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1-Convex Transferable Utility Games
A Reappraisal

Pierre Dehez

Center for Operations Research and Econometrics (CORE),
University of Louvain
34 Voie du Roman Pays, 1348 Louvain-la-Neuve, Belgium
pierre.dehez@uclouvain.be

Abstract

1-convex games have been introduced by Theo Driessen in his 1985 PhD dissertation. They form an interesting class of games for at least one reason: the core of a 1-convex n-player game is a regular simplex of dimension $n - 1$ or a single point. As a consequence, its nucleolus is the center of gravity of the core. We recall and extend the results obtained by Driessen and provide examples and applications.

Keywords: transferable utility games, core, nucleolus, Shapley value

1. Introduction

1-convex transferable utility games were introduced in 1985 by Theo Driessen in his PhD dissertation.[1] They form a class of balanced games whose core is a full-dimensional regular simplex or a single point. Driessen has published several papers developing further the concept of 1-convex game and providing applications.[2] One remarkable consequence of the simple geometric structure of the core of a 1-convex game is that the nucleolus coincides with its center of gravity, simplifying considerably its computation.

In the present paper, we recall the major propositions proved by Driessen and add some new results. Most of these new results are related to the duals of 1-convex games that are games in which forming a coalition other than the grand coalition is not rational from an individual point of view: players may gain only by getting all together. Alongside, it explains why only the marginal contributions to the grand coalition enter into the definition of the core of a 1-convex game.

The paper is organized as follows. Basic definitions applying to transferable utility games are recalled in Section 2. It is followed by a section devoted to solution concepts. Section 4 and 5 are devoted to the definition of 1-convex games and their properties. The relation between 1-convexity and convexity is the object of Section 6 while Section 7 is devoted to the dual of a 1-convex games. The core, the nucleolus and the Shapley value of 1-convex games are the subject of Sections 7 and 8. The last section reviews a number of applications.

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[1] "*Contributions to the theory of games: the* τ*-value and k-convex games*", University of Nijmegen.
[2] Driessen has actually developed the concept of *k-convex game*. Here, we limit ourselves to the most interesting case, namely the case where $k = 1$.

2. Transferable Utility Games

Given a set of players $N = \{1,\ldots,n\}$, $n \geq 2$, a *transferable utility game* is defined by a set function v, called *characteristic function*, that associates real numbers to subsets: $v(S)$ is the "worth" of coalition S. It could be a gain or a loss that measures what coalition S *could* or *should* obtain if it decided to form. As pointed out by Aumann (2010), there are two points of view: "could" applies to transferable utility games that are derived from games in strategic form, while "should" relies on some given criterion. By convention, $v(\emptyset) = 0$.

Notation. Finite sets are denoted by upper-case letters. Lower-case letters are used to denote their cardinals: $t = |T|$, $s = |S|,\ldots$ For a vector x, $x(S)$ denotes the sum of its coordinates over the subset S. By convention, a sum over an empty set is zero. Inclusion and strict inclusion are denoted by $\subset$ and $\subsetneq$ respectively. Braces are sometimes omitted for coalitions: $v(\{i,j,k,\ldots\})$ is written $v(i,j,k)$ instead. The characteristic function can be written as a list of numbers. Games with 3 or 4 players will be written in the following compact and self-explanatory form:

v = (1, 2, 3 | 12, 13, 23 | 123) for $n = 3$,

v = (1, 2, 3, 4 | 12, 13, 14, 23, 24, 34 | 123, 124, 134, 234 | 1234) for $n = 4$.

The theory of transferable utility games is concerned with the division of the payoff that results from the joint and coordinated actions of all players. The set of allocations associated to a game (N,v) is given by: $X(N,v) = \{x \in \mathbb{R}^n \mid x(N) = v(N)\}$.

We denote by $G(N)$ the set of all characteristic functions on a set of player N. It can be identified to the vector space $\mathbb{R}^{2^n - 1}$. It is assumed that there is no loss in forming the grand coalition:

$$\sum_{i \in N} v(i) \leq v(N), \qquad (1)$$

Given (1), it is possible to allocate the total value of the game in such a way that no one *receives less* than his stand-alone utility. Games for which (1) holds with a strict inequality are said to be *essential*.

A game (N,v) is *additive* if $v(S) = \sum_{i \in S} v(i)$ for all $S \subset N$.

Additive games are inessential. *Quasi-additive games* are games that satisfy the above equalities for all $S \neq N$.

A game (N,v) is *symmetric* if there exists a real-valued function f such that $f(0) = 0$ and $v(S) = f(s)$ for all $S \subset N$.

Two games (N,v) and (N,w) on a common set of player are *strategically equivalent* if there exists a real number $\alpha > 0$ and a vector $\beta \in \mathbb{R}^n$ such that $w(S) = \alpha v(S) + \beta(S)$. Strategic equivalence defines an *equivalence relation*.

There is no loss of generality in considering games where individual worth's are set to zero. The 0-*normalization* of a game (N,v) is the game (N,v_0) defined by:

$$v_0(S) = v(S) - \sum_{i\in S} v(i) \quad \text{for all } S \subset N.$$

A game (N,v) and its 0-normalization (N,v_0) are strategically equivalent.

A game (N,v) is *monotonic* if $S \subset T \Rightarrow v(S) \le v(T)$. It is *0-monotonic* if the 0-normalized game (N,v_0) is monotonic.

Superadditivity is commonly assumed. A game (N,v) is *superadditive* if $v(S)+v(T) \le v(S \cup T)$ for all pairs of disjoint subsets $S,T \subset N$: splitting a coalition cannot be beneficial.

Proposition 1 A superadditive game (N,v) is monotonic if $v(i) \ge 0$ for all $i \in N$.

Proof Consider a coalition *S* and a player $i \notin S$. By superadditivity, we have:

$$v(S \cup i) \ge v(S) + v(i) \ge v(S).$$

Adding another player *j*, we have $v(S \cup i \cup j) \ge v(S \cup i) + v(j) \ge v(S \cup i) \ge v(S)$, and so on. ♦

Weak superadditivity imposes that there is no gain in partitioning the set of players:

$$\sum_{h=1}^{m} v(S_h) \le v(N) \text{ for all partitions } (S_1,\dots,S_m) \text{ of } N.$$

Proposition 2 A game is weakly superadditive *if and only if* it is 0-monotonic.

Proof If (N,v_0) is weakly superadditivity, we have:

$$v_0(S \cup i) - v_0(S) = v(S \cup i) - v(S) + v(i) \ge 0.$$

0-monotonicity follows. If the game (N,v) is 0-monotonic, $v_0(S \cup i) \ge v_0(S)$ for all coalition *S* not containing player *i*. ♦

As a consequence, superadditivity implies 0-monotonicity.

