frac{B(\frac{1}{\lambda}+\alpha,\beta)}{B(\alpha,\beta)}\right]$$

5.3. Libby and Novick's Generalized Beta-binomial distribution

The probability function of Libby and Novick's generalized Beta (LNBB) distribution is given by

$$g(\vartheta)=\frac{c^{\alpha}\vartheta^{\alpha-1}(1-\vartheta)^{\beta-1}}{B(\alpha,\beta)[1-(1-c)\vartheta]^{\alpha+\beta}},\ (0<\vartheta<1;\alpha,\beta,c>0)$$

(see Libby and Novick (1982); Nadarajah and Kotz (2007))

From 1, the probability function of Libby and Novick's Generalized Beta-binomial (LNBB) distribution is given by

$$\begin{aligned} f(x)&=\int_0^1\binom{n}{x}\frac{\vartheta^{x+\alpha-1}(1-\vartheta)^{n-x+\beta-1}c^{\alpha}}{B(\alpha,\beta)[1-(1-c)\vartheta]^{\alpha+\beta}}d\vartheta \\ &=\binom{n}{x}\frac{c^{\alpha}B(\alpha+x,n-x+\beta)}{B(\alpha,\beta)}\int_0^1\frac{\vartheta^{x+\alpha-1}(1-\vartheta)^{n-x+\beta-1}c^{\alpha}}{B(\alpha+x,n-x+\beta)\left[1-(1-c)\vartheta\right]^{\alpha+ta}}d\vartheta \\ &=\binom{n}{x}{}_2\tilde{F}_1(x+\alpha,\alpha+\beta\alpha+\beta+n;1-c)\frac{c^{\alpha}B(\alpha+x,n-x+\beta)}{B(\alpha,\beta)}, \end{aligned} \quad (16)$$

$$(x=0,1,\ldots,n;\alpha,\beta,c>0;\ {}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;z)=\int_0^1\frac{\vartheta^{\alpha-1}(1-\vartheta)^{\beta-1}}{B(\alpha,\beta)(1-z\vartheta)^{\gamma}}d\vartheta)$$

(16) reduces to the standard Beta-binomial distribution 7, when $c=1$.

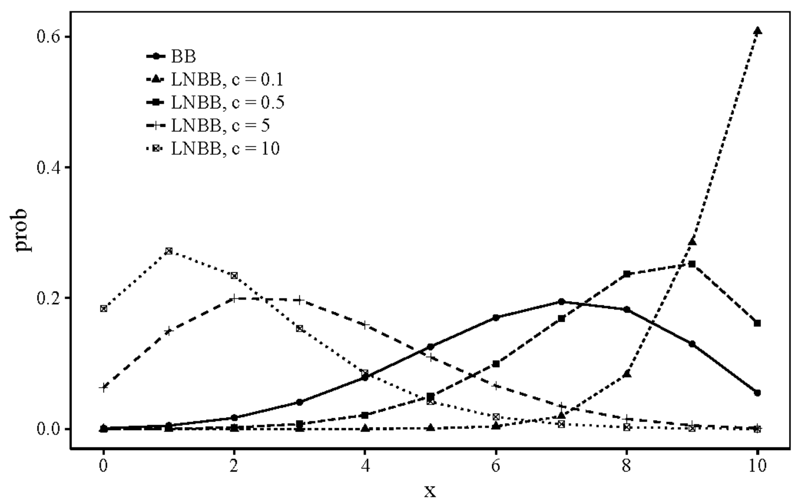

Figure 2. Probability function plots of the Beta-binomial (BB) for $\alpha=8, \beta=4, n=10$ and the Libby and Novick Beta-binomial (LNBB) for $\alpha=8, \beta=4$ and varying values of c

Figure 2 compares plots of the BB with the LNBB distribution for varying values of the additional parameter c. The LNBB is left skewed for $0<c<1$ with kurtosis increasing as c tends towards 1. For $c>1$, the LNBB is positively skewed, while its kurtosis increases with an increase in the value of c within this interval. The LNBB approaches the BB at $c=1$.

