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On Various Properties of δ-Compactness in Bitopological Spaces

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Abstract

By introducing the notion of δ-compact Anjali Srivastava and Sandhya Gupta in paper (A. Srivastava and S. Gupta 2005) obtained the generalization of various results of Park in a paper (Herrington and Long 1975 and Park 1988). In this paper, we introduce the concept on various properties of δ-compactness in bitopological spaces.

Keywords: H-closed spaces, net, ultranet, w-limit point, w-closure, δ-compact bitopological spaces, bitopological spaces, δ-Hausdorff bitopological spaces, θ-compactness.

1. Introduction

Jong Suh Park in the paper "H-closed spaces and W-Lindelof spaces "has got various interesting results related with H-closed spaces. Moreover Park has introduced the concept of W-Lindelof spaces which is a generalization of Lindelof spaces. By using the notions of σ-continuous maps, w-closure, w-limit point etc. Park has proved various results concerned with these concepts.

Anjali Srivastava and Sandhya Gupta in the paper "On various properties of δ-compact spaces" have introduced the concept of δ-compact spaces and have got many theorems giving a generalization of Park's theorems by using the tools of δ-continuous maps, w^*-closure, δ-convergence of nets and δ- cluster points of nets etc.

In this paper we have introduced the concept of δ-compactness in bitopological spaces and have got many theorems giving a generalization of Park's theorem by using the tools of δ-continuous maps, w^*-closure, δ-convergence of nets and δ- cluster points of nets etc. in bitopological spaces.

In section 2 of this paper we obtain a characterization of δ-Hausdorff bitopological spaces and discuss various properties of δ-Hausdorff bitopological spaces which compare (i, j)-Hausdorff spaces and Hausdorff spaces. Further the notion of δ-compactness in bitopological spaces is introduced and it is shown that δ-compactness in bitopological spaces is preserved by δ-continuous surjections, arbitrary products and w^*-closed sets in bitopological spaces, which is a generalization of A. Srivastava and S. Gupta (2005).

In section 3 of this paper we study θ-compactness in bitopological spaces a generalization of quasi-H-closed sets and its applications to some forms of continuity using θ-open and δ-open sets in bitopological spaces. Among other results, it is shown that a weakly θ-retract of a Hausdorff spaces X is a δ-closed subset of X in bitopological spaces, which is a generalization of some results of Mohammad Saleh (2004).

2. δ-Compactness in Bitopological Spaces

The section beings with the following of δ-compactness in bitopological spaces.

Definition 2.1: Let (X, T_1, T_2) be a bitopological space. A subset A of X is called a (i, j)-H-closed set ((i, j)-H-set) in X (J. Vermeer, 1985) if every pairwise open cover $\mathfrak{C}$ of A then $\exists$ a finite subfamily $\{U_i\}\subset\mathfrak{C}$, i = 1, 2, .., n such that $A \subset \bigcup_{i=1}^{n} (j\mathrm{Cl}(U_i))$.

Definition 2.2: A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called (i, j)-δ-compact if for each pairwise open cover $\{U_n\}$ of X there are finitely many n_k such that $X = \bigcup_{k=1}^{\infty} \mathrm{iInt}(\mathrm{jCl}(U_{nk}))$ where $i \neq j$, $i, j = 1,2$.

Obviously (i, j)-δ-compact space is (i, j)-H-closed. But the converse is not true.

Example 2.1: Let $X = \mathfrak{R}$,

T_1= The usual topology on $\mathfrak{R}$, T_2= The discrete topology on $\mathfrak{R}$.

Let $A = [m, m+r \mid m, r \in \boldsymbol{Z}, r>1]$.

Then clearly A is (i, j)-H-closed. Now consider

C = $\{(n-1, n) | n \in \boldsymbol{Z}\} \cup \{\{s\} | s \in \boldsymbol{Z}\} \cup$ {the unions of these subsets}. Then C is a pairwise open cover of A in (X, T_1, T_2).

Let $C' = \{(m+r-1, m+r), \{\{m+r-1\} \cup \{m+r\}\}, (m+r-2, m+r-1), \{\{m+r-2\} \cup \{m+r-1\}\}, \ldots\ldots.., (m,$

m +1), {{m} ∪ {m+1}}}. Therefore,

T_2-Cl(m+i−1, m+i) = (m +i−1, m+i), i= 1, 2, …., r−1 and

T_1-Cl{m+i} = {m+i}, i= 1, 2, …., r−1.

Clearly, these two closures together cover A.

Now, T_1−Int(T_2-Cl(m+i−1, m+i))= (m+i−1, m+i), i= 1, 2, …., r−1 and

T_2-Int(T_1−Cl({m+i})=ϕ, i= 1, 2, …., r−1. Therefore, the two classes of sets together do not cover A. Hence A is not (i, j)-δ-compact.

