"Science Stays True Here"
Journal of Mathematics and Statistical Science (ISSN 2411-2518, USA), Vol.10, Issue 7, 119-151 | Science Signpost Publishing

The Strongly Solutions of Nonlinear Parabolic Partial Differential Equations Problems in Sobolev Spaces

Omar Mohammed Abdullah Al-haj and Mohammed Al-Hawmi

Department of Mathematics
Faculty of Education and Science
University of Saba Region, Marib, yemen
omaralhajmohammed@gmail.com
m.alhomi2011@gmail.com

Abstract:

In this paper, we study the existence of a strong solutions for the initial-boundary value problems of the nonlinear degenerated parabolic equation

$$\frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = \text{f in } Q$$

where $A(u)$ =- div $a(x,t,u,\nabla u)$, is a Leary lions operator acted from the weighted Sobolev Space $L^P(0,T,W_0^{1,p}(\Omega,w))$ in to its dual $L^{P'}(0,T,W_0^{-1,p'}(\Omega,w^*))$ and $g(x,t,u,\nabla u)$ is a nonlinear term with critical growth condition with respect to u. The source term f is assumed to belong to $L^{p'}(0,T,W_0^{-1,p'}(\Omega,w^*))$.

Key words and phrases : Weighted Sobolev Spaces, Boundary Value problems, parabolic problems, nonlinear equation, Compactness, a time mollification sequence, measurable, Compact set, Continuous, a symmetric bilinear, weak convergence, Hölders inequality, strongly solution, integral, continuity, weak solution, converges strongly, positive constant.

1 Introduction:

Let Ω be a bounded open subset of R^N, $N \geq 2$, and let Q be the cylinder $(0,T)\times\Omega, T>0$, we consider the parabolic initial- boundary value problem

$$(P) = \begin{cases} \dfrac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f & in\ Q \\ u(x,0) = u_0(x) & in\ \Omega \\ u(x,t) = 0 & on \sum = \partial\Omega \times (0,T) \end{cases}$$

where $A(u)$ =-div $a(x,t,u,\nabla u)$ is a classical divergence operator of leray-lions type with respect to the weighted Sobolev space $L^P(0,T,W_0^{1,p}(\Omega,w))$ for some $p \in (1,\infty)$.

The right-hand side f is supposed lying in $L^{P'}(0,T,W_0^{-1,p'}(\Omega,w^*))$. and $\frac{1}{p}+\frac{1}{p'}=1$ or $p'=\frac{p}{p-1}$.

and where the perurbation $g(x,t,u,\nabla u)$ satisfies the growth condition

$$|g(x,t,u,\nabla u)| \leq h(|u|)[c(x,t)+|\nabla u|^p],$$

for some continuous function $h : R^+ \to R^+$,

and the condition

$$g(x,t,u,\nabla u)u \geq 0,$$

R.Landes, V.Mustonen have proved in [13] the existence of solution for the parabolic initial-boundary value problems (P), with $g \equiv g(x,t,u)$, when 1<p<2. Their result generalizes the analogous ones of Lions [16], Landes [12] with $g \equiv 0$, when 1<p<2. in this setting, problems of the form (P) where solved by Brezis-Browder [7] in the case $p \leq 2$. (see[5],[6],[8]).

In all the previous works, the principal part a of the operator A is supposed to satisfy the classical coercivity,

$$a(x,t,s,\xi)\xi \geq \beta \ |\xi|^p \tag{1.1}$$

where β is some strictly positive constat or $\beta \geq 0$ and $(x,t) \in Q, s \in R, \xi \in R^N$ when the operator A becomes generated on the variable space x, that is (P) is replaced by the following (called degeneracy)

$$\sum_{i=1}^{N} a_i(x,t,s,\xi)\xi_i \geq \sum_{i=1}^{N} w_i(x) \ |\xi_i|^p \tag{1.2}$$

where $\xi' \in R^N, \xi \in R^N, \xi \neq \xi'$ and $w = \{w_i, 1 \leq i \leq N\}$ function on Ω, the operator A is not coercive in the classical sobolev space thus we must change this setting by the more general one, called weighted sobolev space $W_0^p(\Omega, w)$,[2][9]. it is our purpose in this paper to study the existence result for the strongly parabolic problem (P), in the setting of weighted sobolev space where the principal part a and the nonlinearity g satisfy some general growth conditions (see (A_2) and (A_3) below). The simplest model our, problem is the following,

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f & in \ Q \\ u(x,0) = u_0(x) & in \ \Omega \\ u(x,t) = 0 & on \sum = \partial\Omega \times (0,T) \end{cases}$$

for that, we must extend the classical sopolev techniques of problems, on the general settings of weighted soblev space.

2 Abstract framework:

let Ω be abounded open set of $\mathcal{R}^{\mathcal{N}}$, p be a real number such that $1<p<\infty$ and $w = \{w_i(x), 1 \le i \le N\}$ be a vector of weight functions, i.e, every component $w_i(x)$ is a measurable function which is strictly positive in Ω. Further, we suppose in all this section that, there exists

$$r_0 > max(N, p) \;\; such\ that \;\; w_i^{\frac{r_0}{r_0-p}} \in L^1_{loc}(\Omega), \tag{2.1}$$

and

$$w_i^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega), \tag{2.2}$$

for any $1 \le i \le N$. We denote by $W^{1,p}(\Omega, w)$ the space of all real-valued functions $u \in L^p(\Omega, w)$. Such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \text{ for all } i = 1, ..., N,$$

which is a Banach Space under the norm

$$\|u\|_{1,p,w} = \left[\int_\Omega |u(x)|^p w_0 dx + \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u(x)}{\partial x_i} \right|^p w_i(x) dx \right]^{\frac{1}{p}}. \tag{2.3}$$

The condition (2.1) implies that C_0^∞ is a subset of $W^{1,p}(\Omega, w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega, w)$ of $W^{1,p}(\Omega, w)$ as the closure of C_0^∞ with respect to the norm (2.3). Moreover, the condition (2.2) implies that $W^{1,p}(\Omega, w)$ as well as $W_0^{1,p}(\Omega, w)$ are reflexive Banach Spaces. We recall that the dual Space of weighted Sobolev Spaces, $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,P'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-P'}, i = 0, ..., N\}$ and where p' is the conjugate of p. i.e. $p' = \frac{p}{p-1}$. and for more details, we refer the reader to [10]. Now, since we have not some general compactness result in the compacted Sobolev Spaces, in order to deal with time derivative, we introduce a time mollification of a function u belonging in some weighted Lebesgue Space. Thus we define for all $\mu \ge 0$ and all $(x, t) \in Q$:

$$u_\mu = \mu \int_{-\infty}^{t} \tilde{u}(x, s)\ exp(\mu(s - t))\ ds \;\; where \;\; \tilde{u}(x, s) = u(x, s)\chi_{(0,T)} \tag{2.4}$$

Proposition(2.1)

If $u \in L^P(Q, w_i)$ then u_μ is measurable in Q and $\dfrac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and

$$\Big(\int_Q |u_\mu|^p w_i(x) dx dt\Big)^{\frac{1}{p}} \leq \Big(\int_Q |u|^p w_i(x) dx dt\Big)^{\frac{1}{p}}$$

$$\|u_\mu\|_{L^P(Q,w_i)} \leq \|u\|_{L^P(Q,w_i)}.$$

Proof.

Since $(x, t, s) \to u(x, s)$ exp $(\mu(s - t))$ is measurable in $\Omega \times [0, T] \times [0, T]$,

u_μ then is measurable by Fubinis theorem. By jensens integral inequality and since $\displaystyle\int_{-\infty}^{0} \mu \ \exp\ (\mu s)\ ds = 1$, we have

$$|u_\mu|^p = \Big|\int_{-\infty}^{t} \mu\ \tilde{u}(x, s)\ exp(\mu(s - t))\ ds\Big|^p = \Big|\int_{-\infty}^{0} \mu\ exp(\mu s)\ \tilde{u}(x, s + t) ds\Big|^p$$

$$\leq \int_{-\infty}^{0} \mu\ exp(\mu s)\ |\tilde{u}(x, s + t)|^p ds$$

which implies

$$\|u\|_{L^P(Q,w_i)} = \int_Q |u_\mu|^p\ w_i(x)\ dx\ dt \leq \int_{\Omega\times R} \Big(\int_{-\infty}^{0} \mu\ exp(\mu s)|\tilde{u}(x, s{+}t)|^p ds\Big) w_i(x)\ dx\ dt$$

$$\leq \int_{-\infty}^{0} \mu\ exp(\mu s)\Big(\int_{\Omega\times R} |\tilde{u}(x, s + t)|^p\ w_i(x)\ dx\ dt\Big) ds$$

$$\leq \int_{-\infty}^{0} \mu\ exp(\mu s)\Big(\int_Q |u(x, t)|^p\ w_i(x)\ dx\ dt\Big) ds$$

$$= \int_Q |u(x, t)|^p\ w_i(x)\ dx\ dt = \|u\|^p_{L^P(Q_T,w_i)}$$

Furthermore,

$$\frac{\partial u_\mu}{\partial t} = \lim_{\theta\to 0} \frac{1}{\theta}(e^{-\mu\theta} - 1)\ u_\mu(x, t) \tag{2.5}$$

$$+ \lim_{\theta\to 0} \frac{1}{\theta} \int_t^{t+\theta} u(x, s)\ e^{\mu(s-(t+\theta))} ds = -\mu\ u_\mu + \mu\ u.$$

Proposition(2.2)

If $u \in W_0^{1,p}(Q, w_i)$ then $u_\mu \to u$ in $W_0^{1,p}(Q, w_i)$ as $\mu \to +\infty$.

proof.