5.3.1. Factorial Moments

The jth raw moment of the LNBB distribution is given by

$$E(\vartheta^j)=\int_0^1 \frac{c^\alpha \vartheta^{j+\alpha-1}(1-\vartheta)^{\beta-1}}{B(\alpha,\beta)\left[1-(1-c)\vartheta\right]^{\alpha+\beta}}d\vartheta$$
$$=\frac{c^\alpha B(\alpha+j,\beta)}{B(\alpha,\beta)}\,{}_2\tilde{F}_1(j+\alpha,\alpha+\beta;\alpha+\beta+j;1-c)$$

Thus the jth factorial moment of the LNBB distribution is

$$E\left[X_{(j)}\right]=\frac{n_{(j)}c^{\alpha}B(\alpha+j,\beta)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha+j,\alpha+\beta;\alpha+\beta+j;1-c)} \tag{17}$$

5.3.2. Raw and Central Moments

From 5, the raw moments of 16 are given by

$$\begin{aligned} E\left[X^k\right] &= \sum_{j=0}^{k} S(k,j)E\left[X_{(j)}\right] \\ &= \sum_{j=0}^{k} S(k,j)\frac{n_{(j)}c^{\alpha}B(\alpha+j,\beta)}{B(\alpha,\beta)}\,{}_2\tilde{F}_1(\alpha+j,\alpha+\beta;\alpha+\beta+j;1-c) \end{aligned} \tag{18}$$

Thus its mean m_1, is $\frac{n_{(j)}c^{\alpha}B(\alpha+j,\beta)}{B(\alpha,\beta)}\,{}_2\tilde{F}_1(\alpha+j,\alpha+\beta;\alpha+\beta+j;1-c)$.

m_2 is given by

$$\begin{aligned} E\left[X^2\right] &= S(2,1)n_{(1)}\frac{c^{\alpha}B(\alpha+1,\beta)}{B(\alpha,\beta)}\,{}_2\tilde{F}_1(\alpha+1,\alpha+\beta;\alpha+\beta+1;1-c) \\ &+S(2,2)n_{(2)}\frac{c^{\alpha}B(\alpha+2,\beta)}{B(\alpha,\beta)}\,{}_2\tilde{F}_1(\alpha+2,\alpha+\beta;\alpha+\beta+2;1-c) \\ &= \frac{nc^{\alpha}}{B(\alpha,\beta)}B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\alpha+\beta;\alpha+\beta+1;1-c) \\ &\left[1+\frac{(n-1)B(\alpha+2,\beta)\,{}_2\tilde{F}_1(\alpha+2,\alpha+\beta;\alpha+\beta+2;1-c)}{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\alpha+\beta;\alpha+\beta+1;1-c}\right] \end{aligned}$$

and the variance is

$$\begin{aligned} Var(X) &= \frac{nc^{\alpha}}{B(\alpha,\beta)}B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\alpha+\beta;\alpha+\beta+1;1-c) \\ &\left[1+\frac{(n-1)B(\alpha+2,\beta)\,{}_2\tilde{F}_1(\alpha+2,\alpha+\beta;\alpha+\beta+2;1-c)}{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\alpha+\beta;\alpha+\beta+1;1-c)}\right]- \\ &\left[\frac{nc^{\alpha}B(\alpha+1,\beta)}{B(\alpha,\beta)}\,{}_2\tilde{F}_1(\alpha+1,\alpha+\beta;\alpha+\beta+1;1-c\right]^2 \end{aligned}$$