Definition 2.3: Let (X, $\mathcal{T}_1$, $\mathcal{T}_2$) be a bitopological space. A net (x_n) in X is said to be (i, j)-δ-accumulate to a point x of X denoted by $x_n \overset{\delta}{\infty} x$ if for any i-neighbourhood U of x and n there is an $n_1 \geq n$ such that $x_{n_1} \in i$Int(jCl(U)) where $i \neq j$, i, j =1, 2.

Definition 2.4: Let (X, $\mathcal{T}_1$, $\mathcal{T}_2$) be a bitopological space. A net (x_n) in X is said to be (i, j)-δ-converge to a point x of X denoted by $x_n \overset{\delta}{\longrightarrow} x$ if for each i-neighbourhood U of x there is an $n_1 \geq n$ such that $x_{n_1} \in$ iInt(jCl(U)) where $i \neq j$, i, j =1, 2.

Definition 2.5: A net (x_n) on a set X is called universal, or an ultranet (From wikipedia) if for every subset A of X, either (x_n) is eventually in A or (x_n) is eventually in X−A. (By eventually in A we mean, $\exists$**N** such that for all $n \geq$ **N**, $x_n \in A$).

Lemma 2.1: Let (X, $\mathcal{T}_1$, $\mathcal{T}_2$) be a bitopological space. If a ultranet (x_n) of X (i, j)-δ-accumulate to a point x of X then (x_n) is (i, j)-δ-converge to x.

Note: If (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is a bitopological space, then for any $A \subset X$, we define

(1) Cl(A) = $\cap$ {$F_1 \cup F_2$ where $A \subset F_1 \cup F_2$ and F_1, F_2 are respectively $\mathcal{T}_1$ and $\mathcal{T}_2$ closed}, then Cl(A) is called a pairwise closure of A.

(2) We also define a pairwise closure in abitopological space (X, $\mathcal{T}_1$, $\mathcal{T}_2$) by

Cl(A) = {$x \in X$: $A \cap (U \cup V) \neq \phi$, where $A \subset X$ and $x \in U \in \mathcal{T}_1$, $x \in V \in \mathcal{T}_2$}.

Note that the closure of a subset A w.r.to $\mathcal{T}_1$ and w.r.to $\mathcal{T}_2$ is a subset of the pairwise closure of A.

Now we show that the above two definitions are equivalent.

Proof: Let $x \notin$ Cl(A) in (2). This implies that $A \cap (U \cup V) = \phi \Rightarrow A \cap U = \phi$ and $A \cap V = \phi$. Let U_0 is the union of all $\mathcal{T}_1$ neighborhood U of x and V_0 is the union of all $\mathcal{T}_2$ neighborhood V of x. Then $A \cap U_0 = \phi$ and $A \cap V_0 = \phi$. Therefore $A \subseteq (U_0)^c$ and $A \subseteq (V_0)^c$. Hence $\mathcal{T}_1$-Cl(A)$\subseteq (U_0)^c$ and $\mathcal{T}_2$-Cl(A)$\subseteq (V_0)^c$ where $(U_0)^c$ is closed in $\mathcal{T}_1$ and $(V_0)^c$ is closed in $\mathcal{T}_2$. Therefore $x \notin (U_0)^c \cup (V_0)^c \Rightarrow x \notin$ Cl(A) in (1).

Conversely, let $x \notin$ Cl(A) in (1) then there exist $\mathcal{T}_1$-closed set F_1 and $\mathcal{T}_2$-closed set F_2 such that $x \notin F_1 \cup F_2$ this implies that $x \notin F_1$ and $x \notin F_2$. Therefore $x \in (F_1)^c$ and $x \in (F_2)^c$ implies $x \in (F_1)^c \cup (F_2)^c$ where$(F_1)^c$ is $\mathcal{T}_1$-open and $(F_2)^c$ is $\mathcal{T}_2$-open and $A \cap ((F_1)^c \cup (F_2)^c) = \phi$. Hence $x \notin$ Cl(A) in (2).

Definition 2.6: Let (X, $\mathcal{T}_1$, $\mathcal{T}_2$) be a bitopological space. For a subset A of X the (i, j)-weak closure of A denoted by (i, j)-$\text{Cl}_w^*(A)$ is defined by the set (i, j)-$\text{Cl}_w^*(A) = \{x \in X: A \cap i\text{Int}(j\text{Cl(U)}) \neq \phi$ for all i-open neighborhood U of $x\}$ where $i \neq j$, $i, j = 1,2$.

Example 2.2: Consider the topologies on $X = \{a, b, c\}$ be $\mathcal{T}_1 = \{X, \phi, \{a\}, \{a, c\}\}$ and $\mathcal{T}_2 = \{X, \phi, \{b\}, \{a, b\}\}$. Let $a \in X$ and $A = \{b, c\}$ be a subset of X, then A is (1,2)-weak closure of A, since for all $\mathcal{T}_1, \mathcal{T}_2$-open neighborhood U of a, we have $A \cap \mathcal{T}_1$-Int($\mathcal{T}_2$-Cl(U))$\neq \phi$.