First, by applying the statement of Proposition(2.1) for u_μ and for $\dfrac{\partial u_\mu}{\partial x_i}$ (remark

that $\dfrac{\partial u_\mu}{\partial x_i} = (\dfrac{\partial u}{\partial x_i})_\mu$, we can easily prove that $u_\mu \in W^{1,p}_\circ(Q,w)$
Now we can prove that $u_\mu \to u$ in $W^{1,p}_\circ(Q,w)$ as $\mu \to +\infty$

Let $\phi_k \subset D(Q)$ such that $\phi_k \to u$ in $W^{1,P}_\circ(Q,w)$.

in virtue of Proposition(2.1) we have,

$$|(\phi_k)_\mu(x,t) - \phi_k(x,t)| = \frac{1}{\mu}\left|\frac{\partial \phi_k}{\partial t}(x,t)\right| \leq \frac{1}{\mu}\left|\left|\frac{\partial \phi_k}{\partial t}\right|\right|_\infty \tag{2.6}$$

On other hand,
$\int_Q |u_\mu - u|^p w_i(x)\ dx\ dt$

$\leq \int_Q |u_\mu - (\phi_k)_\mu|^p w_i(x)\ dx\ dt + \int_Q |(\phi_k)_\mu - \phi_k|^p\ w_i(x)\ dx\ dt + \int_Q |\phi_k - u|^p w_i(x)\ dx\ dt.$

This implies that
$\int_Q |u_\mu - u|^p\ w_i(x)\ dx\ dt$

$\leq \int_Q |(u - \phi_k)_\mu|^p\ w_i(x)\ dx\ dt + \int_Q |(\phi_k)_\mu - \phi_k|^p\ w_i(x)\ dx\ dt + \int_Q |\phi_k - u|^p\ w_i(x)\ dx\ dt$

Thinks to Proposition(2.1) and the equality (2.6) we get

$$\begin{aligned}\int_Q |u_\mu - u|^p\ w_i(x)\ dx\ dt &\leq 2\int_Q |(u - \phi_k)|^p\ w_i(x)\ dx\ dt + \frac{T}{\mu^p}\int_{K_k}\left|\frac{\partial \phi_k}{\partial t}\right|^p w_i(x)\ dx \\ &\leq 2\int_Q |u - \phi_k|^p\ w_i(x)\ dx\ dt + \frac{T}{\mu^p}\ C_k,\end{aligned}$$

where

$$C_k = \left|\left|\frac{\partial \phi_k}{\partial t}\right|\right|_\infty^p \int_{K_k} w_i(x)\ dx$$

and where K_k is a compact set such that Sup $\phi_k \subset K_k$ let $\epsilon > 0$, there exists K such that $\int_Q |u - \phi_k|^p w_i(x)\ dx\ dt \leq \dfrac{\epsilon}{3}$ and there exists μ_0 such that $\frac{T}{\mu^p}C_k \leq \dfrac{\epsilon}{3}$ for all $\mu \geq \mu_0$. Hance $\int_Q |u_\mu - u|^p w_i(x)\ dx\ dt \leq \epsilon$ which implies that

$$\|u_\mu - u\|_{L^P(Q,w_i)} \leq \epsilon$$

Since $D^\alpha_x(u_\mu) = (D^\alpha_x u)_\mu$ (for all $|\alpha| \leq 1$), then applying the same argument as above for each $D^\alpha_x u$ we conclude the disired result.

Proposition(2.3)

If $u_n \to u$ in $W^{1,P}_{\circ}(Q,w)$, then $(u_n)_\mu \to u_\mu$ in $W^{1,P}_0(Q,w)$.

Proof.

Using proposition2.1 and

$$D^\alpha_x((u_n)_\mu) - D^\alpha_x(u_\mu) = (D^\alpha_x(u_n))_\mu - (D^\alpha_x(u))_\mu = (D^\alpha_x(u_n) - D^\alpha_x(u))_\mu,$$

we have,

$$\int_Q |D^\alpha_x((u_n)_\mu) - D^\alpha_x(u_\mu)|^p w_i(x)dxdt \leq \int_Q |D^\alpha_x(u_n) - D^\alpha_x(u)|^p w_i(x)dxdt \to 0$$

as $n \to \infty$.

Then $(u_n)_\mu \to u_\mu$ in $W^{1,p}_{\circ}(Q,w)$ as $n \to \infty$.

Now, we give some imbedding and compactness results in weighted sobolev spaces which allow in particular to extend in the setting of weighted sobolev spaces.

Let $V = W^{1,P}_{\circ}(\Omega,w)$, $H = L^2(\Omega,\sigma)$ (where σ is a weight function on Ω such that $\sigma \in L^1(\Omega)$ and $\sigma^{-1} \in L^1(\Omega)$) and Let $V^* = W^{-1,p'}(\Omega,w^*)$, with $(2 \leq p \leq \infty)$.

Let $X = L^p(0,T,V)$. The dual Space of X is $X^* = L^{p'}(0,T,V^*)$ where $\frac{1}{p'} + \frac{1}{p} = 1$ and denoting the Space $W^1_p(0,T,V,H) = \{v \in X : v' \in X^*\}$ endowed with the norm

$$\|u\|_{w^{1,p}} = \|u\|_X + \|u'\|_{X^*}, \tag{2.7}$$

which is Banach Space. Here u' stands for the generalized derivative of u, i.e.,

$$\int_0^T u'(t)\ \phi(t)dt = -\int_0^T u(t)\ \phi'(t)dt \text{ for all } \phi \in C^\infty_0(0,T).$$

Lemma 2.4.

The Banach Space H is an Hilbert Space and its dual H' can be identified with himself, i.e., $H' \simeq H$.

Proof.

Indeed, let

$$F : H \times H \to R$$

$$(f,g) \to \int_\Omega fg\sigma dx$$

Remark that F is asymmetric bilinear from which is also continuous and defined positively, since

$$\int_\Omega fg\ \sigma\ dx = \int_\Omega f\ \sigma^{\frac{1}{2}} g\ \sigma^{\frac{1}{2}} dx \leq \left(\int_\Omega |f|^2 \sigma dx\right)^{\frac{1}{2}} \left(\int_\Omega |g|^2 \sigma dx\right)^{\frac{1}{2}}.$$

Then the Banach Space H is a Hilbert Space.

Finally by a standard argument, H is identified with its dual H' i.e., $H' \simeq H$.

Lemma 2.5.

The evolution triple $V \subseteq H \subseteq V^*$ is verified.

Indeed, by the embedding assumption (3.4) below and because $2 \leq p \leq \infty$, and $\sigma \in L^1(\Omega)$, we can write

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^p(\Omega, \sigma) \hookrightarrow H' \simeq H \hookrightarrow W^{-1,p'}(\Omega, w^*).$$

Lemma 2.6.

Assume that,

$$\frac{\partial u_n}{\partial t} = h_n + k_n \ in\ D'(\Omega),$$

where h_n and k_n are bounded respectively in $L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))$ and in $L^1(Q)$. If u_n is bounded in $L^p(0,T,W_0^{1,p}(\Omega,w))$, then $u_n \to u$ in $L^p_{loc}(Q,\sigma)$. Further $u_n \to u$ strongly in $L^1(Q)$.

proof.

Consider $\phi(x,t) = \psi(x)\eta(t)$ with in $D(\Omega)$ and η in $D(0,T)$ and set

$$v_n = \phi\ u_n,\ \ \alpha_n = \phi\ h_n + \frac{\partial \phi}{\partial t}\ u_n,\ \ \beta_n = \phi\ k_n.$$

Then for any bounded open subset K with supp $\psi \subset K \subset \Omega$ we have,

$$\begin{cases} \dfrac{\partial v_n}{\partial t} = \alpha_n + \beta_n & in\ D'(K \times (0,T)); \\ v_n\ bounded & in\ L^P(0,T,W_0^{1,p}(K,w)); \\ \alpha_n\ bounded & in\ L^{P'}(0,T,W^{-1,p'}(K,w^*)); \\ \beta_n\ bounded & in\ L^1(K \times (0,T)). \end{cases}$$

Let ρ_n be a mollification sequence such that

$$||\bar{v}_n - v_n||_{L^P(Q,\sigma)} \leq \frac{1}{n}, \tag{2.8}$$

where

$$\bar{v}_n = v_n * \rho_n, \quad \bar{\alpha}_n = \alpha_n * \rho_n, \quad \bar{\beta}_n = \beta_n * \rho_n.$$

The function $\bar{\beta}_n$ belongs to $L^1(0,T,L^1(K))$ and is bounded in this space while $\bar{\alpha}_n$ is bounded in $L^{p'}(0,T,W^{-1,P'}(K,w^*))$. Since $1 \leq r_0' \leq \frac{N}{N-1}$, we can write

$$L^1(K) \hookrightarrow W_0^{-1,r_0'}(K).$$

Then $\bar{\beta}_n$ is bounded in $L^1(0,T,W^{-1,r_0'}(K))$, we have $\bar{\alpha}_n$ also bounded in $L^1(0,T,W^{-1,r_0'}(K))$. Thus we get,

$$\begin{cases} \dfrac{\partial \bar{v}_n}{\partial t} = \bar{\alpha}_n + \bar{\beta}_n & in\ D'(K \times (0,T)); \\ \bar{v}_n\ bounded & in\ L^P(0,T,W_0^{1,p}(K,w)); \\ \dfrac{\partial \bar{v}_n}{\partial t}\ bounded & in\ L^1(0,T,W^{-1,r_0'}(K)) \end{cases}$$

We deduce that $\bar{v}_n$ is relatively compact in $L^p(0,T,L^p(K,\sigma))$. In view of (2.8) this implies that u_n is relatively compact in $L^p_{loc}(Q_T,\sigma)$. Finally by using Holders inequality and $\sigma^{-1} \in L^1(\Omega)$, it is easy to deduce that $u_n \to u$ strongly in $L^1(Q_T)$.

Lemma2.7.