Convexity is instead a stronger assumption. A game (N,v) is *convex* if

$$v(S)+v(T) \le v(S \cup T) - v(S \cap T) \text{ for all subsets } S, T \subset N.$$

In a game (N,v), the *marginal contribution* of player *i* to a coalition *S* is defined by $v(S) - v(S \setminus i)$. Convexity can be characterized in terms of marginal contributions. A game (N,v) is convex *if and only if* marginal contributions are *non-decreasing* with coalition size:

$$v(S) - v(S \setminus i) \le v(T) - v(T \setminus i) \text{ for all } i, S \text{ and } T \text{ such that } i \in S \subset T.$$

A game (N,v) is (weakly) *subadditive* (resp. *concave*) if the game $(N,-v)$ is (weakly) superadditive (resp. convex): inequalities are simply reversed. In the case of a concave game, marginal contributions are *non-increasing* with coalition size.

If Π_N denotes the set of permutations of the players in *N*, the *marginal contributions vector* associated to the permutation $\pi = (i_1, i_2, ..., i_n) \in \Pi_N$ is the vector $\mu^\pi(N,v)$ defined by:

$$\mu^\pi_{i_k}(N,v) = v(i_1,...,i_k) - v(i_1,...,i_{k-1}) \quad (i = 1,\ldots,n).$$

In particular, $\mu^\pi_{i_1}(N,v) = v(i_1)$ and $\mu^\pi_{i_n}(N,v) = v(N) - v(N \setminus i_n)$. There are $n!$ marginal contribution vectors, not necessarily distinct. By construction, marginal contributions vectors are allocations: $\mu^\pi(N,v) \in X(N,v)$ for all $\pi \in \Pi_N$.

The *dual* (N, v^d) of a game (N,v) is defined by $v^d(S) = v(N) - v(N \setminus S)$, for all $S \subset N$ where the difference $v(N) - v(N \setminus S)$ is the contribution of coalition *S* to the grand coalition.

Simple games are superadditive games (N,v) such that $v(S) \in \{0,1\}$ and $v(N) = 1$. In a simple game, a coalition *S* is *winning* if $v(S) = 1$ and a player is *veto* is he is member of *all* wining coalitions: the set of veto players coincides with the intersection of all wining coalitions. It is easily seen that, by superadditivity, player *i* is veto in the simple game (N,v) *if and only if* $v(N \setminus i) = 0$.

Consider a situation where the characteristic function associates a *cost* $c(S)$ to coalition *S*, instead of a gain. This define a *cost-sharing game* (N,c) that can be analyzed through the corresponding surplus-sharing game (N,v) defined by:

$$v(S) = \sum_{i \in S} c(i) - c(S) \text{ for all } S \subset N. \tag{2}$$

Indeed, $v(S)$ measures the gain that coalition *S* would make if it decided to form. It is equivalent to the game $(N, -c_0)$. An allocation *x* of the total surplus $v(N)$ and the associated allocation *y* of the total cost $c(N)$ are related by the equations $x_i + y_i = c(i)$, $i = 1,\ldots,n$.

A cost-sharing game is (weakly) *subadditive* (resp. *concave*) if the associated surplus sharing game is (weakly) superadditive (resp. convex).

3. Solution Concepts: the Core, the Nucleolus and the Shapley Value

Two conditions must be imposed on allocations. *Efficiency* requires that the entire value of a game is distributed to the player. *Individual rationality* requires that players should obtain at least their individual worth. Together, these two requirements define *imputations*. For a given game (N,v), the *imputation set* is defined by $I(N,v) = \{x \in X(N,v) \mid x_i \geq v(i) \text{ for all } i \in N\}$.

(1) ensures that the imputation set is nonempty and it reduces to the status quo $(v(1),...,v(n))$ if the game is inessential. Geometrically, the imputation set of an essential game is a *regular simplex* of dimension $n - 1$.

The core extends individual rationality to coalitions. A coalition may indeed raise an objection against an allocation that gives to the coalition as a whole less than its worth. The *core* of a game (N,v) is then defined by:

$$C(N,v)=\left\{x\in\mathbb{R}^n \,\middle|\, x(N)=v(N) \text{ and } x(S)\geq v(S) \text{ for all } S\subset N\right\}.$$

This definition applies to surplus-sharing situations where (1) holds. In a cost-sharing situations, the notion of anticore is used:

$$AC(N,c)=\{x\in\mathbb{R}^n \mid x(N)=c(N) \text{ and } x(S)\leq c(S) \text{ for all } S\subset N\}.$$

The core of a game can be viewed equivalently as the set of allocations that give to each coalition *not more* than its contribution to the grand coalition:

$$x(S)\geq v(S) \quad\Leftrightarrow\quad x(S)\leq v(N)-v(N\setminus S)=v^d(S) \quad \text{for all } x\in X(N,v).$$

Hence, the core of a game coincides with the anticore of its dual:

$$C(N,v)=\{x\in\mathbb{R}^n \mid x(N)=v(N) \text{ and } x(S)\leq v^d(S) \text{ for all } S\subset N\}=AC(N,v^d).$$

Given an allocation *x*, the *excess* (or loss) of a coalition *S* is defined by $e(x,S)=v(S)-x(S)$, the difference between its worth and what it gets. The *ε-core* introduced by Shapley and Shubik (1966) allows for allocations that induce an excess that does not exceed ε. The *least-core* introduced by Maschler, Peleg and Shapley (1979) is the set of allocations that minimize the largest excess:

$$Min_{x\in X(N,v)} Max_{\substack{S\subset N\\ S\neq\emptyset,N}} e(x,S)=\varepsilon_{min}.$$

The *nucleolus* introduced by Schmeidler (1969) goes beyond the least-core by comparing the excesses lexicographically to eventually retain a *single* allocation, an allocation that belongs to the core if nonempty.

Like the nucleolus, the *Shapley value* introduced in Shapley (1953) is an allocation rule. It is given by the average marginal contribution vector, taking into account all *n*! vectors (Shapley, 1971):[3]

$$SV_i(N,v)=\frac{1}{n!}\sum_{\pi\in\Pi_N}\mu_i^\pi(N,v) \quad (i=1,...,n).$$

4. 1-Convex Games

For a given game (N,v), we denote the *marginal contribution* of player *i* to the grand coalition by $b_i(N,v)=v(N)-v(N\setminus i)$. They define the *upper-vector* $b(N,v)$. The *gap function* g_v associated to a game (N,v) is defined by:

$$g_v(S)=\sum_{i\in S}b_i(N,v)-v(S) \quad \text{for all } S\subset N.$$

It measures the gain (or loss if negative) of a coalition for being assigned its members' marginal contributions to the grand coalition.

[3] Alternatively, the Shapley value allocates to each player a weighted average of his marginal contributions.