Further simplified to give

$$
\begin{aligned}
Var(X) = {} & \frac{nc^{\alpha}}{B(\alpha,\beta)} B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\alpha+\beta;\alpha+\beta+1;1-c) \\
& \left[1+\frac{(n-1)B(\alpha+2,\beta)\,{}_2\tilde{F}_1(\alpha+2,\alpha+\beta;\alpha+\beta+2;1-c)}{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\alpha+\beta;\alpha+\beta+1;1-c)}\right. \\
& \left.-\frac{nc^{\alpha}B(\alpha+1,\beta)}{B(\alpha,\beta)}\,{}_2\tilde{F}_1(\alpha+1,\alpha+\beta;\alpha+\beta+1;1-c)\right]
\end{aligned}
$$

5.4. Gauss Hypergeometric-Binomial distribution

The probability function of the Gauss Hypergeometric distribution as suggested by Armero and Bayarri (1994); Nadarajah and Kotz (2007) is as follows

$$
g(\vartheta) = \frac{\vartheta^{\alpha-1}(1-\vartheta)^{\beta-1}}{B(\alpha,\beta)[1+z\vartheta]^{\gamma}\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)},
$$
$$
(0<\vartheta<1;\ \alpha,\beta>0;\ -\infty<\gamma<\infty;\ |z|<1 \text{ for convergence})
$$

The probability function of the Gauss Hypergeometric-Binomial (GHB) is thus derived as

$$
\begin{aligned}
f(x) &= \int_0^1 \binom{n}{x} \frac{\vartheta^{x+\alpha-1}(1-\vartheta)^{n-x+\beta-1}}{B(\alpha,\beta)(1+z\vartheta)^{\gamma}\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)}\,d\vartheta \\
&= \binom{n}{x} \frac{B(\alpha+x,n-x+\beta)\,{}_2\tilde{F}_1(\alpha+x,\gamma;\alpha+\beta+n;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)},
\end{aligned}
$$

$$
(x=0,1,\ldots,n;\ \alpha,\beta>0;\ -\infty<\gamma<\infty) \qquad (19)
$$

(19) reduces to the standard Beta-binomial distribution when z or γ equals 0.

For $\gamma=\alpha+\beta$ and $z=-(1-c)$, the GHB distribution simplifies to the Libby and Novick Beta-binomial distribution 16

Figure 3 shows the graph of the Beta-binomial distribution compared to that of the GHB distribution for varying values of γ and z. As the value of γ increases from zero, the graph of the GHB becomes more positively skewed. The reverse happens as γ decreases from zero.