Let $g \in L^r(Q,\gamma)$ and let $g_n \in L^r(Q,\gamma)$, with $||g_n||_{L^r(Q,\gamma)} \leq c, 1<r<\infty$. if $g_n(x) \to g(x)$ a.e in Q, then $g_n \rightharpoonup g$ in $L^r(Q,\gamma)$, where $\rightharpoonup$ denotes weak convergence and γ is a weight function on Q

Proof. Since $g_n \gamma^{\frac{1}{r}}$, is bounded in $L^r(Q)$ and $g_n(x)\,\gamma^{\frac{1}{r}}(x) \to g\,\gamma^{\frac{1}{r}}$, a.e. in Q,then by Lemma 2.5[15], we have

$$g_n\,\gamma^{\frac{1}{r}} \rightharpoonup g\,\gamma^{\frac{1}{r}} \text{ in } L^r(Q).$$

Moreover, for all $\phi \in L^{r'}(Q,\gamma^{1-r'})$, we have $\phi\,\gamma^{\frac{-1}{r}} \in L^{r'}(Q)$. then

$$\int_Q g_n\,\phi\,dx \to \int_Q g\,\phi\,dx, \quad i.e.\ g_n \rightharpoonup g \text{ in } L^r(Q,\gamma).$$

Lemma 2.8. [18]

Let $V \subseteq H \subseteq V^*$ be an evolution triple, then the embedding

$$W_p^1(0,T,V,H) \subseteq C([0,T],H)$$

is continuous.

3 Basic assumptions and main result:

3.1 Some weighted embedding and compactness results.

we suppose in all our consideration that, there exists $r_o>$ max (N,P) such that $w_i^{\frac{r_0}{r_0-p}} \in L^1_{loc}(\Omega)$ and $w_i^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$, for any $0 \le i \le N$. Now we state our basic assumptions :

Assumption (A_1)**.** for $2 \le p<\infty$, we suppose that the expression:

$$|||u||| = \left(\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x)\ dx \right)^{\frac{1}{p}} \tag{3.1}$$

is a norm on $W_0^{1,p}(\Omega,w)$ and its equivalent to (2.2). There exists a weight function σ on Ω such that

$$\sigma^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega) \quad , \ \sigma \in L^1(\Omega) \tag{3.2}$$

The Hardy inequality,

$$\left(\int_\Omega |u(x)|^p \sigma dx \right)^{\frac{1}{p}} \le c \left(\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x) dx \right)^{\frac{1}{p}}, \tag{3.3}$$

holds for every $u \in W_0^{1,p}(\Omega,w)$ with a constant $c >o$ independent of u. moreover, the embedding

$$W_0^{1,p}(\Omega,w) \hookrightarrow L^p(\Omega,\sigma) \tag{3.4}$$

expressed by the inequality (3.3) is compact, note that $(W_0^{1,p}(\Omega,w), |||.|||)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 3.1.

Assume that $w_o(x) \equiv 1$ and there exists $v \in]\frac{N}{p}, +\infty[\cap[\frac{1}{p-1}, +\infty[$ such that

$$w_i^{\frac{N}{N-1}} \in L^1_{loc}(\Omega), w_i^{-v} \in L^1(\Omega) \quad \forall\ i = 1, ..., N \tag{3.5}$$

Note that the assumptions(2.1) and (3.5) imply that

$$|||u||| = \left(\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x) dx \right)^{\frac{1}{p}} \tag{3.6}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and it is equivalent to (2.3) and that, the embedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^p(\Omega) \tag{3.7}$$

is compact $[see[11], p.46]$.

Thus the hypotheses (A_1) is satisfied for $\sigma \equiv 1$.

Assumption A_2

for $i = 1, ..., N$,

$$a_i(x, t, s, \xi) \leq \beta \sigma^{\frac{1}{p}}(x) \left(c_0(x, t) + \sigma^{\frac{1}{p}} |s|^{p-1} + \sum_{i=1}^{N} w_i^{\frac{1}{p'}}(x) |\xi_i|^{p-1} \right) \tag{3.8}$$

$$|a_i(x, t, s, \xi)| \leq \beta w_i^{\frac{1}{p}}(x) \left(c_1(x, t) + \sigma^{\frac{1}{p'}} |s|^{p-1} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x) |\xi_j|^{p-1} \right) \tag{3.9}$$

$$\left(a(x, t, s, \xi) - a(x, t, s, \eta) \right) (\xi - \eta) > 0 \ \xi \neq \eta \in R^N, \tag{3.10}$$

$$a_0(x, t, s, \xi).s + \sum_{i=1}^{N} a_i(x, t, s, \xi) \xi_i \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p, \tag{3.11}$$

$$a(x, t, s, \xi).\xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p \tag{3.12}$$

where $c_0(x, t)$ and $c_1(x, t)$ are some positive function in $L^{p'}(Q)$, and α and β are some strictly positive constants.

Assumption (A_3)

$$|g(x, t, s, \xi)| \leq b(|s|) \left(\sum_{i=1}^{N} w_i(x) |\xi_i|^p + c(x, t) \right) \tag{3.13}$$

$$g(x,t,s,\xi)s \geq 0. \tag{3.14}$$

where $b : R^+ \to R^+$ is a continuous increasing function and c is a positive function in $L^1(Q)$.

we recall that for $k>1$ and s in R the truncation is defined as

$$T_k(s) = \begin{cases} s & if \quad s \leq k \\ k\frac{s}{|s|} & if \quad |s|>k \end{cases}$$

Lemma 3.1.

Assume that (A_1) and (A_2) are satisfied and let (u_n) be a sequence in $L^p(0,T,W_0^{1,P}(\Omega,w))$ such that $u_n \to u$ weakly in $L^P(0,T,W_0^{1,p}(\Omega,w))$ and

$$\int_Q [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u] \;\; dx\ dt \to 0 \tag{3.15}$$

Then, $u_n \to u$ in $L^p(0,T,W_0^{1,p}(\Omega,w))$.

proof,

Let $D_n = [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]$. Then by (3.10), D_n is a positive function and by (3.15), $D_n \to 0$ in $L^1(Q)$.

Extracting a subsequence, still denoted by u_n, and using (3.4) we can write $u_n \to u$ a.e. in Q, $D_n \to 0$ a.e in Q.

Then, there exists a subset B of Q, of zero measure such that, for $(x,t) \in Q, |u_n(x,t)|<\infty, |\nabla u(x,t)|<\infty,\ |c_1(x,t)|<\infty, w_i(x)>0$ and $u_n(x,t) \to u(x,t), D_n(x,t) \to 0$. We set $\epsilon_n = \nabla u_n(x,t)$ and $\epsilon = \nabla u(x,t)$. From (3.8),(3.9) and (3.12), then

$$\begin{aligned} D_n(x,t) &= [a(x,t,u_n,\epsilon_n) - a(x,t,u_n,\epsilon)](\epsilon_n - \epsilon) \\ &\geq \alpha \sum_{i=1}^{N} w_i|\epsilon_n^i|^p + \alpha \sum_{i=1}^{N} w_i|\epsilon^i|^p \\ &- \sum_{i=1}^{N} \beta w_i^{\frac{1}{p}} \left[c_1(x,t) + \sigma^{\frac{1}{p'}}|u_n|^{p-1} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}|\epsilon_n^j|^{p-1} \right] |\epsilon^i| \\ &- \sum_{i=1}^{N} \beta w_i^{\frac{1}{p}} \left[c_1(x,t) + \sigma^{\frac{1}{p'}}|u_n|^{p-1} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}|\epsilon^j|^{p-1} \right] |\epsilon_n^i| \end{aligned}$$

i.e,

$$D_n(x,t) \geq \alpha \sum_{i=1}^{N} w_i |\epsilon_n^i|^p - c_{x,t}\Big[1 + \sum_{j=1}^{N} w_j^{\frac{1}{p'}} |\epsilon_n^j|^{p-1} + \sum_{i=1}^{N} w_i^{\frac{1}{p}} |\epsilon_n^i|\Big], \tag{3.16}$$

where $C_{x,t}$ is a constant which depends on x and t, but dose not depend on n. Since $u_n(x,t) \to u(x,t)$, we have $|u_n(x,t)| \leq M_{x,t}$ where $M_{x,t}$ is some positive constant. Then by a standard argument $|\epsilon_n|$ is bounded uniformly with respect to n. Indeed,(3.16) becomes

$$D_n(x,t) \geq \sum_{i=1}^{N} |\epsilon_n^i|^p \bigg(\alpha\ w_i - \frac{C_{x,t}}{N|\epsilon_n^i|^p} - \frac{C_{x,t} w_i^{\frac{1}{p'}}}{|\epsilon_n^i|} - \frac{C_{x,t} w_i^{\frac{1}{p}}}{|\epsilon_n^i|^{p-1}}\bigg).$$

If $|\epsilon_n| \to \infty$(for a subsequence) there exists at least one i_0 such that $|\epsilon_n^{i_0}| \to \infty$, which implies that $D_n(x,t) \to \infty$, which gives a contradiction.

Let now ϵ^* be a cluster point of ϵ_n. We have $|\epsilon^*|<\infty$ and by the continuity of a with respect to the last variables we obtain

$$(a(x,t,u(x,t),\epsilon^*) - a(x,t,u(x,t),\epsilon))(\epsilon^* - \epsilon) = 0.$$

In view of (3.10) we have $\epsilon^* = \epsilon$. The uniqueness of the cluster point implies

$$\nabla u_n(x,t) \to \nabla u(x,t) \text{ a.e. in } Q.$$

since the sequence $a(x,t,u_n,\nabla u_n)$ is bounded in $\prod_{i=1}^{N} L^{p'}(Q, w_i^*)$ and $a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u)$ a.e in Q Lemma 2.7 implies $a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u)$ in $\prod_{i=1}^{N} L^{p'}(Q, w_i^*)$ and a.e in Q.