Remark 1 Notice that $g_v(\varnothing)=0$: the gap function defines a game (N,g_v). Furthermore, the gaps associated to coalitions of size $n-1$ coincide with the gaps associated to the grand coalition: $g_v(N\setminus i)=\sum_{j\in S} b_j(N,v)-v(N\setminus i)=\sum_{j\in N} b_j(N,v)-v(N)=g_v(N).$

A game (N,v) is 1-*convex* if its gaps are nonnegative and attain a *minimum* at the grand coalition:

$$0\le g_v(N)\le g_v(S) \text{ for all } S\subset N,\ S\ne\varnothing \tag{3}$$

i.e.

$$\sum_{i\in S} b_i(N,v)\ge v(S) \text{ for all } S\subset N,\ S\ne\varnothing, \tag{4}$$

$$v(S)\le v(N)-\sum_{i\in N\setminus S} b_i(N,v) \text{ for all } S\subset N, S\ne\varnothing. \tag{5}$$

In particular, $g_v(i)=b_i(N,v)-v(i)\ge 0$ implies $v(i)+v(N\setminus i)\le v(N)$. Notice that (5) is an identity for the coalitions of sizes n and $n-1$.

Example 1 The 3-player game given by v = (0, 0, 0 | 21, 25, 28 | 41) is 1-convex: its gap function g_v = (13, 16, 20 | 8, 8, 8 | 8) verifies (3).

For a simple game (N,v), $b_i(N,v)=1$ *if and only if* player i is a veto player. The following proposition is a corollary of a proposition of Driessen (1988, Lemma 7.2, p.79).

Proposition 3 A simple game is 1-convex *if and only if* there is a single veto player.

Proof Indeed, in the absence of veto players, $b_i(N,v)=0$ for all $i\in N$ and, as a consequence, $g_v(N)=-1$: the game is not 1-convex. If 1 is the unique veto player, we have $b_1(N,v)=1$ and $b_i(N,v)=0$ for all $i\ne 1$, implying $g_v(S)=g_v(N)=0$ for all $S\subset N$: the game is 1-convex. If there are more than one veto player, say k, we have $g_v(N)=k-1>0$ and $g_v(S)=0$ for all S not containing the k veto players: the game is not 1-convex. ♦

A symmetric game (N,v) where $v(S)=f(s)$ is 1-convex *if and only if*:

$$f(n)-f(s)\ge(n-s)\Delta_n(f) \text{ and } f(n)\le n\Delta_n(f) \text{ for all } s,\ 0<s<n,$$

where $\Delta_n(f)=f(n)-f(n-1)$.

A game that is *strategically equivalent* to a 1-convex games is 1-convex: if the games (N,v) and (N,w) are related by the equations $w(S)=\alpha v(S)+\beta(S)$, we have:

$$b_i(N,w)=\alpha b_i(N,v)+\beta_i \text{ and } g_w(S)=\alpha\ g_v(S).$$

In particular, the 0-normalization of a 1-convex game produces a 1-convex game: the two gap functions coincide. Furthermore, adding two 1-convex game on a common player set results in a 1-convex game. Hence, the set of 1-convex games is a *convex cone* within $G(N)$.

A game (N,v) is 1-*concave* if the game $(N,-v)$ is 1-convex: $g_v(S) \le g_v(N) \le 0$ for all $S \subset N$, $S \ne \emptyset$. If the game (N,v) is 1-convex, the game (N,g_v) defined by its gap function is 1-concave. Indeed, $b_i(N,g_v) = g_v(N) - g_v(N \setminus i) = 0$ for all $i \in N$ and, as a consequence, $g_{g_v}(S) = -g_v(S)$ for all $S \subset N$.

Considering a cost-sharing game (N,c) and the associated surplus-sharing game (N,v), we observe that the two gap functions add-up to zero: $g_v(S) + g_c(S) = 0$ for all $S \subset N$. This follows from the fact that the surplus-sharing game (N,v) associated to the cost-sharing game (N,c) defined in (2) can be written as $v(S) = -c_0(S)$.

5. Superadditivity and 1-convexity

1-convex games are not necessarily superadditive. They are however *weakly* superadditive.

Proposition 4 1-convex games are weakly superadditive.

Proof Consider a 1-convex game (N,v) and assume that the reversed inequality holds for some nontrivial partition $(S_1,...,S_m)$ of *N*. Then an impossibility results if $g(N) \ge 0$:

$$g_v(N) = \sum_{i\in N} b_i(N,v) - v(N) > \sum_{h=1}^{m}\left(\sum_{i\in S_h} b_i(N,v) - v(S_h)\right) = \sum_{h=1}^{m} g_v(S_h) \ge m\, g_v(N). \quad \blacklozenge$$

Obviously, 2-player 1-convex games are superadditive. This is also true for 3-player games, and for 4-player games under a restriction.

Proposition 5 1-convex 3-player games are superadditive.

Proof Consider a 3-player 1-convex game (N,v). In view of Proposition 3, we only have to prove that $v(i) + v(j) \le v(i,j)$ for all $i \ne j$. Applying (5) to the coalitions {1} and {2}, we get:

$$v(1) + v(2) \le 2v(N) - b_1(N,v) - b_2(N,v) - 2b_3(N,v) = v(N) - \sum_{i=1}^{3} b_i(N,v) + v(1,2).$$

Superadditivity then follows from (4) applied to *N*. ♦

Proposition 6 1-convex 4-player games (N,v) are superadditive if the characteristic function of their 0-normalization takes *non-negative values*.

Proof Let (N,v) be a 4-player 1-convex game. In terms of the normalized game (N,v_0), we have $v_0(i) + v_0(j) = 0 \le v_0(i,j)$ for all $i \ne j$. Given what we know from Proposition 3, we only have to prove superadditivity for pairs of disjoint coalitions of the form {1,2}-{3}. The normalized game being 1-convex, we can apply (5) to coalition {1,2}:

$$v(1,2) \le v(N) - b_3(N,v) - b_4(N,v) = v(1,2,3) + v(1,2,4) - v(N)$$

or

$$v(1,2,3) - v(1,2) \ge v(N) - v(1,2,4) = b_3(N,v)$$

where $b_3(N,v) \geq v(3) = 0$ by (4). The 0-normalized game is therefore superadditive and so is the original game (N,v). ♦

The 4-player game defined by $v(S) = s$ except $v(1,2) = 1$ is 1-convex but not superadditive. Its gaps are given by $g(S) = 0$ for all $S \neq \{1,2\}$ and $g(1,2) = 1$. The 5-player game defined by $v(i) = 0$ for all i, $v(S) = s$ for all S of sizes 2 or more, except $v(1,2,3) = 1$, is 1-convex but fails to be superadditive.

Remark 2 It is easily verified that superadditivity of the game (N,v) implies subadditivity of the game (N,g_v), and vice-versa.

6. Convexity and 1-Convexity

Convexity of a game can be assessed in terms of the game defined by its gap function.

Lemma 1 (Driessen, 1988, p.112) A game (N,v) is convex *if and only if* the game (N,g_v) is concave.

Indeed, the marginal contributions of the game (N,v) are related to the marginal contributions of the game (N,g_v) by the equations $g_v(S) - g_v(S \setminus i) = b_i(N,v) - \left(v(S) - v(S \setminus i)\right)$. Moreover, the right-hand side being non-negative by convexity, the gap function of a convex game is non-decreasing in terms of set inclusion. The following proposition then follows from (3).