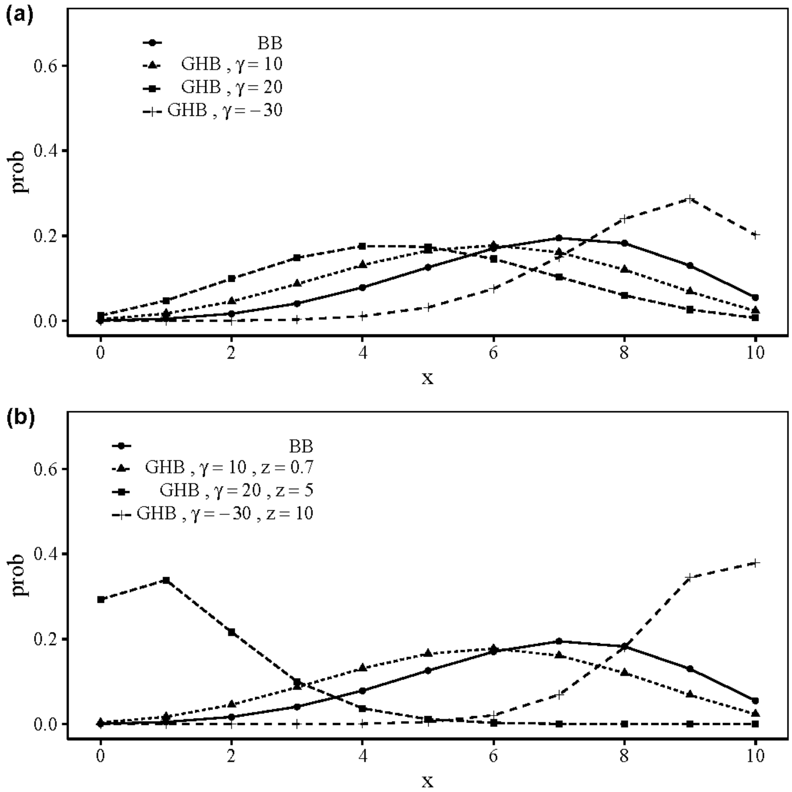

Figure 3. Probability function plots of the BB and the GHB distributions (a) BB for $\alpha=8,\beta=4,n=10$ and GHB for $\alpha=8,\beta=4,n=10,z=0.7$ and varying values of γ (b) BB for $\alpha=8,\beta=4,n=10$ and GHB for $\alpha=8,\beta=4,n=10$ and varying values of γ and z

5.4.1. Factorial Moments

From 4, the jth factorial moments of the GHB distribution are given by

$$E\left[X_{(j)}\right]=n_{(j)}E\left[\vartheta^{j}\right]$$

We then evaluate the jth moment of the Gauss Hypergeometric distribution as follows

$$\begin{aligned}E\left[\vartheta^{j}\right]&=\int_{0}^{1}\frac{\vartheta^{\alpha+j-1}\left(1-\vartheta\right)^{\beta-1}}{B(\alpha,\beta)\left(1+z\vartheta\right)^{\gamma}{}_{2}\tilde{F}_{1}(\alpha,\gamma;\alpha+\beta;-z)}d\vartheta\\&=\frac{B(\alpha+j,\beta)\,{}_{2}\tilde{F}_{1}(\alpha+j,\gamma;\alpha+\beta+j;-z)}{B(\alpha,\beta)\,{}_{2}\tilde{F}_{1}(\alpha,\gamma;\alpha+\beta;-z)}\end{aligned}$$

To give us

$$E\left[X_{(j)}\right]=n_{(j)}\frac{B(\alpha+j,\beta)\,{}_2\tilde{F}_1(\alpha+j,\gamma;\alpha+\beta+j;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)} \tag{20}$$

5.4.2. Raw and Central Moments

From 5 the kth raw moment of (19) is

$$\begin{aligned}E\left[X^k\right]&=\sum_{j=0}^{k}S(k,j)E\left[X_{(j)}\right]\\&=\sum_{j=0}^{k}S(k,j)n_{(j)}\frac{B(\alpha+j,\beta)\,{}_2\tilde{F}_1(\alpha+j,\gamma;\alpha+\beta+j;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)}\end{aligned} \tag{21}$$

so that the mean m_1, is $n\frac{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\gamma;\alpha+\beta+1;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)}$.

m_2 is

$$\begin{aligned}E\left[X^2\right]&=S(2,1)n\frac{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\gamma;\alpha+\beta+1;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)}\\&\quad+S(2,2)n_{(2)}\frac{B(\alpha+2,\beta)\,{}_2\tilde{F}_1(\alpha+2,\gamma;\alpha+\beta+2;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)}\\&=n\frac{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\gamma;\alpha+\beta+1;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)}\Bigg[1+\\&\quad\frac{(n-1)B(\alpha+2,\beta)\,{}_2\tilde{F}_1(\alpha+2,\gamma;\alpha+\beta+2;-z)}{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\gamma;\alpha+\beta+1;-z)}\Bigg]\end{aligned}$$

The variance, $m_2-m_1^2$ is then

$$\begin{aligned}Var(X)&=n\frac{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\gamma;\alpha+\beta+1;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)}\Bigg[1+\\&\quad\frac{(n-1)B(\alpha+2,\beta)\,{}_2\tilde{F}_1(\alpha+2,\gamma;\alpha+\beta+2;-z)}{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\gamma;\alpha+\beta+1;-z)}\Bigg]-\\&\quad\left[n\frac{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\gamma;\alpha+\beta+1;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)}\right]^2\end{aligned}$$