We set $\bar{y}_n = a(x,t,u_n,\nabla u_n)\nabla u_n$ and $\bar{y} = a(x,t,u,\nabla u)\nabla u$. As in the proof of Lemma 5 in [5] we can write $\bar{y}_n \to \bar{y}$ in $L^1(Q)$. By (3.12), we have $\alpha \sum_{i=1}^{N} w_i \Big|\frac{\partial u_n}{\partial x_i}\Big| \leq a(x,t,u_n,\nabla u_n)\nabla u_n$.

Let $z_n = \sum_{i=1}^{N} w_i \Big|\frac{\partial u_n}{\partial x_i}\Big|^p$, $z = \sum_{i=1}^{N} w_i \Big|\frac{\partial u}{\partial x_i}\Big|^p$, $y_n = \frac{\bar{y}}{\alpha}$ and $y = \frac{\bar{y}}{\alpha}$.

Then, by Fatous Lemma we obtain

$$\int_Q 2y\ dx\ dt \leq \lim_{n\to\infty} \inf \int_Q y + y_n - |z_n - z|\ dx\ dt,$$

i.e

$$0 \leq \lim_{n\to\infty} \text{ sup } \int_Q |z_n - z| \; dx \; dt,$$

hence,

$$0 \leq \lim_{n\to\infty} inf \int_Q |z_n - z| \; dx \; dt \leq \lim_{n\to\infty} sup \int_Q |z_n - z| \; dx \; dt \leq 0.$$

This implies

$$\nabla u_n \to \nabla u \text{ in } \prod_{i=1}^{N} L^P(Q, w_i),$$

which with (3.1) completes the present proof,

Now we recall the well-known general Sobolev embedding theorems for evolution equations (see [18]).

Lemma 3.2.[18]Let Z_1, Y_1, Z_2 be real reflexive Banach space. Assume that the embedding $Z_1 \subseteq Y \subseteq Z_2$ are continuous, and the embedding $Z_1 \subseteq Y$ iscompact. $0<T<\infty, 1<p, q<\infty$ Then $W = u \in L^p(0, T, Z_1) : u' \in L^q(0, T, Z_2)$ equipped with the norm $\|u\| = \|u\|_{L^p(0,T,Z_1)} + \|u'\|_{L^q(0,T,Z_2)}$ is a Banach space and the embedding $W \subseteq L^p(0, T, Y)$ is compact.

Definition 3.1. A monotone map $T : D(T) \to X^*$ is called maximal monotone if its graph

$$G(T) = \{(u, T(u)) \in X \times X^* \text{ for all } u \in D(T)\}$$

is not a proper subset of any monotone set in $X \times X^*$.
Let us consider the operator $\dfrac{\partial}{\partial t}$ which induces a linear map L from the subset

$$D(L) = \{v \in X : v^{'} \in X^*, v(0) = 0\} \text{ of } X \text{ into } X^* \text{ by}$$

$$\langle Lu, v\rangle_X = \int_0^T \langle u^{'}(t), v(t)\rangle_V \; dt \;, u \in D(L), \; v \in X. \tag{3.17}$$

Definition 3.2.A mapping S is called pseudo-monotone with respect to D(L), if for any sequence $\{u_n\}$ in $D(L)$ with $u_n \rightharpoonup u$ and $Lu_n \rightharpoonup Lu$ and $\lim_{n\to\infty}\sup$ $\langle S(u_n), u_n - u\rangle \leq 0$, we have $\lim_{n\to\infty} \langle S(u_n), u_n - u\rangle = 0$ and $S(u_n) \rightharpoonup S(u)$ as

$n \to \infty$.

Consider the following non linear parabolic problem

$$(P):\begin{cases} \dfrac{\partial u}{\partial t}+A(u)+g(x,t,u,\nabla u)=f & in\ Q \\ u(x,0)=u_0(x) & in\ \Omega \\ u(x,t)=0 & on \sum=\partial\Omega\times(0,T) \end{cases}$$

where u_0 is a given function in $L^2(\Omega,\sigma)$.

Definition 3.3. A function u is said to be a weak solution of the initial-boundary value problem (P) if $u \in C([0,T],H)\bigcap L^p(0,T,V)$, $\dfrac{\partial u}{\partial t} \in L^{p'}(0,T,V^*)$ and u satisfies the equation,

$$\frac{\partial u}{\partial t}+Au+G(u)=f \quad 0<t<T\ ,\ u(0)=u_0,$$

where the operator $A+G : X \to X^*$ is defined by:
$\langle (A+G)(u),v\rangle = \displaystyle\int_Q a(x,t,u,\nabla u)\nabla v\ dx\ dt + \int_Q g(x,t,u,\nabla u)v\ dx\ dt$

3.2 The approximate problem.

We consider the sequence of approximate equations,

$$(P_n):\begin{cases} \dfrac{\partial u_n}{\partial t}+A(u_n)+g_n(x,t,u_n,\nabla u_n)=f \\ u_n(0)=u_0^n \end{cases}$$

where

$$g_n(x,t,s,\xi)=\frac{g(x,t,s,\xi)}{1+\frac{1}{n}|g(x,t,s,\xi)|}\chi_{\Omega_n},\ \ u_0^n \in W_0^{1,p}(\Omega,w).$$

Note thate Ω_n is a sequence of compacts covering the bounded open set Ω and χ_{Ω_n} denotes a characteristic function of Ω_n
and $g_n(x,t,s,\xi)$ satisfies the following conditions
$g_n(x,t,s,\xi).s \ \geq\ 0,\ \ |g_n(x,t,s,\xi)| \leq\ g(x,t,s,\xi)$ and $|g_n(x,t,s,\xi)| \leq\ n.$
We define the operator $G_n : X \to X^*$ by $\langle G_n u,v\rangle = \displaystyle\int_Q g_n(x,t,u,\nabla u)v\ dx\ dt.$

Remark 3.2. Note that in the remainder of this section we will consider approximate problem (P_n) with $u_0^n = 0$, without losing the generalities, since if

$u_0^n \neq 0$, we will change $a(x,t,u,\nabla u)$ by $\bar{a}(x,t,u,\nabla u) = a(x,t,u+u_0,\nabla u+\nabla u_0)$ and $g(x,t,u,\nabla u)$ by $\bar{g}(x,t,u,\nabla u) = g(x,t,u+u_0,\nabla u+u_0)$.

Lemma 3.3. The operator $A + G_n : X \to X^*$ is :

a) bounded and demicontinuous

b) pseudo-monotone with respect to $D(L)$

proof.

a) We set $B_n = A + G_n$. Using (3.9) and Hölders inequality we can show that A is bounded. Thanks of Hölders inequality, for all $u \in X$ and all $v \in X$ we have

$$\left| \int_Q g_n(x,t,u,\nabla u) v \, dx \, dt \right| \leq C_n \|v\|^p_{L^p(Q,\sigma)} \leq C'_n \|v\|^p_X. \tag{3.18}$$

Then B_n is bounded. In order to show that B_n is demicontinuous, let $v_\epsilon \to v$ in X as $\epsilon \to 0$, and prove that,

$$\langle B_n(v_\epsilon), \phi \rangle \to \langle B_n(v), \phi \rangle \text{ for all } \phi \in X.$$

Since, $a_i(x,t,v_\epsilon,\nabla v_\epsilon) \to a_i(x,t,v_\epsilon,\nabla v)$ as $\epsilon \to 0$, for a.e. $x \in \Omega$, then by the growth conditions (3.9) and Lemma 2.7 we get

$$a_i(x,t,v_\epsilon,\nabla v_\epsilon) \rightharpoonup a_i(x,t,v,\nabla v) \text{ in } L^{p'}(Q, w_i^{1-p'}) \text{ as } \epsilon \to 0.$$

Finally for all $\phi \in X$,

$$\langle A(v_\epsilon), \phi \rangle \to \langle A(v), \phi \rangle \text{ as } \epsilon \to 0.$$

On the other hand, $g_n(x,t,v_\epsilon,\nabla v_\epsilon) \to g_n(x,t,v,\nabla v)$ as $\epsilon \to 0$ for a.e. (x,t) in Q. Also $g_n(x,t,v_\epsilon,\nabla v_\epsilon)_\epsilon$ is bounded in $L^{p'}(Q,\sigma^{1-p'})$ in fact,

$$\int_Q |g_n(x,t,v_\epsilon,\nabla v_\epsilon)|^{p'} \sigma^{1-p'} dx \, dt \leq (n)^{p'} \; T \int_{\Omega_n} \sigma^{1-p'} dx \leq c_n.$$

Then , Lemma 2.7 gives

$$g_n(x,t,v_\epsilon,\nabla v_\epsilon) \rightharpoonup g_n(x,t,v,\nabla v) \text{ in } L^{p'}(Q,\sigma^{1-p'}) \text{ as } \epsilon \to 0.$$

Since $\phi \in L^p(Q,\sigma)$, for all $\phi \in X$ we have

$$\langle G_n(v_\epsilon), \phi \rangle \to \langle G_n(v), \phi \rangle \text{ as } \epsilon \to 0.$$

b) Suppose that $\{u_j\}$ is any sequence in $D(L)$ with

$i)u_j \rightharpoonup u$ weakly in X

ii) $Lu_j \to Lu$ weakly in X*,

iii) lim sup $\langle A + G_n(u_j), u_j - u\rangle_X \leq 0$.

Through the definition of the operator L, defined in (3.17), $\{u_j\}$ is a bounded sequence in $W_p^1(0, T, V, H)$.