Note: A subset A of X is called (i, j)-regular open if $A = i$Int(jCl(A)) and X is called (i, j)-semi-regular space if it has a base consisting of (i, j)-regular open sets (Biswas, S.K. and Akhter, N. 2015).

Following lemma establishes the similar behaviour of a pairwise closure and (i, j)-weak closure of a set in terms of the (i, j)-δ-convergence of nets.

Lemma 2.2: Let (X, $\mathcal{T}_1$, $\mathcal{T}_2$) be a bitopological space and $A \subseteq X$. Then $x \in (i, j)$-$\text{Cl}_w^*(A)$ iff there is a net (x_n) of points of A, (i, j)-δ-converge to $x \in X$.

Proof: Let $x \in (i, j)$-$\text{Cl}_w^*(A)$. Then $A \cap i\text{Int}(j\text{Cl}(U_n)) \neq \phi$ for all i-neighborhoods U_n of x in X. Consider the family η_x of all i-neighborhood of x with the reverse order inclusion and define a net in X as follows:

S: $\eta_x \rightarrow X$ by

$S(U_n) = x_n$ where $x_n \in A \cap i\text{Int}(j\text{Cl}(U_n))$ then (x_n) is a net of point of A and $x_n \xrightarrow{\delta} x$.

Conversely, assume that $x_n \xrightarrow{\delta} x$. For a i- neighborhoods U of x, $\exists n_1$ such that $x_n \in i\text{Int}(j\text{Cl}(U))$ $\forall n \geq n_1$. Since $x_n \in A\, \forall n$, we have $A \cap i\text{Int}(j\text{Cl}(U)) \neq \phi$. Thus $x \in (i, j)\text{-Cl}_w^*(A)$.

Definition 2.7: Let (X, $\mathcal{T}_1$, $\mathcal{T}_2$) be a bitopological space and $A \subseteq X$. Then A is called (i, j)-w*-closed if $A = (i, j)\text{-Cl}_w^*(A)$.

Definition 2.8: A bitopological space (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is called (i, j)-δ-Hausdorffif for any two distinict points x and y of X there are i-open neighborhood U of x and j-open neighbourhood V of y such that $i\text{Int}(j\text{Cl}(U)) \cap j\text{Int}(i\text{Cl}(V)) = \phi$ where $i \neq j$, $i, j = 1, 2$.

Equivalently, X is said to be (i, j)-δ-Hausdorffif forevery $x \neq y \in X$, $\exists (i, j)$-δ-open set U_x and (j, i)-δ-open set V_y such that $U_x \cap V_y = \phi$.

Definition 2.9: (Noiri and Popa 2007) A bitopological space (X, $\mathcal{T}_1$, $\mathcal{T}_2$) is said to be pairwise Hausdorff if for every $x, y \in X$, $x \neq y$ $\exists$ $U \in \mathcal{T}_1$, $V \in \mathcal{T}_2$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Note that a (i, j)-δ-Hausdorff space is pairwise Hausdorff but the following example shows that X is pairwise Hausdorff but not (i, j)-δ-Hausdorff.

Example 2.3: Consider the following bitopologies on $X = \{a, b, c\}$:

$\mathcal{T}_1 = \{X, \varphi, \{a\}, \{a,c\}\}$ and $\mathcal{T}_2 = \{X, \varphi, \{a,b\}, \{b,c\}, \{b\}, \{c\}\}$.

Then X is pairwise Hausdorff. But X is not (1,2)-δ-Hausdorff since for b, $c \in X$, if we consider $\mathcal{T}_1$-open set $U = \{a, c\}$ and $\mathcal{T}_2$-open set $V = \{$a, b$\}$ we have $\mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl(U)}) = \{a\}$ and $\mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(V)) = X$ i.e, $\mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl(U)}) \cap \mathcal{T}_2\text{-Int}(\mathcal{T}_1\text{-Cl}(V)) \neq \phi$. Similarly, for a, $b \in X$ or c, $a \in X$ we have X is not (1,2)-δ-Hausdorff. Hence X is not (i, j)-δ-Hausdorff.

Following theorem gives a characterization of (i, j)-δ-Hausdorff spaces in terms of diagonal of X.

Theorem 2.1: Let (X, $\mathcal{T}_1$, $\mathcal{T}_2$) be a bitopological space. Then the following statement are equivalent:

(i) X is (i, j)-δ-Hausdorff.

(ii) Every net in X (i, j)-δ-converges to atmost point of X.

(iii) The diagonal $\Delta = \{(x,x): x \in X\}$ is a (i, j)-w*-closed set of $X \times X$.

Proof: (i)$\Longrightarrow$(ii). Assume that a net (x_n) in X (i, j)-δ-converges to distinict points x and y of X. Since X is (i, j)-δ-Hausdorff there are i-open neighbourhood U of x and j-open neighbourhood V of y such that $i\text{Int}(j\text{Cl}(U)) \cap j\text{Int}(i\text{Cl}(V)) = \phi$. Since $x_n \overset{\delta}{\rightarrow} x$, $\exists n_1$ such that $x_n \in i\text{Int}(j\text{Cl}(U))$ $\forall n \geq n_1$. Since $x_n \overset{\delta}{\rightarrow} y$, $\exists n_2$ such that $x_n \in j\text{Int}(i\text{Cl}(V))$ $\forall n \geq n_2$.