Proposition 7 (Driessen, 1988, p.182) A 1-convex game is convex *if and only if* its gap function is constant: $g_v(S) = g_v(N) \geq 0$ for all $S \subset N$, $S \neq \varnothing$.

Proof Consider a 1-convex game (N,v) and assume that there exists a real $a \geq 0$ such that:

$$\sum_{i \in S} b_i(N,v) - v(S) = a \quad \text{for all } S \subset N.$$

Furthermore, we have:

$$v(S) + v(T) = \sum_{i \in S \cup T} b_i(N,v) + \sum_{i \in S \cap T} b_i(N,v) - 2a = v(S \cup T) + v(S \cap T) \text{ if } S \cap T \neq \varnothing,$$

and

$$v(S) + v(T) = \sum_{i \in S \cup T} b_i(N,v) - 2a \leq \sum_{i \in S \cup T} b_i(N,v) - a = v(S \cup T) \text{ if } S \cap T = \varnothing,$$

for all $S,T \subset N$. Convexity then follows. Now, assume that the game (N,v) is both convex and 1-convex. Then, for all $S \subset N$, we have $g_v(S) \geq g_v(N) \geq 0$ by (3) and $g_v(S) \leq g_v(N)$ by Lemma 1. Gaps are therefore constant and non-negative. ♦

Remark 3 For games with constant gaps, convexity is *equivalent* to 1-convexity. Indeed, if the gaps are equal to some arbitrary real a, 1-convexity is equivalent to $a \geq 0$ and convexity follows then from the above proof. On the other hand, if the game (N,v) is convex, it is also superadditive. Hence,

$g_v(i) = b_i(N,v) - v(i) = v(N) - v(N \setminus i) - v(i) \geq 0.$ As a consequence, $a \geq 0$ and 1-convexity follows. In terms of marginal contributions, if gaps are constant and non-negative, we have:

$$v(S) - v(S \setminus i) = \sum_{j \in S} b_j(N,v) - \sum_{j \in S \setminus i} b_j(N,v) = b_i(N,v) \quad \text{for all } S \neq \{i\},$$

and

$$v(i) = b_i(N,v) - a \leq b_i(N,v).$$

Marginal contributions are increasing and convexity then follows. We observe that the marginal contributions of a player to a nonempty coalition are constant.

Additive games are convex *and* 1-convex: marginal contributions are constant and the gap function is identically zero. Quasi-additive games involving more than two players and verifying (1) are convex but not 1-convex. Indeed, if (N,v) is a quasi-additive game verifying (1), we have:

$$v(S) - v(S \setminus i) = v(i) \text{ for all } S \subsetneq N \quad \text{and} \quad v(N) - v(N \setminus i) = v(N) - \sum_{j \neq i} v(j) \geq v(i),$$

and

$$g_v(N) = (n-1)\left(v(N) - \sum_{i \in N} v(i) \right) \geq s \left(v(N) - \sum_{i \in N} v(i) \right) = g_v(S) \geq 0 \text{ for all } S \subsetneq N.$$

By Proposition 3, simple games are 1-convex if (and only if) there is a single veto player. These games are actually convex as well: the gap function is identically zero.

7. Dual of a 1-Convex Game

How do properties of a game translate into its dual? Superadditivity or subadditivity, even weak, do *not* translate into similar properties of the dual of a game, as the following examples show. The game v = (1, 2, 3 | 5, 7, 9 | 10) is superadditive while its dual v^d = (1, 3, 5 | 7, 8, 9 | 10) is neither subadditive nor superadditive. The game v = (2, 3, 3 | 3, 5, 6 | 6) is subadditive while its dual v^d = (0, 1, 3, | 3, 3, 4 | 6) is neither subadditive nor superadditive. It goes differently for convexity or concavity.

Proposition 8 The dual of a convex game is a concave game (and vice versa).

Proof Consider a convex game (N,v) and its dual (N,v^d). Given two arbitrary subsets S and T, we have successively:

$$v^d(S) + v^d(T) = 2v(N) - v(N \setminus S) - v(N \setminus T)$$

and

$$v(N \setminus S) + v(N \setminus T) \leq v((N \setminus S) \cup (N \setminus T)) + v((N \setminus S) \cap (N \setminus T))$$

where $(N \setminus S) \cup (N \setminus T) = N \setminus (S \cap T)$ and $(N \setminus S) \cap (N \setminus T) = N \setminus (S \cup T)$. Concavity of the dual game then follows:

$$v^d(S) + v^d(T) \geq 2v(N) - v(N \setminus (S \cup T)) - v(N \setminus (S \cap T)) = v^d(S \cup T) + v^d(S \cap T). \quad \blacklozenge$$

The dual of a 1-convex game is generally not 1-concave (nor vice versa), as the following examples show.

Example 1 (continued) The dual of the 1-convex game $v = (0, 0, 0 \mid 21, 25, 28 \mid 41)$ is given by $v^d = (13, 16, 20 \mid 41, 41, 41 \mid 41)$. Its gaps satisfy the inequalities $g_{v^d}(N) \leq g_{v^d}(S) \leq 0$ for all $S \subset N$.

Example 2 The *unanimity game* is the simple game (N,u) defined by $u(S) = 1$ *if and only if* $S = N$. It is not 1-convex. Indeed, its gap function takes nonnegative values and attains a *maximum* at the grand coalition. All coalitions have worth 1 in the dual and the gaps are all equal to –1: the game (N,u^d) is 1-concave.

1-convexity of a game can be characterized in terms of its dual.

Proposition 9 A game (N,v) is 1-convex *if and only if* its dual verifies the following inequalities:

$$\sum_{i \in S} v^d(i) \leq v^d(S) \text{ for all } S \neq N \text{ and } \sum_{i \in N} v^d(i) \geq v^d(N). \tag{6}$$

Proof The gap function of the game (N,v) is given by:

$$g_v(S) = \sum_{i \in S} b_i(N,v) - v(S) = \sum_{i \in S} v^d(i) - v^d(N) + v^d(N \setminus S).$$

We then have successively:

$$g_v(N) = \sum_{i \in N} v^d(i) - v^d(N)$$

and

$$g_v(N) - g_v(S) = \sum_{i \in N \setminus S} v^d(i) - v^d(N \setminus S) \text{ for all } S \subsetneq N. \quad \blacklozenge$$

Example 3 The game $v = (4, 2, 3, 6 \mid 7, 9, 10, 8, 10, 9 \mid 12, 12, 13, 12 \mid 13)$ satisfies (6). Its dual is the super-additive 1-convex game $v^d = (1, 0, 1, 1 \mid 4, 3, 5, 3, 4, 6 \mid 7, 10, 11, 9 \mid 13)$. The gap function of the game (N,v) satisfies the reversed inequalities $g_v(N) \leq g_v(S) \leq 0$ for all $S \subset N$.