By virtue of Lemma 3.2, we get,

$$u_j \to u \text{ strongly in } L^P(Q, \sigma).$$

On the other hand,

$$\langle G_n u_j, u_j - u\rangle = \int_Q g_n(x, t, u_j, \nabla u_j)(u_j - u)\ dx\ dt.$$

Thus Holders inequality and (i)imply,

$$\langle G_n u_j, u_j - u\rangle \leq \left(\int_Q |g_n|^{p'} \sigma^{1-p'} dx\ dt\right)^{\frac{1}{p'}} \|u_j - u\|_{L^P(Q,\sigma)}$$

$$\leq C_n \|u_j - u\|_{L^P(Q,\sigma)},$$

i.e, $\langle G_n u_j, u_j - u\rangle \to 0$ as $j \to \infty$. Combing the last convergence with (iii), we obtain

$$\lim_{j\to\infty} sup\langle Au_j, u_j - u\rangle \leq 0.$$

Also, by the pseudo-monotonicity of A (see Proposition 2.1, [9]), we have

$$Au_j \rightharpoonup Au \text{ in X* and } \lim_{j\to\infty} \langle Au_j, u_j - u\rangle = 0.$$

Then,

$$\lim_{j\to\infty} \langle Au_j + G_n(u_j), u_j - u\rangle = 0.$$

On the other hand, $\lim_{j\to\infty} \langle Au_j, u_j - u\rangle = 0$, which implies that

$$0 = \lim_{j\to\infty} \int_Q a(x, t, u_j, \nabla u_j)\ \nabla(u_j - u)\ dx\ dt$$

$$= \lim_{j\to\infty} \int_Q [a(x, t, u_j, \nabla u_j) - a(x, t, u_j, \nabla u)][\nabla u_j - \nabla u]\ dx\ dt$$

$$+\lim_{j\to\infty}\int_Q a(x,t,u_j,\nabla u)(\nabla u_j-\nabla u)\ dx\ dt.$$

The last integral in the right hand tend to zero, since by the continuity of the Nemytskii operator, $a(x,t,u_j,\nabla u)\to a(x,t,u,\nabla u)$ in $\prod_{i=1}^{N} L^{p'}(Q,w_i^{1-p'})$ as $j\to+\infty$. so,

$$\lim_{j\to\infty}\int_Q [a(x,t,u_j,\nabla u_j)-a(x,t,u_j,\nabla u)][\nabla u_j-\nabla u]dx\ dt=0.$$

By Lemma 3.1 we have

$$\nabla u_j\to\nabla u \text{ a.e. in Q.}$$

Hence $g_n(x,t,u_j,\nabla u_j)\to g_n(x,t,u,\nabla u)$ a.e. in Q as $j\to+\infty$ and since

$$|g_n(x,t,u_j,\nabla u_j)|\leq n\chi_{\Omega_n}\in L^{p'}(Q,\sigma^{1-p'}),$$

by Lebesques dominated convergence theorem, we obtain

$$g_n(x,t,u_j,\nabla u_j)\to g_n(x,t,u,\nabla u) \text{ in } L^{p'}(Q,\sigma^{1-p'}).$$

Finally,

$$(A+G_n)(u_j)\rightharpoonup(A+G_n)(u) \text{ in X*}.$$

Definition 3.4. A function u is said to be a weak solution for the problem (P_n) iff $u\in C([0,T],H)\bigcap D(L)$ and u satisfies the evolution equation

$$Lu+(A+G_n)(u)=f. \tag{3.19}$$

Theorem 3.4 Assume that the conditions $(A_1)-(A_3)$ hold, then the problem (P_n) admits a weak solution for any $f\in X^*$.

Proof . By virtue of Lemma 3.3, the operator $A+G_n:X\to X^*$ is pseudo-monotone with respect to $D(L)$, and the operator $A+G_n$ satisfies the strong coercivity condition, which impiies that both of the condition (i) and (ii) in theorem 4 of [3] hold. So all the condition of theorem 4 in [3] are met. Therefore, there exists a solution $u_n\in D(L)$ of the evolution equation (3.19) for any $f\in X^*$. In order to prove that u_n is also a weak solution of the problem (P_n), we have to show that $u_n \in C([0,T],H)$. By the definition of $D(L)$ and Lemma 2.8, we obtion

$$D(L) \subseteq W_p^1(0,T,V,H) \subseteq C([0,T],H).$$

This implies that $u_n \in C([0,T],H)$.

3.3 Existence result of the general problem.

Theorem3.5. Assume that the conditions $(A_1)-(A_3)$ hold true.
Then the problem (P) admits at least one weak solution
$u \in D(A)\bigcap L^p(0,T,W_0^{1,p}(\Omega,w))\bigcap C([0,T],L^2(\Omega,\sigma))$ such that $g(x,t,u,\nabla u) \in L^1(Q)$, $g(x,t,u,\nabla u)u \in L^1(Q)$.
Furthermore $u(x,0)=u_0(x)$ for a.e. $x\in\Omega$ and we have

$$-\int_Q u\,\frac{\partial\phi}{\partial t}\,dx\,dt + \left[\int_\Omega u(t)\,\phi(t)\,dx\right]_0^T + \int_Q a(x,t,u,\nabla u)\,\nabla\phi\,dx\,dt$$
$$+\int_Q g(x,t,u,\nabla u)\,\phi\,dx\,dt = \langle f,\phi\rangle$$

for all $\phi \in L^p(0,T,W_0^{1,p}(\Omega,w))\bigcap L^\infty(Q)\bigcap C^1([0,T],L^2(\Omega,\sigma))$, for any $f\in X^*$.

proof. Step 1: A priori estimates.

First, for all τ in $[0,T]$, we choose $u_n\chi_{0,\tau}$ as test function in (P_n), we have

$$\frac{1}{2}\int_\Omega u_n^2(\tau)\,dx - \frac{1}{2}\int_\Omega u_0^2\,dx + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)u_n\,dx\,dt$$
$$=\int_{Q_\tau} f\,u_n\,dx\,dt + C.$$

Since g_n verifies sign condition, by using (3.12), for $\tau=T$ we obtain

$$\alpha\sum_{i=1}^N\int_Q w_i\left|\frac{\partial u_n}{\partial x_i}\right|^p \le \int_Q f\,u_n\,dx\,dt,$$

i.e.,

$$\alpha\|u_n\|^p_{L^p(0,T,V)} \le \|f\|_{L^{p'}(0,T,V^*)}\|u_n\|_{L^p(0,T,V)} \le c\|u_n\|_{L^p(0,T,V)}.$$

Then

$$\|u_n\|_{L^p(0,T,V)} \le \beta_1, \qquad (3.20)$$

where β_1 is some positive constant. Then by (3.20) and (3.9) we conclude that $A(u_n)$ is bounded in $L^{P'}(0,T,V^*)$, and there exists $h \in \prod_{i=1}^{N} L^{p'}(Q,w_i)$, such that

$$a(x,t,u_n,\nabla u_n) \rightharpoonup h \ in \prod_{i=1}^{N} L^{p'}(Q,w_i), \tag{3.21}$$

and,

$$\int_Q g_n(x,t,u_n,\nabla u_n)u_n \ dxdt \leq \beta_2, \tag{3.22}$$

where β_2 is some positive constant.

Moreover,

$$\|G_n(u_n)\|_{L^1(Q)} \leq \beta_3 \tag{3.23}$$

Indeed, let $Q^{k,n} = (x,t)\epsilon Q/|u_n(x,t)| \leq k$. We get by (3.13) and (3.14),

$$\int_Q |g_n(x,t,u_n,\nabla u_n)| \ dx \ dt$$

$$= \int_{|u_n|\leq k} |g_n(x,t,u_n,\nabla u_n)| \ dx \ dt + \int_{|u_n|>k} |g_n(x,t,u_n,\nabla u_n)| \ dx \ dt$$

$$\leq b(k)\Bigg(\int_Q \Bigg[\sum_{i=1}^{N} w_i|\frac{\partial u_n}{\partial x_i}|^p + c(x,t)\Bigg] \ dx \ dt\Bigg) + \frac{1}{k}\int_Q g_n(x,t,u_n,\nabla u_n)u_n \ dx \ dt \leq \beta_3.$$

Finally, denoting $u_n' = (f +$ div $a(x,t,u_n,\nabla u_n) + (-G_n(u_n))$ we observe that

$h_n = f + div \ a(x,t,u_n\nabla u_n)$ is bounded in $L^{p'}(0,T,V^*)$ and $k_n = -G_n(u_n)$ is bounded in $L^1(Q)$. Thus we can invoke a result of Lemma 2.6 to conclude that u_n is relatively compact in $L^p_{loc}(Q,\sigma)$, and we can deduce $u_n \to u$ in $L^p_{loc}(Q,\sigma)$, and $u_n \to u$ strongly in $L^1(Q)$.

Step 2:Basic convergence results

Fix $k>0$ and let $\phi(s) = s \ e^{\delta} s^2, \delta > 0$. it is well known that when $\left(\delta \geq \frac{b(k)}{2\alpha}\right)^2$ one has

$$\phi'(s) - \frac{b(k)}{\alpha} \ \phi(s)| \geq \frac{1}{2} \qquad \forall \ \ s \in \ R. \tag{3.24}$$

Let $\phi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^2(\Omega,\sigma)$. Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t}T_k(\phi_i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$ see proposition (2.1) and proposition (2.2). Note that w^i_μ is a smooth function having the following properties :

$$\left\{ \begin{array}{l} \dfrac{\partial}{\partial t}(w^i_\mu) = \mu(T_k(u) - w^i_\mu), \quad w^i_\mu(0) = T_k(\psi_i), \quad |w^i_\mu| \geq k \\ w^i_\mu \to T_k(u) \quad in \quad L^p(0,T,W^{1,P}_0(\Omega,w)) \ as \quad \mu \to \infty \end{array} \right\}$$

using in (P_n) the test function $z^{\mu,i}_n = \phi(T_k(u_n) - w^i_\mu)$, we get for $\tau = T$.

$$\langle u^{'}_n, z^{\mu,i}_n\rangle + \int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla w^i_\mu)\ \phi^{'}(T_k(u_n) - w^i_\mu)\ dx\ dt$$

$$+ \int_Q g_n(x,t,u_n,\nabla u_n)\ \phi(T_k(u_n) - w^i_\mu)\ dx\ dt = \langle f, \phi(T_k(u_n) - w^i_\mu)\rangle$$

which implies since $g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n) - w^i_\mu) \geq 0$ on $|u_n|>k$:

$$\langle u^{\prime}_n, z^{\mu,i}_n\rangle + \int_Q a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla w^i_\mu)\ \phi^{'}(T_k(u_n) - w^i_\mu)\ dx\ dt$$

$$+ \int_{|u_n|\leq k} g_n(x,t,u_n,\nabla u_n)\ \phi(T_k(u_n) - w^i_\mu)\ dx\ dt \leq \langle\ f\ ,\ \phi(T_k(u_n) - w^i_\mu)\ \rangle. \quad (3.25)$$