Choose $m \geq n_1$ and $m \geq n_2$.Then $x_m \in i\text{Int}(j\text{Cl}(U)) \cap j\text{Int}(i\text{Cl}(V))$. This is a contradiction. Thus $x = y$.

(ii)$\Longrightarrow$(iii). Let $(x, y) \in (i, j)\text{-Cl}_w^*(\Delta)$.Then there is a net (x_n) in X such that $(x_n, x_n) \overset{\delta}{\rightarrow} (x, y)$.Since $x_n \overset{\delta}{\rightarrow} x$ and $x_n \overset{\delta}{\rightarrow} y$ by (ii) $x = y$. Thus $(x, y) \in \Delta$.

(iii)$\Longrightarrow$(i). Let x, y $\in X$ with $x \neq y$. Then $(x, y) \in \Delta = (i, j)\text{-Cl}_w^*(\Delta)$. Hence there is a i-neighborhood W of (x, y) such that $\Delta \cap i\text{Int}(j\text{Cl}(W)) = \phi$. Choose i-open set U and j-open set V of X with $x \in U$, $y \in V$ and $U \times V \subset W$. Then $i\text{Int}(j\text{Cl}(U)) \cap j\text{Int}(i\text{Cl}(V)) = \phi$.

Lemma 2.3: Let X be (i, j)-δ-compact space.Then for each net (x_n) in X there is an $x \in X$such that $x_n \overset{\delta}{\infty} x$.

Proof: Suppose that (x_n) has no (i, j)-δ-limit point in X. Then (x_n) is not (i,j)-δ-accumulate to a point x in X. For each $x \in X$ there is a i-neighborhood U_x of x and n_x such that $x_n \notin i\text{Int}(j\text{Cl}(U_x))$ $\forall n \geq n_x$. Then $\{U_x : x \in X\}$ is a pairwise open cover of X. Since X is (i,j)-δ-compact, there are finitely many x_k such that $X = \bigcup_{k=1}^{n} i\text{Int}(j\text{Cl}(U_{x_k}))$. Choose m such that $m \geq n_x\ k \forall k = 1, 2, \ldots, n$. Conclude from above $x_m \notin \bigcup_{k=1}^{n} i\text{Int}(j\text{Cl}(U_x k)) \forall k = 1, 2, \ldots, n$. This contradiction shows that (x_n) has necessarily a (i, j)-δ-cluster point in X.

Theorem 2.2: If abitopological space X is (i, j)-δ-compact then every net in X has a (i, j)-δ-convergent subnet.

Proof: Let (x_n) be a net in X. Since every net has a ultra subnet, (x_n) has a ultra subnet (x_{n_k}). Then by above lemma 2.3 there is an $x \in X$ such that $x_{n_k} \overset{\delta}{\infty} x$. Therefore we have $x_{n_k} \overset{\delta}{\rightarrow} x$.

Theorem 2.3: Let X be a (i, j)-δ-compact space. If A is $(i, j)\text{-Cl}_w^*$-closed subset of X, then A is (i, j)-δ-compact.

Proof: Let (x_n) be a net in A. Then (x_n) is a net in X. Since X is (i, j)-δ-compact (x_n) has a (i, j)-δ-convergent subnet. Let $x_n \overset{\delta}{\rightarrow} x$. Since $x \in (i, j)\text{-Cl}_w^*(A)$ and A is a $(i, j)\text{-Cl}_w^*$-closed we conclude that x

$\in A$. It shows that A is (i, j)-δ-compact.

Theorem 2.4: Let X be a (i, j)-δ-Hausdorff space. Then every (i, j)-δ-compact subset of X is (i, j)-w^*-closed.

Proof: Let $x \in (i, j)$-$\mathrm{Cl}_w^*(A)$. Then there is a net (x_n) in A such that $x_n \xrightarrow{\delta} x$. Then x is a (i, j)-δ-limit point of (x_n). Since A is (i, j)-δ-compact, $x \in A$. Hence (i, j)-$\mathrm{Cl}_w^*(A) = A$ i.e, A is a (i, j)- w^*-closed set of X.

Definition 2.10: A function f: $(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-δ-continuous at a point x if for each σ_i-neighborhood U of $f(x)$ there are $\mathcal{T}_i$-neighborhood V of x such that $f(i\mathrm{Int}(j\mathrm{Cl}(V))) \subset i\mathrm{Int}(j\mathrm{Cl}(U))$ where $i \neq j$, $i, j = 1,2$. If f is (i, j)-δ-continuous at every $x \in X$, then f is called (i, j)-δ-continuous.