This is further simplified to give

$$Var(x) = n\frac{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\gamma;\alpha+\beta+1;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)}\left[1+\frac{(n-1)B(\alpha+2,\beta)\,{}_2\tilde{F}_1(\alpha+2,\gamma;\alpha+\beta+2;-z)}{B(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\gamma;\alpha+\beta+1;-z)}\right.$$
$$\left.-\frac{nB(\alpha+1,\beta)\,{}_2\tilde{F}_1(\alpha+1,\gamma;\alpha+\beta+1;-z)}{B(\alpha,\beta)\,{}_2\tilde{F}_1(\alpha,\gamma;\alpha+\beta;-z)}\right]$$

5.5. Confluent Hypergeometric-Binomial distribution

Given the probability function of the confluent hypergeometric distribution as suggested in Gordy (1988); Nadarajah and Kotz (2007)

$$g(\vartheta) = \frac{\vartheta^{\alpha-1}(1-\vartheta)^{\beta-1}e^{-\vartheta\lambda}}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)},(0<\vartheta<1;\ \alpha,\beta>0;\ -\infty<\lambda<\infty)$$

We derive the Confluent hypergeometric-Binomial (CHB) distribution as

$$f(x) = \int_0^1 \binom{n}{x}\frac{\vartheta^{x+\alpha-1}(1-\vartheta)^{n-x+\beta-1}}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha\alpha+\beta-\lambda)}e^{-\vartheta\lambda}d\vartheta$$
$$=\binom{n}{x}\frac{B(x+\alpha,n-x+\beta)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)}\int_0^1\frac{\vartheta^{x+\alpha-1}(1-\vartheta)^{n-x+\beta-1}}{B(x+\alpha,n-x+\beta)}e^{-\vartheta\lambda}d\vartheta$$
$$=\binom{n}{x}\frac{B(x+\alpha,n-x+\beta)\,{}_1\tilde{F}_1(x+\alpha;\alpha+\beta+n;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)},$$

$$(x=0,1,\ldots,n;\ \alpha,\beta>0;\ -\infty<\lambda<\infty) \tag{22}$$

(22) reduces to the Beta-binomial when $\lambda = 0$

Figure 4 shows the graph of the Beta-binomial distribution compared to that of the CHB distribution for varying values of the additional parameter λ. As the value of λ increases from zero, the graph of the CHB becomes more negatively skewed. The reverse happens as λ decreases from zero.

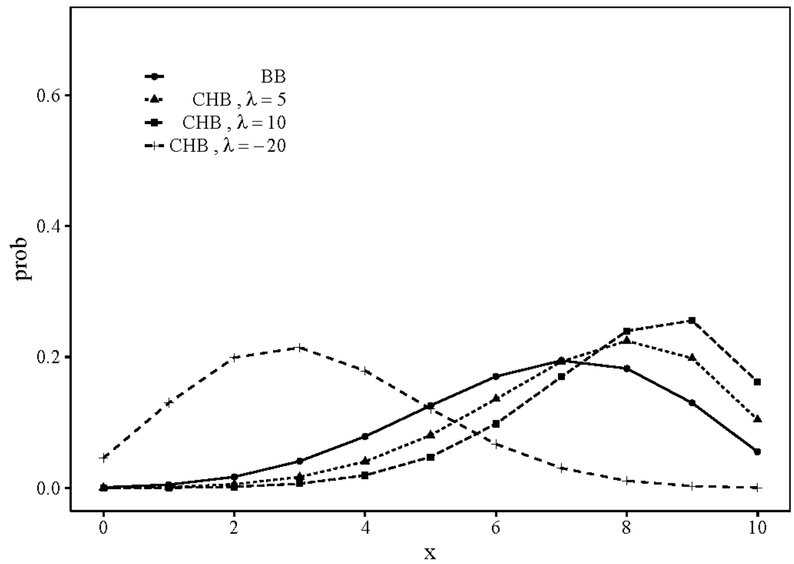

Figure 4. Probability function plots of the BB for $\alpha = 8, \beta = 4, n = 10$ and CHB for $\alpha = 8, \beta = 4, n = 10$ and varying values of λ