In the sequel and throughout the paper, we will omit for simplicity to denote $\epsilon(n,\mu,i)$ all quantities (possibly different) such that $\lim\limits_{i\to\infty}\lim\limits_{\mu\to\infty}\lim\limits_{n\to\infty}\epsilon(n,\mu,i) = 0$ and this will be the order in which the parameters we use will tend to infinity,that is,first n, then μ and finally i. Similary we will write only $\epsilon(n)$, or $\epsilon(n,\mu)$, ... to mean that the limits are made only on the specified parameters. We will deal with each term of (3.25). First of all, observe that

$$\langle\ f\ ,\ \phi(T_k(u_n) - w^i_\mu)\rangle = \epsilon(n,\mu), \quad (3.26)$$

since $T_k(u_n) - w^i_\mu \rightharpoonup T_k(u) - w^i_\mu$ weakly in $L^p(0,T,V)$ as $n \to \infty$, and $T_k(u) - w^i_\mu \to 0$ in $L^p(0,T,V)$ as $\mu \to +\infty$.

Let us recall that for $u_n \in W^{1,p}_0(\Omega,w)$, there exists a smooth function $u_{n\theta}$ such that $u_{n\theta} \to u_n$ for the modular convergence in $W^{1,p}_0(\Omega,w)$,

$\frac{\partial u_{n\theta}}{\partial t} \to \frac{\partial u_n}{\partial t}$ for the modular convergence in $W^{-1,p}_0(\Omega,w^*) + L^1(Q)$, we have,

$$\langle\frac{\partial u_n}{\partial t}, z^{\mu,i}_n\rangle = \lim_{\theta\to\infty}\int_Q \frac{\partial u_{n\theta}}{\partial t}\ \phi(T_k(u_{n\theta}) - w^i_\mu)\ dx\ dt$$

$$= \lim_{\theta\to\infty}\int_Q [(T_k(u_{n\theta}))^{'} + (G_k(u_{n\theta}))^{'}]\ \phi(T_k(u_{n\theta}) - w^i_\mu)\ dx\ dt,$$

where $G_k(s) = s - T_k(s)$. Hence

$$\langle \frac{\partial u_n}{\partial t}, z_n^{\mu,i} \rangle = \lim_{\theta\to\infty} \int_Q (T_k(u_{n\theta}) - w_\mu^i)' \phi(T_k(u_{n\theta}) - w_\mu^i)\ dx\ dt$$

$$+ \int_Q (w_\mu^i)'\ \phi(T_k(u_{n\theta}) - w_\mu^i)\ dx\ dt$$

$$+ \int_Q (G_k(u_{n\theta}))'\ \phi(T_k(u_{n\theta}) - w_\mu^i)\ dx\ dt$$

$$= \lim_{\theta\to\infty} (I_1(\theta) + I_2(\theta) + I_3(\theta)).$$

Setting $\Phi(s) = \displaystyle\int_0^s \phi(r)\ dr$, it is easy to see that, since $\Phi(s) \geq 0$

$$I_1(\theta) = \int_\Omega \left[\int_0^T \phi(T_k(u_{n\theta}) - w_\mu^i)(T_k(u_{n\theta}) - w_\mu^i)' dt \right] dx$$

$$= \left[\int_\Omega \Phi(T_k(u_{n\theta})(t) - w_\mu^i(t))\ dx \right]_0^T$$

and

$$I_1(\theta) \geq -\int_\Omega \Phi(T_k(u_{n\theta}(0)) - w_\mu^i(0))\ dx.$$

Since, as $\theta \to \infty$ the last side goes to

$$-\int_\Omega \Phi(T_k(u_0) - T_k(\psi_i))\ dx,$$

$$\to 0 \text{ as } i \to \infty$$

which is of the form $\epsilon(i)$, we get

$$\limsup_{\theta\to\infty} I_1(\theta) \geq \epsilon(i) \qquad .$$

About $I_2(\theta)$, we have, since $(w_\mu^i)' = \mu(T_k(u) - w_\mu^i)$

$$I_2(\theta) = \mu \int_Q (T_k(u) - w_\mu^i)\ \phi(T_k(u_{n\theta}) - w_\mu^i)\ dx\ dt$$

$$= \mu \int_Q \Big((T_k(u) - T_k(u_{n\theta})) + (T_k(u_{n\theta}) - w_\mu^i) \Big) \phi(T_k(u_{n\theta}) - w_\mu^i)\ dx\ dt$$

since $\phi(s)s \geq 0$ then

$$I_2(\theta) \geq \mu \int_Q (T_k(u) - T_k(u_{n\theta}))\ \phi(T_k(u_{n\theta}) - w_\mu^i)\ dx\ dt.$$

Since, as $\theta \to \infty$,

$$\mu \int_Q (T_k(u) - T_k(u_n))\ \phi(T_k(u_n) - w_\mu^i)\ dx\ dt, \quad \to 0 \ \ as\ n \to +\infty$$

which is of form $\epsilon(n)$, hence

$$\limsup_{\theta\to\infty} I_2(\theta) \geq \epsilon(n).$$

Through integration, for $I_3(\theta)$ we have,

$$I_3(\theta) = -\int_Q G_k(u_{n\theta})\ \phi^{'}(T_k(u_{n\theta}) - w_\mu^i)(T_k(u_{n\theta}) - w_\mu^i)'\ dx\ dt$$
$$+\Big[\int_\Omega G_k(u_{n\theta})(t)\ \phi(T_k(u_{n\theta}) - w_\mu^i)(t)\ dx\Big]_0^T$$

since $(T_k(u_{n\theta}))' = 0$ on $\{|u_{n\theta}|>k\}$ and $G_k(u_{n\theta}) = 0$ on $\{|u_{n\theta}| \leq k\}$. since

$$\Big[\int_\Omega G_k(u_{n\theta})(t)\ \phi(T_k(u_{n\theta}) - w_\mu^i)(t)\ dx\Big]_0^T \geq -\int_\Omega G_k(u_{n\theta})(0)\ \phi(T_k(u_{n\theta})(0) - w_\mu^i(0))\ dx$$

we have

$$I_3(\theta) \geq \int_Q G_k(u_{n\theta})\ \phi'(T_k(u_{n\theta}) - w_\mu^i)(w_\mu^i)'\ dx\ dt$$
$$- \int_\Omega G_k(u_{n\theta}(0))\ \phi(T_k(u_{n\theta}(0)) - T_k(\psi_i))\ dx,$$
$$= \mu \int_Q G_k(u_{n\theta})\ \phi'(T_k(u_{n\theta}) - w_\mu^i)(T_k(u) - w_\mu^i)\ dx\ dt$$
$$- \int_\Omega G_k(u_{n\theta}(0))\ \phi(T_k(u_{n\theta}(0)) - T_k(\psi_i))dx,$$

which implies that

$$\limsup_{\theta\to\infty} I_3(\theta) \geq \mu \int_Q G_k(u_n)\ \phi'(T_k(u_n) - w_\mu^i)(T_k(u) - w_\mu^i)\ dx\ dt$$
$$- \int_\Omega G_k(u_0)\ \phi(T_k(u_0) - T_k(\psi_i))\ dx$$

as $n \to \infty$ hence

$$\limsup_{\theta\to\infty} I_3(\theta) \geq \mu \int_Q G_k(u)\ \phi'(T_k(u) - w_\mu^i)(T_k(u) - w_\mu^i)\ dx\ dt$$
$$- \int_\Omega G_k(u_0)\ \phi(T_k(u_0) - T_k(\psi_i))\ dx + \epsilon(n),$$

$$\geq \int_\Omega G_k(u_0)\ \phi(T_k(u_0) - T_k(\psi_i))\ dx \to 0 \text{ as } i \to \infty$$

where we have used (recall $|w^i_\mu| \leq k$)

$$\int_\Omega G_k(u)\ \phi'(T_k(u) - w^i_\mu)(T_k(u) - w^i_\mu)\ dx\ dt$$

$$= \int_{\{u>k\}} (u-k)\ \phi'(k - w^i_\mu)(k - w^i_\mu)\ dx\ dt$$

$$+ \int_{\{\ u<-k\}} (u+k)\ \phi'(-k - w^i_\mu)(-k - w^i_\mu)\ dx\ dt \geq 0$$

we deduce then that

$$\limsup_{\theta\to\infty}\ I_3(\theta) \geq \epsilon(n,i)$$

. combining these estimates, we get

$$\langle (\frac{\partial u_n}{\partial t}), \phi(T_k(u_n) - w^i_\mu)\rangle \geq \epsilon(n,i). \tag{3.27}$$

On the other hand, splitting the second term of the left hand side of (3.25) where $|u_n| \leq k$ and $|u_n|>k$, For $s>0$, set $Q = \{(x,t) \in Q : |\nabla T_k(u)| \geq s\}$ and $Q^s = \{(x,t) \in Q : |\nabla T_k(u)| \geq s\}$. We can write

$$\int_Q a(u_n, \nabla u_n)[\nabla T_k(u_n) - \nabla w^i_\mu]\phi^{'}(T_k(u_n) - w^i_\mu)\ dx\ dt \tag{3.28}$$

$$\geq \int_{\{|u_n|\leq k\}} a(T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u_n) - \nabla w^i_\mu]\ \phi^{'}(T_k(u_n) - w^i_\mu)\ dx\ dt$$

$$- C_k \int_{\{|u_n|>k\}} |a(u_n, \nabla u_n)||\nabla w^i_\mu|\ dx\ dt$$

$$= J_1 - C_k J_2$$

where $C_k = \phi^{'}(2K)$.