Notice that games that satisfy (6) are weakly *subadditive*. Indeed, for all nontrivial partitions $(S_1, ..., S_m)$ of N, we have:

$$\sum_{h=1}^{m} v(S_h) \geq \sum_{h=1}^{m} \sum_{i \in S_h} v(i) = \sum_{i \in N} v(i) \geq v(N).$$

Remark 4 Proposition 9 applies to 1-concavity as well: a game is *if and only if* its dual verifies the opposite inequalities (6).

Together with Proposition 9, Proposition 7 and Remark 3 imply that a game is 1-convex *and* convex *if and only if* its dual is a quasi-additive game. Indeed, we have:

$$g_v(S) = g_v(N) \Rightarrow \sum_{i \in S} v^d(i) = v^d(S) \text{ for all } S \neq N.$$

8. The Core of 1-Convex Games

The core of a n-player 1-convex game can be given a simple characterization that involves only $n+1$ constraints instead than 2^n–1.

Lemma 2 (Driessen, 1988, p.172) Given a game (N,v) verifying (1), an allocation x belongs to $C(N,v)$ *if and only if*

$$x_i \le b_i(N,v) \text{ for all } i \in N, \tag{7}$$

and

$$x(S) \ge v(S) \text{ for all } S \ne \varnothing \text{ such that } g_v(S) < g_v(N). \tag{8}$$

By (8), the core of a 1-convex game is entirely determined by its upper-vectors and the following proposition follows.

Proposition 10 (Driessen, 1988, p.73) The core of a 1-convex games is defined by:

$$C(N,v) = \left\{ x \in \mathbb{R}^n \mid x(N) = v(N),\ x_i \le b_i(N,v) \text{ for all } i \in N \right\}.$$

Proof Consider a 1-convex game (N,v). Assuming that $x_i \le b_i(N,v)$ for all $i \in N$ and using (5) we have $v(S) \le v(N) - b(N \setminus S) \le v(N) - x(N \setminus S) = x(S)$. ♦

Notice that the non-negativity of the gaps follows from the nonemptiness of the core. Indeed, (7) implies $v(S) \le x(S) \le \sum_{i \in S} b_i(N,v)$.

By Proposition 10, the core of a 1-convex game has a simple and regular geometric structure.

Proposition 11 The core of a 1-convex game (N,v) is a *regular simplex* of dimension $n-1$ if $g_v(N) > 0$. In the case where $g_v(N) = 0$, it reduces to the single point $b(N,v)$.

Proof Consider a 1-convex game (N,v). We have $C(N,v) = b(N,v) - Z$ where Z is given by:

$$Z = \{ y \in \mathbb{R}^n_+ \mid y(N) = \sum_{i \in N} b_i(N,v) - v(N) = g_v(N) \}. \tag{10}$$

Z is a full-dimensional simplex if $g_v(N) > 0$ while it coincides with the {0} if $g_v(N) = 0$. ♦

Remark 5 It follows from Proposition 11 that the core of a 1-convex game (N,v) reduces to the singleton $\{b(N,v)\}$ *if and only if* $g_v(N) = 0$. It is the case of additive games.

The set defined by (10) is the core of the game defined by the gap function, $C(N, g_v)$, and its n vertices are of the form $g_v(N) \cdot e^j$, $i = 1,\dots,n$. Hence, the vertices $(\theta^1,...,\theta^n)$ of the core $C(N,v)$ are given by $\theta^j = b(N,v) - g_v(N) e^j$:

$$\theta_i^j(N,v) = b_i(N,v) - g_v(N) \quad \text{if } i = j,$$
$$= b_i(N,v) \quad \text{if } i \ne j.$$

These are the *efficient upper-vectors* that are obtained from the upper-vectors in such a way that the resulting vector is efficient: by construction, $\theta^j \in X(N,v)$ for all $j \in N$.

Example 1 (continued) We have seen that the game *v* = (0, 0, 0 | 21, 25, 28 | 41) is 1-convex. The core is the regular simplex represented in the imputation triangle by Figure 1.[4] Its vertices are the efficient upper-vectors (5, 16, 20), (13, 8, 20) and (13, 16, 12).

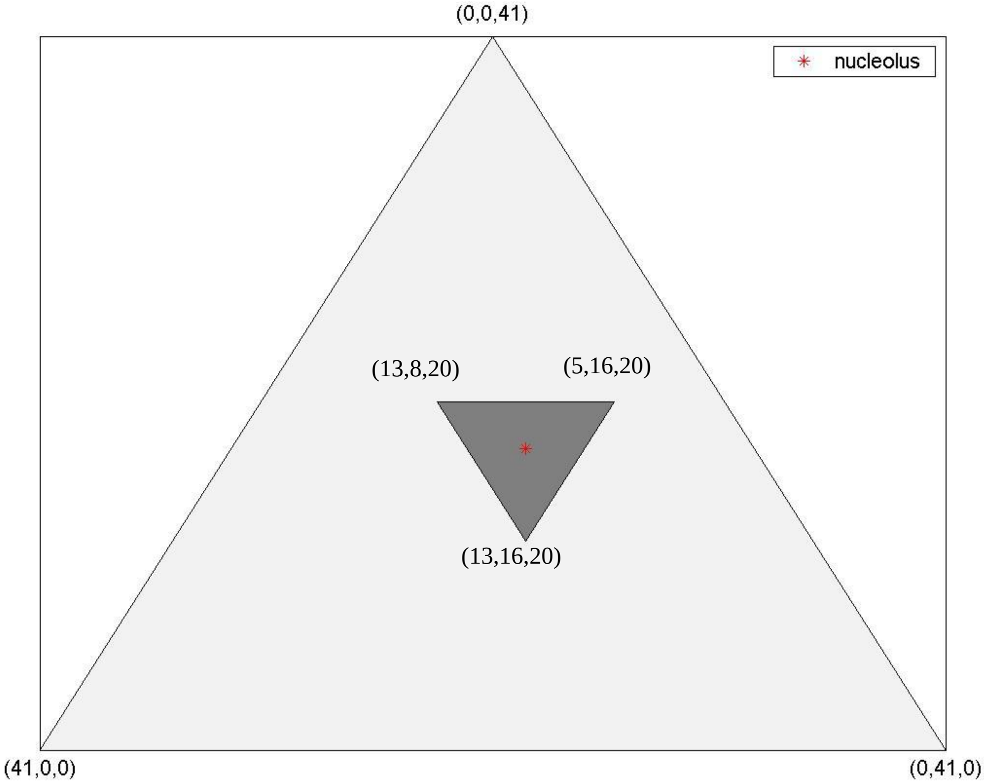

Figure 1 Core of the 1-convex game of Example 1

It is well known that the core of a game is a subset (possibly empty) of the convex hull of the marginal contribution vectors, known as the *Weber set* (Weber, 1988). Here is a similar proposition involving the efficient upper-vectors.

Proposition 11 (Driessen, 1988, p.171) The core of a game is a subset of the convex hull of its efficient upper-vectors.