Definition 2.11: (Khedr and AL-Areefi, 1992) Amapping f: $(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise continuous if inverse image of every σ_1-open (resp. σ_2-open) set in Y is $\mathcal{T}_1$-open (resp. $\mathcal{T}_2$-open) in X.

Note that the concepts of pairwise continuous maps and (i, j)-δ-continuous maps are different.

Example 2.4: Consider the topologies on $X = \{a, b, c\}$ and $Y = \{p, q, r\}$ respectively by

$\mathcal{T}_1 = \{X, \phi, \{a, b\}, \{a\}\}$, $\mathcal{T}_2 = \{X, \phi, \{b\}, \{b, c\}\}$ and $\sigma_1 = \{Y, \phi, \{p\}, \{p, r\}\}$, $\sigma_2 = \{Y, \phi, \{q\}\}$.

Let, f:$(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a map defined as $f(a) = p$, $f(b) = q$, $f(c) = r$. Then f is (1, 2)-δ-continuous since if $a \in X$ and σ_1-neighborhood $U = \{p, r\}$ we have $\mathcal{T}_1$-neighborhood $V = \{a\}$ such that $f(\mathcal{T}_1\text{-Int}(\mathcal{T}_2\text{-Cl}(V))) \subset \sigma_1\text{-Int}(\sigma_2\text{-Cl}(U))$. But it is not a pairwise continuous maps since inverse image of σ_1-open set $f^{1}(p, r) = \{a, c\}$ in Y which is not $\mathcal{T}_1$-open in X.

Following theorem gives a characterization of (i, j)-δ-continuous maps between two spaces.

Theorem 2.5: A mapping f:$(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-δ-continuous at $x \in X$ iff for any net (x_n) in X satisfying $x_n \xrightarrow{\delta} x$, the net $f((x_n)) \xrightarrow{\delta} f(x)$ in Y.

Proof: Given any σ_i-neighborhood U of $f(x)$, there is a $\mathcal{T}_i$-neighborhood V of x such that $f(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(V))) \subset \sigma_i\text{-Int}(\sigma_j\text{-Cl}(U))$ where $i \neq j$, $i, j = 1,2$.

Also there is an n_1 such that $x_n \in i\mathrm{Int}(j\mathrm{Cl}(V))$ for all $n \geq n_1$. Since $f(x_n) \in f(i\mathrm{Int}(j\mathrm{Cl}(V))) \subset i\mathrm{Int}(j\mathrm{Cl}(U))$

for all $n \geq n_1$, we have $f((x_n)) \overset{\delta}{\rightarrow} f(x)$.

Conversely, assume that f is not (i, j)-δ-continuous at x. Then there is a σ_i-neighborhood U of $f(x)$ such that $f(\mathcal{T}_i\text{-Int}(\mathcal{T}_j\text{-Cl}(V))) \not\subset \sigma_i\text{-Int}(\sigma_j\text{-Cl}(U))$ where $i \neq j$, $i, j = 1, 2$ for all $\mathcal{T}_i$-neighborhood V of x. Let (V_n) be the family of $\mathcal{T}_i$-neighborhoods of x with the reverse inclusion order. For each n, since $f(i\text{Int}(j\text{Cl}(V_n))) \not\subset i\text{Int}(j\text{Cl}(\text{U}))$, there is an $x_n \in i\text{Int}(j\text{Cl}(V_n))$ such that $f(x_n) \notin i\text{Int}(j\text{Cl}(U))$. Then the net (x_n) in X (i, j)-δ-converges to x but the net $f(x_n)$ in Y does not (i, j)-δ-converges to $f(x)$. Thus we have a contradiction. Hence f is (i, j)-δ-continuous at x.

Definition 2.12: A mapping $f:(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to have (i, j)-w*-closed graph if its graph $G(f) = \{x, f(x): x \in X\}$ is (i, j)-w*-closed subset of $X \times Y$.

Theorem 2.6: A mapping $f:(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ has a (i, j)-w*-closed graph iff for any net (x_n) in X, $x_n \overset{\delta}{\rightarrow} x \in X$ and $f(x_n) \overset{\delta}{\rightarrow} y \in Y$ implies $y = f(x)$.

Proof: Assume that $f:(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ has a (i, j)-w*-closed graph. Since $(x_n, f(x_n))$ is a net in $G(f)$ and $(x_n, f(x_n)) \overset{\delta}{\longrightarrow} (x, y)$, we have $(x, y) \in (i, j)\text{-Cl}_{\text{w}}^*(G(f)) = G(f)$. Thus $y = f(x)$.

Conversely, assume that $(x, y) \in (i, j)\text{-Cl}_{\text{w}}^*(G(f))$. Then there is a net (x_n) in X such that $(x_n, f(x_n)) \overset{\delta}{\longrightarrow} (x, y)$. Since $x_n \overset{\delta}{\longrightarrow} x$ and $f(x_n) \overset{\delta}{\longrightarrow} f(x)$, $y = f(x)$. Thus $(x, y) \in G(f)$. Hence $G(f)$ is (i, j)-w*-closed.