Now observe that

$$J_1 = \int_Q [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))]$$

$$\times\ [\ \nabla T_k(u_n) - \nabla T_k(u)\ ]\ \phi^{'}(T_k(u_n) - w^i_\mu)\ dx\ dt \tag{3.29}$$

$$+ \int_Q a(T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u) - \nabla w^i_\mu]\ \phi^{'}(T_k(u_n) - w^i_\mu)\ dx\ dt$$

$$+\int_Q a(T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]\ \phi^{'}(T_k(u_n)-w^i_\mu)\ dx\ dt$$

by the continuity of the Nymetskii operator, we have for all $i=1,....,N$ $a_i(T_k(u_n),\nabla T_k(u))\phi^{'}(T_k(u_n)-w^i_\mu)\to a_i(T_k(u),\nabla T_k(u))\phi^{'}(T_k(u)-w^i_\mu)$ strongly in $L^{p'}(Q,w_i^{1-p'})$ and since $\frac{\partial T_k(u_n)}{\partial x_i}\rightharpoonup\frac{\partial T_k(u)}{\partial x_i}$ weakly in $L^p(Q,w_i)$, the third term of the right hand side of (3.29) tends 0 as $n\to\infty$.

Thanks to (3.21) the second term of the right hand side of (3.29)tends to

$$\int_Q h_k[\nabla T_k(u)-\nabla w^i_\mu]\ \phi^{'}(T_k(u)-w^i_\mu)\ dx\ dt$$

so that, $\int_Q h_k[\nabla T_k(u)-\nabla w^i_\mu]\ \phi^{'}(T_k(u)-w^i_\mu)\ dx\ dt\to 0$ as $\mu\to\infty$
then we have,

$$J_1=\int_Q[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))]$$

$$\times[\nabla T_k(u_n)-\nabla T_k(u)]\ \phi^{'}(T_k(u_n)-w^i_\mu)\ dx\ dt+\epsilon(n,\mu)$$

and

$$J_2=\int_{|u_n|\geq k}|a(u_n,\nabla u_n)|\ |\nabla w^i_\mu|\ (\phi^{'}(T_k(u_n)-w^i_\mu))\ dx\ dt$$

$$\to\int_Q h|\nabla w^i_\mu|\ \phi^{'}(T_k(u)-w^i_\mu)\chi_{|u|>k}\ dx\ dt \text{ as } n\to\infty$$

$$\to\int_Q h|\nabla T_k(u)|\ \phi^{'}(0)\chi_{\{|u|>k\}}\ dx\ dt=0 \text{ as } \mu\to\infty.$$

Therefore (3.25) yieids

$$\int_Q a(u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla w^i_\mu]\ \phi^{'}(T_k(u_n)-w^i_\mu)\ dx\ dt \tag{3.30}$$

$$\geq\int_Q[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))]$$

$$\times[\ \nabla T_k(u_n)-\nabla T_k(u)\ ]\ \phi^{'}(T_k(u_n)-w^i_\mu)\ dx\ dt+\epsilon(n,\mu)$$

For the third term of the left hand side of (3.25)

$$\Big|\int_{|u_n|\leq k}g_n(x,t,u_n,\nabla u_n)\ \phi(T_k(u_n)-w^i_\mu)\ dx\ dt\Big| \tag{3.31}$$

$$\leq b(k)\int_Q\left(c(x,t)+\sum_{i=1}^{N}w_i\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p\right)|\phi(T_k(u_n)-w_\mu^i)|\;dx\;dt$$

$$\leq b(k)\int_Q c(x,t)\;|\phi(T_k(u_n)-w_\mu^i)|\;dx\;dt$$

$$+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\;\nabla T_k(u_n)\;|\phi(T_k(u_n)-w_\mu^i)|\;dx\;dt$$

since $c(x,t)$ belongs to $L^1(Q)$. Furthermore,

$$b(k)\int_Q c(x,t)\;|\phi(T_k(u_n)-w_\mu^i)|\;dx\;dt=\epsilon(n,\mu) \tag{3.32}$$

On the other hand, note that

$$\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\;\nabla T_k(u_n)\;|\phi(T_k(u_n)-w_\mu^i)|\;dx\;dt \tag{3.33}$$

$$=\frac{b(k)}{\alpha}\int_Q\Big[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))\Big]$$

$$\times[\nabla T_k(u_n)-\nabla T_k(u)]\;|\phi(T_k(u_n)-w_\mu^i)|\;dx\;dt$$

$$+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\;\nabla T_k(u)\;|\phi(T_k(u_n)-w_\mu^i)|\;dx\;dt$$

$$+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]\;|\phi(T_k(u_n)-w_\mu^i)|\;dx\;dt$$

By Lebesque is Theorem, we have

$$\nabla T_k(u)\;|\phi(T_k(u_n)-w_\mu^i)|\to\nabla T_k(u)\;|\phi(T_k(u)-w_\mu^i)|\text{ in }\prod_{i=1}^{N}L^p(Q,w_i).$$

Moreover, in view of (3.21) the second term of the right side of (3.33)tends to

$$\frac{b(k)}{\alpha}\int_Q\;h_k\;\nabla T_k(u)\;|\phi(T_k(u)-w_\mu^i)|\;dx\;dt$$

the third term of the right hand side of (3.33) tends to 0 since for all $i=1,....,N$

$$a_i(T_k(u_n),\nabla T_k(u))\;\phi(T_k(u_n)-w_\mu^i)\to a_i(T_k(u),\nabla T_k(u))\;\phi(T_k(u)-w_\mu^i)$$

strongly in $L^{p'}(Q,w_i^{1-p'})$, while

$$\frac{\partial(T_k(u_n))}{\partial x_i}\rightharpoonup\frac{\partial(T_k(u))}{\partial x_i}\text{ weakly in }L^p(Q,w_i).$$

From (3.31),(3.32) and (3.33), we obtain

$$\int_{\{|u_n|<k\}} g_n(x,t,u_n,\nabla u_n)\ \phi(T_k(u_n)-w_\mu^i)\ dx\ dt \tag{3.34}$$

$$\leq \frac{b(k)}{\alpha}\int_Q [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))]$$

$$\times[\nabla T_k(u_n)-\nabla T_k(u)]\ |\phi(T_k(u_n)-w_\mu^i)|\ dx\ dt + \epsilon(n,\mu).$$

By combining (3.25),(3.26),(3.27),(3.30)and (3.34)we get

$$\int_Q [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))] \times [\nabla T_k(u_n)-\nabla T_k(u)]$$

$$\times[\phi'(T_k(u_n)-w_\mu^i) - \frac{b(k)}{\alpha}\ \ |\phi(T_k(u_n)-w_\mu^i)|]\ dx\ dt \leq \epsilon(n,\mu,i).$$

and so, because (3.24)

$$\int_Q [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))] \times [\nabla T_k(u_n)-\nabla T_k(u)]dxdt$$

$$\leq 2\epsilon(n,\mu,i)$$

and by passing to the limit sup over n, we get

$$0 \leq \lim_{n\to\infty} sup \int_Q [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))] \times [\nabla T_k(u_n)-\nabla T_k(u)]\ dxdt$$

$$\leq \lim_{n\to\infty} 2\epsilon(n,\mu,i)$$

in which we let successively $\mu \to \infty$ and $i \to \infty$ to obtain

$$\int_Q [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))] \tag{3.35}$$

$$\times[\nabla T_k(u_n)-\nabla T_k(u)]\ dxdt \to 0 \tag{3.36}$$

as $n \to \infty$

which implies that by using Lemma 3.1

$$T_k(u_n) \to T_k(u)\ \ strongly\ \ in\ \ L^p(0,T,W_0^{1,p}(\Omega,w))\ \ \forall\ k \geq 0 \tag{3.37}$$

an thus, there exists a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u\ \ a.e.\ in\ Q. \tag{3.38}$$

We then deduce that, for all $k > 0$ $a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))$

and $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u)$ weakly in $\prod_{i=1}^{N} L^{p'}(Q,w_i)$.

Step 3:Equi-integrability of the nonlinearities

since g_n verifies the sign condition, then by (3.22) we deduce,

$$0 \leq \int_Q g_n(x,t,u_n,\nabla u_n)u_n \; dx \; dt \leq \gamma.$$

For any measururable subset E of Q and any $m > 0$, we have

$$\int_E |g_n(x,t,u_n,\nabla u_n)| \, dx \; dt = \int_{E \bigcap \chi_m^n} |g_n(x,t,u_n,\nabla u_n)| \, dx \; dt + \int_{E \bigcap Y_m^n} |g_n(x,t,u_n,\nabla u_n)| \, dx \; dt$$

where

$$\chi_m^n = \{(x,t) \in [0,T] \times \Omega, |u_n(x,t)| \leq m\},$$

and

$$Y_m^n = \{(x,t) \in Q, |u_n(x,t)| \geq m\}.$$

From these expressions,

$$\int_E |g_n(x,t,u_n,\nabla u_n)| \; dx \; dt \leq \int_{E \bigcap \chi_m^n} |g_n(x,t,T_m(u_n),\nabla T_m(u_n))| \; dx \; dt$$

$$+\frac{1}{m} \int_Q g_n(x,t,u_n,\nabla u_n)u_n \; dx \; dt$$

$$\leq b(m) \int_E \left(\sum_{i=1}^{N} w_i \left| \frac{\partial T_m(u_n)}{\partial x_i} \right|^p + \int_E c(x,t) \right) dxdt + \gamma \frac{1}{m}.$$

Since the sequence $((T_m)(u_n))$ converge strongly and fact that $c(x,t) \epsilon L^1(Q)$, there exists θ>0 such that

$$|E| < \theta \Rightarrow \int_E |g_n(x,t,u_n,\nabla u_n)| \; dx \; dt \leq \epsilon \; \forall \; n.$$

This shows that $g_n(x, t, u_n, \nabla u_n)$ is uniformly equi-integrable in Q as required.