A game is convex *if and only if* its core is the polyhedra whose vertices are the marginal contribution vectors. This is a well-known result due to Shapley (1971) and Ichiishi (1981). Here is a similar proposition for 1-convex games that offers a characterization of 1-convexity in terms of the core.

Proposition 12 A game is 1-convex *if and only if* its core is the simplex whose vertices are the efficient upper-vectors.

It results from Proposition 11 combined with the following lemma.

[4] All pictures have been obtained using the platform *www.tuglabweb.uvigo.es* maintained by Miguel Angel Miras Calvo and Estela Sanchez Rodriguez (Universidad de Vigo).

Lemma 3 (Driessen, 1985, p.21) A game is 1-convex *if and only if* its efficient upper-vectors *all* belong to its core.

We have seen that the core of a game coincides with the *anticore* of its dual. The following proposition then follows.

Proposition 13 The core of a game, whose *dual* is 1-convex, is a regular simplex of dimension $n - 1$ or a single point.

Proof Consider a game (N, v) and assume that its dual (N, v^d) is 1-convex. Then, by Proposition 10, we have:

$$C(N, v^d) = \left\{ x \in \mathbb{R}^n \mid x(N) = v^d(N),\ x_i \le v^d(N) - v^d(N \setminus i) \text{ for all } i \in N \right\}$$
$$= \left\{ x \in \mathbb{R}^n \mid x(N) = v(N),\ x_i \le v(i) \text{ for all } i \in N \right\} = AC(N, v).$$

The n vertices of $AC(N, v)$ are given by:

$$\omega_i^j(N, v) = v(i) + h_v(N) \quad \text{if } i = j,$$
$$= v(i) \quad \text{if } i \ne j,$$

where $h_v(S) = \sum_{i \in S} v(i) - v(S) = \sum_{i \in S} b_i(N, v^d) - v^d(S) = g_{v^d}(S)$. Hence, $\omega_i^j(N, v) = \theta_i^j(N, v^d)$.

If the game (N, v) is inessential, $h_{v^*}(N) = g_{v^*}(N) = 0$ and the core reduces to the allocation $\{v(1), ..., v(n)\}$. ♦

Remark 6 Because $C(N, v) = AC(N, v^d)$, Proposition 13 applies to the case where the dual game (N, v^d) is 1-concave.

Quasi-additive games are not 1-convex but their duals are 1-convex or 1-concave. Indeed, the gaps of their duals are constant and the gap associated to the grand coalition is nonnegative or non-positive:

$$g_{v^*}(S) = \sum_{i \in S} v(i) - v(N) + v(N \setminus S) = \sum_{i \in N} v(i) - v(N).$$

Hence, the core of a quasi-additive game is a regular simplex.

Example 3 (continued) The game v = (4, 2, 3, 6 | 7, 9, 10, 8, 10, 9 | 12, 12, 13, 12 | 13) satisfies (6). Its dual v^d is a 1-convex game whose core is the simplex characterized by the vertices (2,2,3,6), (4,0,3,6), (4,2,1,6) and (4,2,3,4). It coincides with the core of the original game v.

Example 4 The game v = (2, 3, 7 | 5, 8, 9 | 15) satisfies the opposite inequalities in (6). Its dual is the 1-concave game v^d = (6, 7, 10 | 8, 12, 13 | 15) whose core is the simplex characterized by the vertices are (5,3,7), (2,6,7) and (2,3,10). It coincides with the core of the original game v represented by Figure 2.

9. The Nucleolus and the Shapley Value of a 1-Convex Game

The situation turns out to be simple when considering a 1-convex game because the least-core is a singleton and it therefore coincides with the nucleolus. Driessen (1988, p.200) proves the following proposition.

Proposition 14 The nucleolus of a 1-convex game is the *center of gravity* of its core given by:[5]

$$NUC_i(N,v) = b_i(N,v) - \frac{1}{n}\left(\sum_{i\in N} b_i - v(N)\right) \quad (i = 1,\ldots,n). \tag{11}$$

Proof Consider a 1-convex game (N,v) and its core:

$$C(N,v) = \left\{x \in \mathbb{R}^n \,\middle|\, x(N) = v(N) \text{ and } x_i \le v(N) - v(N\setminus i) \text{ for all } i \in N\right\}.$$

The inequalities $x_i \le v(N) - v(N\setminus i)$ are equivalent to $x(N\setminus i) \ge v(N\setminus i)$. Hence, the excesses that matter are those of the coalitions $N\setminus i$: $e(x, N\setminus i) = x_i - b_i(N,v)$. The problem

$$Min_{x\in X(N,v)}\ Max_{i\in N}\ e(x,N\setminus i) = Min_{x\in X(N,v)}\ Max_{i\in N}(x_i - b_i(N,v)).$$

has a unique solution and it the solution, the differences $x_i - b_i(N,v)$ are equal: $x_i - b_i(N,v) = a$ for all $i \in N$ where the real a satisfies the equation $x(N) - \sum_{i\in N} b_i(N,v) = na$. The least-core is then a singleton and it defines the nucleolus as the average of the core vertices. ♦

Example 1 (continued) The nucleolus of the game v = (0, 0, 0 | 21, 25, 28 | 41) is the allocation (31/3, 40/3, 52/3), as shown in Figure 1.

Example 3 (continued) Because the game v = (4, 2, 3, 6 | 7, 9, 10, 8, 10, 9 | 12, 12, 13, 12 | 13) satisfies (6), its dual is 1-convex. Proposition 13 then applies: its core is a simplex whose four vertices are (4,0,3,6), (4,2,1,6), (4,2,3,4) and (2,2,3,6). Its nucleolus is the center of gravity: (3.5, 1.5, 2.5, 5.5).

Example 4 (continued) The game v = (2, 3, 7 | 5, 8, 9 | 15) satisfies the opposite inequalities (6). Proposition 13 then applies: its core is a simplex whose vertices are (5,3,7), (2,6,7) and (2,3,10). Its nucleolus is the allocation (3,4,8).

Example 5 The game v = (5, 6, 7, 8 | 10, 12, 12, 12, 13, 15 | 16, 18, 17, 20 | 30) satisfies the oppoosite inequalities (6). Its dual is therefore 1-concave. As a consequence, its core is the simplex whose vertices are (9,6,7,8), (5,10,7,8), (5,6,11,8) and (5,6,7,12), and its nucleolus is the allocation (6, 7, 8, 9).

As a consequence of Proposition 12, convexity combined with 1-convexity implies that the efficient upper-vectors coincide with the *distinct* marginal contribution vectors. Because gaps are constant, we have:

$$v(S) = \sum_{i\in S} b_i(N,v) - g_v(N) \text{ for all } S \subset N.$$

[5] The center of gravity has been proposed as a core selection by González-Díaz and Sánchez-Rodríguez (2007). In the 1-concave case, it corresponds to the *egalitarian non-separable cost method*. It also coincides with the τ-*value* introduced by Tijs (1981) and further developed in Driessen and Tijs (1983).