Theorem 2.7: Let (Y, σ_1, σ_2) be a (i, j)-δ-Hausdorff space. Then every (i, j)-δ-continuous mapping $f:(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ has a (i, j)-w*-closed graph.

Proof: Let $(x, y) \in (i, j)\text{-Cl}_{\text{w}}^*(G(f))$. Then there is a net (x_n) in X such that $(x_n, f(x_n)) \overset{\delta}{\longrightarrow} (x, y)$. Then $x_n \overset{\delta}{\longrightarrow} x$ and $f(x_n) \overset{\delta}{\longrightarrow} y$. Since f is (i, j)-δ-continuous at x, $f(x_n) \overset{\delta}{\longrightarrow} f(x)$. Since Y is (i, j)-δ-Hausdorff, $y = f(x)$. This implies $(x, y) \in G(f)$. Hence $G(f)$ is (i, j)-w*-closed.

Theorem 2.8: Let (Y, σ_1, σ_2) be a (i, j)-δ-compact space. If amapping $f:(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ has a (i, j)-w*-closed graph then f is (i, j)-δ-continuous.

Proof: Let (x_n) be a net in X and $x_n \overset{\delta}{\longrightarrow} x$. Since Y is (i, j)-δ-compact the net $(f(x_n))$ in Y is has a $(i,$

j)-δ-convergent subnet by Theorem 2.2. Let $f(x_n) \xrightarrow{\delta} y \in Y$. Since $(x_n, f(x_n)) \xrightarrow{\delta} (x, y)$, $(x, y) \in (i, j)\text{-Cl}_w^*(G(f))$ $= G(f)$. Thus $y = f(x)$ and so $f(x_n) \xrightarrow{\delta} f(x)$. This means that f is (i, j)-δ-continuous at x.

Theorem 2.9: Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a (i, j)-δ-compact space and (Y, σ_1, σ_2) a bitopological space. If $f:(X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-δ-continuous surjection, then Y is a (i, j)-δ-compact.

Proof: Let (y_n) be a net in Y. For each n, there is an $x_n \in X$ such that $y_n = f(x_n)$. Since X is (i, j)-δ-compact, there is a subnet (x_{n_k}) of (x_n) and an $x \in X$ such that $x_{n_k} \xrightarrow{\delta} x$. Since f is (i, j)-δ-continuous at x, $f(x_{n_k}) \xrightarrow{\delta} f(x)$. Thus Y is (i, j)-δ-compact.

3. Hausdorffness and Weak Forms of Compactness in Bitopological Spaces

Definition 3.1: A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be (i, j)-θ-Hausdorff if for every $x \neq y \in X$, $\exists (i, j)$-θ-open set U_x and (j, i)-θ-open set V_y such that $U_x \cap V_y = \phi$.

Definition 3.2: A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be pairwise Urysohn (Bose and Sinha 1982) if for each distinct points x, y, $\exists i$-open set U, j-open set V such that $x \in U$, $y \in V$ and $j\text{Cl}(U) \cap i\text{Cl}(V) = \phi$ for $i \neq j$, $i, j, k = 1, 2$.

It is clear that every (i, j)-θ-Hausdorff space is pairwise Urysohn but a pairwise Urysohn space need not be (i, j)-θ-Hausdorff (M. Saleh, 2003).

Lemma 3.1: A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is (i, j)-R-Hausdorff if for every $x \neq y \in X$, $\exists (i, j)$-regular open set U_x and (j, i)-regular open set V_y such that $U_x \cap V_y = \phi$.

By a (i, j)-weak θ-restriction we mean a (i, j)- weak θ-continuous function $f:(X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow A$ where $A \subset X$ and $f|A$ is the identity function on A. In this case A is said to be a (i, j)- weak θ-restriction of X.

The next theorem is an improvement of Theorem 3.3 of (M. Saleh 2004).

Definition 3.3: Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a bitopological space and $A \subset X$. The set of all (i, j)-δ-adherent point of A is called the (i, j)-δ-closure of A, denoted by (i, j)-δ-Cl(A). A subset A of X is called (i, j)-δ-closed iff $A = (i, j)$-δ-Cl(A). The complement of (i, j)-δ-closed set is called (i, j)-δ-open.

Theorem 3.1: Let $A \subset X$ and f:$(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow$ A be a (i, j)- weak θ-restriction of X onto A. If X (i, j)-R-Hausdorff, then A is an (i, j)-δ-closed subset of X.