Step 4: passage to the limit.

considering the approximate problem P_n one has:

$$-\int_Q u_n \frac{\partial \phi}{\partial t} \; dxdt + \int_Q a(x,t,u_n,\nabla u_n)\nabla \phi \; dxdt + \int_Q g_n(x,t,u_n,\nabla u_n)\phi \; dx \; dt = \langle f, \phi \rangle \tag{3.39}$$

for all $\phi \in D(Q)$, in which, we can easily pass to the limit as $n \to \infty$, to get

$$-\int_Q u\frac{\partial \phi}{\partial t}\, dxdt+\int_Q a(x,t,u,\nabla u)\nabla\phi\, dxdt+\int_Q g(x,t,u,\nabla u)\phi\, dxdt = \langle f,\phi\rangle. \quad (3.40)$$

Let now $(\phi \in C^1([0,T],L^2))(\Omega,\sigma))\bigcap L^\infty(Q)\bigcap L^p)(0,T,W_0^{1,P})(\Omega,w))$, there exists $(\phi_j) \subset D(Q)$ such that $\phi_j \to \phi$ in $\prod_{i=1}^{N} L^p(Q,w_i)$ and weak in $L^\infty(Q)$. Taking $\phi = \phi_j$ and letting $j \to \infty$, yields

$$-\int_Q u\frac{\partial \phi}{\partial t}\, dx\ dt + \left[\int_\Omega u(t)\phi(t)\ dx\right]_0^T + \int_Q a(x,t,u,\nabla u)\nabla\phi\ dx\ dt$$

$$+\int_Q g(x,t,u,\nabla u)\phi\ dx\ dt = \langle f,\phi\rangle$$

for all $\phi \in L^p(0,T,W_0^{1,P}(\Omega,w))\bigcap L^\infty(Q)\bigcap C^1([0,T],L^2(\Omega,\sigma))$.

Step 5: show that $u \in C^1([0,T],H)$.

As in [13], we have for all $\phi \in D(\Omega)$,by letting $\phi(x,t) = \phi(x) \in D(Q)$

$$\langle u_n',\phi\rangle + \int_Q a(x,t,u_n,\nabla u_n)\nabla\phi\ dx\ dt + \int_Q g(x,t,u_n,\nabla u_n)\ \phi\ dx\ dt = \int_Q f\ \phi\ dx\ dt$$

. This implies (see [13]), that $(u_n(t))$ is weakly convergent in H for all t and also that $u(t)$ is weakly continuous in H.

Let now $w_\mu^{i,k} = (T_k(u))\mu - e^{-\mu t}(T_k(\psi_i)$. On the one hand, we have for every $\tau \in [o,T]$

$$\langle (w_\mu^{i,k})', u_n - w_\mu^{i,k})\rangle_{Q_\tau} \to \mu\int_{Q_\tau}(T_k)(u) - w_\mu^{i,k}))(u - w_\mu^{i,k})dxdt \geq 0\ as\ n\to\infty \quad (3.41)$$

On the other hand, by using $(u_n - w_\mu^{i,k})$ as test function in (P_n), we can write,

$$\langle u_n', u_n - w_\mu^{i,k}\rangle = \langle f, u_n - w_\mu^{i,k}\rangle + \int_{Q_\tau} a(u_n,\nabla u_n)\nabla(w_\mu^{i,k} - u_n)\ dx\ dt$$

$$+\int_{Q_\tau} g(x,t,u_n,\nabla u_n)(w_\mu^{i,k} - u_n)\ dx\ dt$$

in which we can use Fatous Lemma and Lebesgue to pass to the limit sup first over n and μ, k, to get

$$\langle u_n', u_n - w_\mu^{i,k}\rangle_{Q_\tau} \leq \epsilon(n,\mu,k) \quad (3.42)$$

Therefore, by writing

$$\frac{1}{2}\|u_n(\tau) - w_\mu^{i,k}(\tau)\|_H^2 = \langle u_n' - (w_\mu^{i,k})', u_n - w_\mu^{i,k}\rangle_{Q_\tau} + \frac{1}{2}\|u_0 - T_k(\psi_i)\|_H^2$$

$$= \langle u_n', u_n - w_\mu^{i,k}\rangle_{Q_\tau} - \langle (w_\mu^{i,k})', u_n - w_\mu^{i,k}\rangle_{Q_\tau} + \frac{1}{2}\|u_0 - T_k(\psi_i)\|_H^2$$

and observing

$$0 \leq \frac{1}{2}\|u(\tau) - w_\mu^{i,k}(\tau)\|_H^2 = \lim_{n\to\infty} \frac{1}{2}\|u_n(\tau) - w_\mu^{i,k}(\tau)\|_H^2$$

we deduce that, in view of (3.41) and (3.42), $\|u(\tau) - w_\mu^{i,k}(\tau)\|_H^2 \leq \epsilon(\mu, k, i)$ not depending on $\tau \in [0, T]$. Implying that $w_\mu^{i,k}$ is a Cauchy sequence in $C([0, T], H)$ converging to u and thus $u \in C([0, T], H)$.
and using (3.41) and (3.43), we deduce that $\|u_n(\tau) - w_\mu^{i,k}(\tau)\|_{L^2(\Omega)} \leq \epsilon(n, k, \mu, i)$ not depending on $\tau \in [0, T]$. This implies that

$$\|u_n(\tau) - u_m(\tau)\|_{L^2(\Omega)} \leq \epsilon(n, m) \text{ not depending on } \tau \in [0, T],$$

and thus, u_n is a Cauchy sequence in $C([0, T], L^2(\Omega))$.
Since the limit of u_n in $L^1(Q)$ is u we deduce that $u_n \to u$ in $C([0, T], \Omega)$, therefore, by letting $n \to \infty$ in the first term of (3.39), we have

$$\left[\int_\Omega u_n(t)\ w_j(t)\ dx\right]_0^\tau \to \left[\int_\Omega u(t)\ w_j(t)\ dx\right]_0^\tau.$$

Consequently, by letting $n \to \infty$ in (3.39),we get

$$\left[\int_\Omega u(t)\ w_j(t)\ dx\right]_0^\tau - \int_{Q_\tau} u\ \frac{\partial w_j}{\partial t}\ dx\ dt + \int_{Q_\tau} a(x, t, u, \nabla u)\nabla w_j\ dx\ dt$$
$$+ \int_{Q_\tau} g(x, t, u, \nabla u) w_j\ dx\ dt = \langle f, w_j\rangle_{Q_\tau}. \qquad (3.43)$$

we shall now go to the limit as $j \to \infty$ in all terms of (3.43). there is problem to pass to the limit in last four terms of (3.43). we have $w_j \to v$ in $C([0, T], L^2(\Omega))$. Therefore, we can let $j \to \infty$ in all terms of (3.43) to get

$$\left[\int_\Omega u(t)\ v(t)\ dx\right]_0^\tau - \langle\ \frac{\partial v}{\partial t}, u\rangle_{Q_\tau} + \int_{Q_\tau} a(x, t, u, \nabla u)\nabla v\ dx\ dt$$
$$+ \int_{Q_\tau} g(x, t, u, \nabla u)v\ dx\ dt = \langle f, v\rangle_{Q_\tau},$$

which shows that u satisfies all properties of Theorem 3.5. It only remains to prove the energy equality. For that, we use, for a given $k>0, T_k(u_n)$, to get

$\langle u_n', T_k(u_n)\rangle_{Q_\tau} =$

$$-\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\,dx\,dt - \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)\,T_k(u_n)\,dx\,dt + \langle f, T_k(u_n)\rangle_{Q_\tau},$$

which gives by setting $S_k(s) = \int_0^s T_k(z)dz$,

$$\int_\Omega S_k(u_n(\tau))\,dx - \int_\Omega S_k(u_0)\,dx =$$

$$-\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\,dx\,dt - \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_k(u_n)\,dx\,dt + \langle f, T_k(u_n)\rangle_{Q_\tau}, \tag{3.44}$$

Recall that $|S_k(u_n(\tau))| \le k\ |u_n(\tau)| \to k\ |u(\tau)|$ in $L^2(\Omega)$ as $n \to \infty$, then, we can pass to the limit as $n \to \infty$ each term of (3.44) to obtain

$$\int_\Omega S_k(u\tau)\,dx - \int_\Omega S_k(u_0)\,dx =$$

$$-\int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u)\,dx\,dt - \int_{Q_\tau} g(x,t,u,\nabla u)T_k(u)\,dx\,dt + \langle f, T_k(u)\rangle_{Q_\tau}. \tag{3.45}$$

Observe that for every $s \in R$,

$$|S_k(s)| \le \frac{s^2}{2}$$

and $S_k(s) \to \frac{s^2}{2}$ as $k \to \infty$, so that, by using Lebesgue theorem and the fact that $u(\tau) \in L^2(\Omega)$,we have, as $k \to \infty$,

$$\int_\Omega S_k(u(\tau))\,dx \to \frac{1}{2}\int_\Omega u^2(\tau) \text{ and } \int_\Omega S_k(u_0)\,dx \to \frac{1}{2}\int_\Omega S_k(u_0)^2\,dx.$$

Remark also that

$$|a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u))| \le a(x,t,u,\nabla u)\nabla u \in L^1(Q)$$

and

$$|g(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u))| \le g(x,t,u,\nabla u)\nabla u \in L^1(Q),$$

therefore, it is easy to pass the limit as $k \to \infty$ in (3.45) to get the energy equality

$$\left[\frac{1}{2}\int_\Omega u(t)^2\,dx\right]_0^\tau + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla u\,dx\,dt + \int_{Q_\tau} g(x,t,u,\nabla u)u\,dx\,dt = \langle f, u\rangle_{Q_\tau}.$$

This completes the proof of Theorem 3.5.

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