Hence, $v(S)-v(S\setminus i)=b_i(N,v)$ for all $S\neq\{i\}$ and all $i\in S$. For the permutation $\pi=(i,i_2,...,i_n)$ where player i is first, we have:

$$\mu_i^\pi(N,v)=v(i)=b_i(N,v)-g_v(N) \text{ and } \mu_j^\pi(N,v)=b_j(N,v) \text{ for all } j\neq i.$$

Hence, all permutations in which a given player is first are identical: there are n distinct vectors, each with multiplicity $(n-1)!$ and they coincide with the efficient upper-vectors. The following proposition then follows.

Proposition 15 If a game is convex *and* 1-convex, its nucleolus and Shapley value coincide.

Notice that Proposition 15 applies to games that are concave and 1-concave.

Example 6 The game v = (0, 3, 7 | 5, 9, 12 | 14) is convex and 1-convex. Its core is the simplex whose vertices are (0,5,9), (2,3,9) and (2,5,7). Its nucleolus is the allocation (1.33, 4.33, 8.33) and it coincides with its Shapley value.

The Shapley value of a 1-convex game does not necessarily define a core allocation, as the following example shows.

Example 7 The game v = (0, 0, 0 | 6, 4, 8 | 10) is 1-convex. Its core vertices are (0,6,4), (2,4,4) and (2,6,2). Its nucleolus (1.33, 5.33, 3.33) differs from the Shapley value (2.33, 4.33, 3.33) as shown in Figure 2.

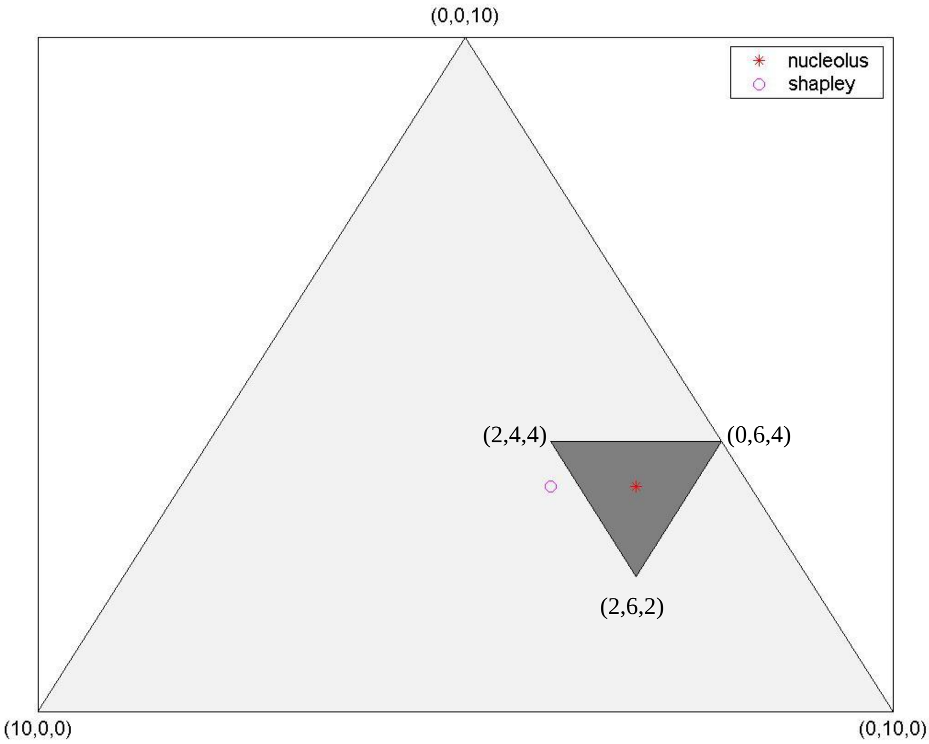

Figure 2 The core of the 1-convex game of Example 7

The Shapley value is an additive sharing rule on the class of superadditive games, while the nucleolus is not. However, the nucleolus is additive on the class of 1-convex games: considering two games (N, v_1) and (N, v_2) on a common set of players *N*, and applying (11), we indeed have $NUC(N, v_1) + NUC(N, v_2) = NUC(N, v_1 + v_2)$.

10. Applications

Here are applications that involve 1-convex or 1-concave games. We have left aside the information games (Driessen et al., 1992 and 1995) and co-insurance games (Driessen et al., 2010) whose cores are single allocations under 1-convexity.

10.1 Bankruptcy resolution (O'Neill, 1982; Aumann and Maschler, 1985)

Consider a situation involving *n* creditors, each holding a claim, $d_i > 0$ for creditor *i*. The value of the estate *E* falls short of the total claim $d(N)$. Reasoning in terms of what a coalition could obtain, the worth of a coalition is equal to what is left for its members, when priority has been given to those outside the coalition:

$$v(S) = Max(0, E - d(N \setminus S)).$$

This define a *bankruptcy games*. It is well known that these games are convex. The upper-vector is given by:

$$b_i(N, v) = v(N) - v(N \setminus i) = Min(E, d_i).$$

Assuming that no individual debt exceeds the estate value, we have $b_i(N, v) = d_i$ for all $i \in N$ and $g_v(S) = d(S) - v(S)$. Therefore, we have:

$$g_v(N) = d(N) - v(N) = d(N) - E \geq 0$$

and

$$g_v(N) - g_v(S) = d(N \setminus S) - E + v(S) = Max\left(0, d(N \setminus S) - E\right) \quad \text{for all } S \neq \emptyset.$$

Hence, a bankruptcy game is 1-convex *if and only if* no proper coalition has a debt exceeding the estate value, i.e. $d(N \setminus i) \leq E \leq d(N)$ for all $i \in S$.[6] If it is the case, Proposition 5 applies: $g_v(S) = g_v(N) = d(N) - v(N) \geq 0$ for all $S \subset N$ and the core is given by:

$$C(N, v) = \left\{ x \in \mathbb{R}^n \mid x(N) = E,\ x_i \leq d_i \text{ for all } i \in N \right\}.$$

Its vertices are the efficient upper-vectors $d + \left(E - d(N)\right) e^i$ and the nucleolus is given by:

$$NUC_i(N, v) = d_i + \frac{1}{n}\left(E - d(N)\right) \quad (i = 1, \ldots, n).$$

It coincides with the Shapley value in accordance with Proposition 15: the excess over the estate value is uniformly distributed among the claimants.

The dual of a bankruptcy game (N, v) has been considered by Aumann (2010). It is given by:

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[6] See Driessen (1995).

$$v^d(S) = Min\big(E, d(S)\big).$$

The worth of a coalition is equal to the sum of its members' claims, as long as it does not exceed the estate value. It is not what coalition S can get. In this formulation, coalition S has a right to $d(S)$. By Proposition 8, (N, v^d) is concave as dual of a convex game. If $d(N\setminus i) \leq E$ for all $i \in S$, the game (N, v) is 1-convex and therefore, its dual is quasi-additive. Indeed, under that assumption, $v^d(S) = d(S)$ for all $S \subsetneq N$ and $v^d(N) = E$.