Proof: Suppose not, then there exists a point $x \in (i, j)$-δ-Cl(A). Since f is a (i, j)-weak θ-restriction, we have $f(x) \neq x$. Since X is (i, j)-R-Hausdorff, there exist $\exists (i, j)$-regular open set U and(j, i)-regular open set V of x and $f(x)$ respectively such that $U \cap V= \phi$. Let W be any open set in X containing x. Then $U \cap i\text{Int}(j\text{Cl}(W))$ is a (i, j)-regular open set containing x and hence $i\text{Int}(j\text{Cl}(U)) \cap i\text{Int}(j\text{Cl}(W)) \cap A \neq \phi$, since $x \in (i, j)$-δ-Cl(A). Therefore, $\exists$a point $y \in i\text{Int}(j\text{Cl}(U)) \cap i\text{Int}(j\text{Cl}(W)) \cap A$. Since $y \in A$, $f(y) = y \in i\text{Int}(j\text{Cl}(U))$ and hence $f(y) \in j\text{Cl}(V)$. This shows that $f(i\text{Int}(j\text{Cl}(W)))$ is not contained in $j\text{Cl}(V)$. This contradicts the hypothesis that f is a (i, j)-weak θ-continuous. Thus A is a (i, j)-δ-closed as claimed.

Definition 3.4: A function f:$(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-weak θ-continuous at $x \in X$ if given any σ_i-open set U in Y containing $f(x)$, $\exists$ a $\mathcal{T}_i$-open set V in X containing x such that $f(i\text{Int}(j\text{Cl}(U))) \subset j\text{Cl}(V)$ where $i \neq j$, i, j = 1, 2. If this condition is satisfied at each point $x \in X$, then f is said to be (i, j)-weak θ-continuous (briefly, (i, j)-w.θ.c).

Theorem 3.2: Let f, g be (i, j)-w. θ.c from a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ into a pairwise Urysohn space (Y, σ_1, σ_2). Then the set $A = \{x \in X\text{: } f(x) = g(x)\}$ is an an (i, j)-δ-closed set.

Proof: We will show that $X \backslash A$ is (i, j)-δ-open. Let $x \in A^c$. Then $f(x) \neq g(x)$. Since Y is a pairwise Urysohn, $\exists \sigma_i$-open set $W_{f(x)}$ and σ_j-open set $V_{g(x)}$ such that $j\text{Cl}(W) \cap i\text{Cl}(V) = \phi$. By (i, j)-w. θ.c of f and g, $\exists (i, j)$-regular open set U_1 and (j, i)-regular open set U_2 of x such that $f(U_1) \subset j\text{Cl}(W)$ and $g(U_2) \subset i\text{Cl}(V)$. Clearly $U = U_1 \cap U_2 \subset X \backslash A$. Thus $X \backslash A$ is (i, j)-δ-open and hence A is (i, j)-δ-closed.

Definition 3.5: Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be a bitopological space, then $A \subset X$ is called (i, j)-θ-dense if (i, j)-Cl$\theta(A) = X$.

Corollary : Let f, g be (i, j)-w.θ.c from a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ into a pairwise Urysohn space (Y, σ_1, σ_2). If f and g agree on a (i, j)-θ-dense subset of X then $f = g$ every where.

Theorem 3.3: Let f: $(X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be (i, j)-w. θ.c map and let $A \subset X$. Then f:$A \longrightarrow Y$ is (i, j)-w. θ.c.

Proof: Straight forward.

Remark 3.1: If f: $(X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j)-w. θ.c map. Then f: $X \rightarrow f(X)$ need not be (i, j)-w. θ.c.

Definition 3.6: A function f: $(X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j)- almost strongly-θ-continuous at $x \in X$ if given any σ_i-open set V in Y containing $f(x)$, $\exists$ a $\mathcal{T}_i$- open set U in X containing x such that $f(j\text{Cl}(U)) \subset j\text{Int}(i\text{Cl}(V))$ where $i \neq j$, $i, j = 1, 2$. If this condition is satisfied at each point $x \in X$, then f is said to be (i, j)- almost strongly-θ- continuous (briefly, (i, j)-a.**s.**c).

Definition 3.7: A function f: $(X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-weak continuous at $x \in X$ if given any σ_i-open set V in Y containing $f(x)$, $\exists$ a $\mathcal{T}_i$- open set U in X containing x such that $f(U) \subset i\text{Cl}(V)$ where $i \neq j$, $i, j = 1, 2$. If this condition is satisfied at each point $x \in X$, then f is said to be (i, j)-weak continuous (briefly, (i, j)-w.c).

Definition 3.8: A function f: $(X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-δ-continuous at $x \in X$ if given any σ_i-open set V in Y containing $f(x)$, $\exists$ a $\mathcal{T}_i$- open set U in X containing x such that $f(i\text{Int}(j\text{Cl}(U))) \subset i\text{Int}(j\text{Cl}(V))$ where $i \neq j$, $i, j = 1, 2$. If this condition is satisfied at each point $x \in X$, then f is said to be (i, j)-δ-continuous (briefly, (i, j)-δ.c).

Example 3.1: Let $X = \Re$ with the usual bitopology, $Y = \Re$ with the countable bitopology and let f: $(X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be defined as f(rationals) = 1 and f(irrationals) = 0 then f is (i, j)-δ.c, (i, j)-a.**s.**c but f: $(X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow f(x)$ is not even (i, j)-w.c.

Definition 3.9: A subset A of a bitopological space X is called pairwise closure compact or pairwise quasi-H-closed if every open cover of A has a finite subcollection whose closures cover A.