5.5.1. Factorial Moments

The jth factorial moment of 22 is given by $E\left[X_{(j)}\right] = n_{(j)}E\left[\vartheta^j\right]$, where $E\left[\vartheta^j\right]$ is the jth moment of the confluent hypergeometric distribution, found as follows

$$E\left[\vartheta^j\right] = \int_0^1 \frac{\vartheta^{\alpha+j-1}\left(1-\vartheta\right)^{\beta-1} e^{-\vartheta\lambda}}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)} d\vartheta$$
$$= \frac{B(x+\alpha, n-x+\beta)\,{}_1\tilde{F}_1(\alpha+j;\alpha+\beta+j;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)}$$

so that the CHB's jth factorial moment is

$$E\left[X^j\right] = \frac{n_{(j)}B(\alpha+j,\beta)\,{}_1\tilde{F}_1(\alpha+j;\alpha+\beta+j;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)} \tag{23}$$

5.5.2. Raw and Central Moments

From 5, the kth raw moment of 22 is given by $E\left[X^k\right]=\sum_{j=0}^{k} S(k,j)E\left[X_{(j)}\right]$.

Further evaluation gives us

$$E\left[X^k\right]=\sum_{j=0}^{k} S(k,j)\frac{n_{(j)}B(\alpha+j,\beta)\,{}_1\tilde{F}_1(\alpha+j;\alpha+\beta+j;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)} \tag{24}$$

So that the mean, m_1 is

$$\begin{aligned}E\left[X\right]&=S(1,1)n_{(1)}\frac{B(\alpha+1,\beta)\,{}_1\tilde{F}_1(\alpha+1;\alpha+\beta+1;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)}\\&=\frac{nB(\alpha+1,\beta)\,{}_1\tilde{F}_1(\alpha+1;\alpha+\beta+1;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)}\end{aligned}$$

m_2 is

$$\begin{aligned}E\left[X^2\right]&=S(2,1)\frac{B(\alpha+1,\beta)\,{}_1\tilde{F}_1(\alpha+1;\alpha+\beta+1;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)}+\\&\quad S(2,2)n_{(2)}\frac{B(\alpha+2,\beta)\,{}_1\tilde{F}_1(\alpha+2;\alpha+\beta+2;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)}\\&=\frac{nB(\alpha+1,\beta)\,{}_1\tilde{F}_1(\alpha+1;\alpha+\beta+1;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta;-\lambda)}\\&\quad\left[1+\frac{(n-1)nB(\alpha+2,\beta)\,{}_1\tilde{F}_1(\alpha+2;\alpha+\beta+2;-\lambda)}{B(\alpha+1,\beta)\,{}_1\tilde{F}_1(\alpha+1;\alpha+\beta+1;-\lambda)}\right]\end{aligned}$$

The variance is thus given by

$$\begin{aligned}Var(X)=\frac{nB(\alpha+1,\beta)\,{}_1\tilde{F}_1(\alpha+1;\alpha+\beta+1;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta+1;-\lambda)}&\left[1+\frac{(n-1)nB(\alpha+2,\beta)\,{}_1\tilde{F}_1(\alpha+2;\alpha+\beta+2;-\lambda)}{B(\alpha+1,\beta)\,{}_1\tilde{F}_1(\alpha+1;\alpha+\beta+1;-\lambda)}\right.\\&\left.-n\frac{B(\alpha+1,\beta)\,{}_1\tilde{F}_1(\alpha+1;\alpha+\beta+1;-\lambda)}{B(\alpha,\beta)\,{}_1\tilde{F}_1(\alpha;\alpha+\beta+1;-\lambda)}\right]\end{aligned}$$

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