Clearly every pairwise compact set is pairwise closure compact but not conversely as it is in the following example.

Example 3.2: Let X be any uncountable bitopological space with the countable bitopology then every subset of X is pairwise closure compact but the only pairwise compact subsets of X are the finite ones.

Definition 3.10: A subset A of a bitopological space X is said to be (i, j)-θ-compact if every cover of (i, j)-θ-open sets has a finite subcover.

Lemma 3.2: A subset A of a bitopological space X is (i, j)-δ-compact iff every cover of (i, j)-δ-open sets has a finite subcover.

Theorem 3.4: Let f: $(X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be (i, j)-w.θ.c and K be an (i, j)-δ-compact subset of X. Then $f(K)$ is a pairwise closure compact subset of Y.

Proof: Let V be an open cover of $f(K)$. For each $k\in K$, $f(k)\in v_k$ for some $v_k\in V$. By (i, j)-w.θ.c of f, $f^{1}(\mathrm{Cl}(v_k))$ is (i, j)-regular open. The collection $\{f^{1}(\mathrm{Cl}(v_k))$: $k\in K\}$is a (i,j)-regular open cover of K and so since K is (i, j)-δ-compact, there is a finite subcollection $\{f^{1}(\mathrm{Cl}(v_k))$: $k\in v_o\}$ where v_o is a finite subset of K and $\{f^{1}(\mathrm{Cl}(v_k))$: $k\in v_o\}$ covers K. Clearly, $\{(\mathrm{Cl}(v_k))$: $k\in v_o\}$ covers $f(K)$ and thus $f(K)$ is a pairwise closure compact subset of Y.

Remark 3.2: It is well known that every closed subset of a pairwise compact space is pairwise compact.

Theorem 3.5: A (i, j)-δ-compact subset of a (i, j)-δ-Hausdorff space is (i, j)-δ-closed.

Proof: Let A be a (i, j)-δ-compact subset of a (i, j)-δ-Hausdorff space X. We will show that $X\backslash A$ is (i, j)-δ-open. Let, $x\in X\backslash A$ then for each $a\in A$, $\exists$ (i, j)-δ-open set $U_{x,a}$ and (j, i)-δ-open set V_a such that $U_{x,a} \cap V_a = \phi$. Thecollection $\{v_a$: $a\in A$ $\}$is a (i, j)-δ-open cover of A. Therefore, $\exists$a finite subcollection v_1, v_2, …, v_n that covers A. Let $U = U_1\cap$……….$\cap U_n$, then $U\cap A = \phi$. Thus $X\backslash A$ is (i, j)-δ-open, proving that A is (i, j)-δ-closed.

Theorem 3.6: Every is (i, j)-δ-closed subset of a (i, j)-δ-compact space is (i, j)-δ-compact.

Proof: Let X be a (i, j)-δ-compact and let A be a (i, j)-δ-closed subset of X. Let C be a (i, j)-δ-open cover of A, then C plus X\A is a (i, j)-δ-open cover of X. Since X is (i, j)-δ-compact, this collection has a finite subcollection that covers X. But then C has a finite subcollection that covers A as we need.

Theorem 3.7: A (i, j)-δ-compact subset of a (i, j)-θ-Hausdorff space is (i, j)-θ-closed.

Proof: Let A be a (i, j)-δ-compact subset of a (i, j)-θ-Hausdorff space X. We will show that X\A is (i, j)-θ-open. Let, $x\in X\backslash A$ then for each $a\in A$, $\exists(i, j)$-θ-open set $U_{x,a}$and (j, i)-θ-open set V_a such that $U_{x,a}\cap V_a = \phi$. Thecollection $\{v_a$: $a\in A$ $\}$is a (i, j)-θ-open cover of A. Therefore, $\exists$a finite subcollection v_1, v_2, …, v_n that covers A. Let $U = U_1\cap$……….$\cap U_n$, then $U\cap A = \phi$. Thus $X\backslash A$ is (i, j)-θ-open, proving that A is (i, j)-θ-closed.

Definition 3.11: A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be pairwise connected (Previn 1967) if it can not be expressed as the union of two non empty disjoint sets U and V such that U is i-open and V is j-open , where $i \neq j$, $i, j = 1, 2$.

Theorem 3.8: Let f: $(X, \mathcal{T}_1, \mathcal{T}_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective (i, j)-w.θ.c and let X be pairwise connected. Then Y is pairwise connected.

Proof: Suppose Y is pairwise disconnected. Then $\exists \sigma_i$-open set V and σ_j-open set W such that $Y = V \cup W$. By (i, j)-w.θ.c of f, $f^1(j\mathrm{Cl}(V)) = f^1(V)$ and $f^1(i\mathrm{Cl}(W)) = f^1(W)$ are open in X. But $X = f^1(V) \cup f^1(W)$ and $f^1(V) \cap f^1(W) = \phi$. Thus X is pairwise disconnected, a contradiction. Therefore, Y is pairwise connected.

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