Example 8 The bankruptcy situation (E | d) = (1000 | 150, 200, 350, 400) is associated to the 1-convex game v = (50, 100, 250, 300 | 250, 400, 450, 450, 500, 650 | 600, 650, 800, 850 | 1000). The four core vertices are given by (50, 200, 350, 400), (150, 100, 350, 400), (150, 200, 250, 400) and (150, 200, 350, 300). The allocation (125, 175, 325, 375) defines the nucleolus and the Shapley value. The dual is the quasi-additive game defined by $v^d(i) = 150$ and $v^d(N) = 1000$.

9.2 Sharing data (Dehez and Tellone, 2013)[7]

A set N of players need to have access to a set M of data that are own by one or several players. A value is attached to each data, $p_h > 0$ for data h, covering the cost of reproducing the data. The subset of data owned by player i is denoted by M_i. If a coalition forms, the cost of acquiring the missing data is given by:

$$v(S) = \sum_{h \in M \setminus M_S} p_h = w_0 - \sum_{h \in M_S} p_h \text{ for all } S \subset N, S \neq \varnothing,$$

where M_S is the data set detained by coalition S and w_0 is the value of the complete data set. This defines a cost-sharing game.[8] Because $v(N) = 0$, an allocation is a set of payments $x_1, x_2, ..., x_n$ summing to zero. Data games are subadditive and 1-concave. They are essential if at least one player does not detain the complete data set.

We denote by $\overline{M} \subset M$ the set of data detained by *single players* and by $\overline{M}_i = M_i \cap \overline{M}$ the set of data that player i is *alone* to detain. Alongside, we denote by $\overline{w}_0$ and $\overline{w}_i$ the value of the data in $\overline{M}$ and the value of the data in $\overline{M}_i$, respectively. We then have:

$$b_i(N, v) = v(N) - v(N \setminus i) = -(w_0 - \sum_{h \in M_{N\setminus i}} p_h) = -\overline{w}_i.$$

The core of a data game is non-empty: it contains the no-compensation allocation $x = 0$. If x is a core allocation, $x(N \setminus i) \leq v(N \setminus i) = \overline{w}_i$ and, because $x(N) = 0$, $x_i \geq -\overline{w}_i$. Hence, if an allocation x satisfies these inequalities for all i, we have:

[7] The background is the EU program REACH that imposes to chemical firms to submit data about their products.
[8] The equivalence between data games and library games (Sales, 2002 and Driessen et al., 2012) has been pointed out by Khmelnitskaya and Driessen (2017).

$$x(S) = -x(N \setminus S) \leq \sum_{i \in N \setminus S} \bar{w}_i \leq \sum_{h \in M \setminus M_S} p_h = v(S) \text{ for all } S \subset N.$$

As a consequence, the core is given by:

$$AC(N,c) = \{x \in \mathbb{R}^n \mid x(N) = 0 \text{ and } x_i \geq -\bar{w}_i \text{ for all } i \in N\}.$$

If $\bar{M} \neq \varnothing$, $\bar{w}_0 > 0$ and the core is the simplex whose vertices are the n efficient upper-vectors $(-\bar{w}_1, -\bar{w}_2, ..., -\bar{w}_n) + \bar{w}_0 e^i$. Therefore, referring to Proposition 14, the game is 1-concave and the nucleolus is given by:

$$NUC_i(N,v) = \frac{1}{n} \sum_{h \in \bar{M}} p_h - \sum_{h \in \bar{M}_i} p_h \quad (i = 1, ..., n)$$

A player is compensated *if and only if* the value of the data he is alone to detain exceeds the *per capita* value of the data detained by single firms. If instead $\bar{M} = \varnothing$, $\bar{w}_0 = 0$ and the core consists of a single allocation, namely the no-compensation allocation $x = 0$. The Shapley value instead makes no difference between data held by single or several players:

$$SV_i(N,v) = \frac{1}{n} \sum_{h \in M} p_h - \sum_{h \in M_i} \frac{p_h}{n(h)} \qquad (i = 1, ..., n)$$

where $n(h)$ denotes the number of players that hold data h. In the particular case where *every* data is held by *one and only one* player, $\bar{M} = M$ and $n(h) = 1$ for all h, the game is concave and the nucleolus coincides with the Shapley value, in accordance with Proposition 15.

Example 9 Consider the 3-player and 3-data situation defined by $M_1 = \varnothing$, $M_2 = \{1,2\}$, $M_3 = \{2,3\}$. The data game corresponding to $p = (6, 9, 12)$ is given by $v = (27, 12, 6 \mid 12, 6, 0 \mid 0)$. The value of the data detained by single players is $\bar{w}_0 = d_1 + d_3 = 18$ and the upper-vector is given by $\bar{w} = -(0,6,12)$. The core vertices are given by (18, -6, -12), (0, 12, -12) and (0, -6, 6). The nucleolus is given then by (6, 0, -6): player 1 pays 6 to player 3, and player 2 stay put. The Shapley value is given by (9, −1.5, −7.5). Here, players 2 and 3 compensate player 1.

10.3 Sharing royalties (Dehez and Poukens, 2013)

Imagine a situation where n firms that operate on different markets are willing to commercialize a product that relies on patents owned by some of them. These patents are assumed to be *weak*: while being essential, they can be invented around at a known cost. The value of the patents owned by firm i is given by $p_i \geq 0$ and the profit it can generate from the commercialization of the product is given by $\pi_i \geq 0$. The associated transferable utility game (N,v) is then defined by:

$$v(S) = \sum_{i \in S} \pi_i - \sum_{i \in N \setminus S} p_i.$$

The question then concerns the redistribution of the total profit $v(N) = \pi(N)$. A redistribution defines transfers (the royalties) between firms. The game (N,v) is both convex and 1-convex. Indeed, gaps are constant and, following Proposition 7, we have:

$$b_i(N,v) = \pi_i + p_i \;\Rightarrow\; g_v(S) = p(N) \text{ for all } S \subset N.$$

The core is therefore defined by:

$$C(N,v) = \left\{ x \in \mathbb{R}^n \,\middle|\, x(N) = \pi(N) \text{ and } x_i \leq \pi_i + p_i \text{ for all } i \in N \right\}.$$

Its vertices are given by $\theta^i = b(N,v) - p(N)e^i$ and, following Proposition 15, the Shapley value and the nucleolus coincide:

$$SV_i(N,v) = NUC_i(N,v) = \pi_i + p_i - \frac{1}{n}\sum_{j \in N} p_j.$$

All firms uniformly support the value of the whole patent set while the value of each patent is entirely redistributed to the firm holding it: patent holders perceives a compensation if the value of the patents they hold is superior to the *per capita* value of the whole patent set.

Example 10 Consider the 3-firm situation where π = (10,20,30) and p = (3,7,9). The associated game is given by v = (-6, 8, 20 | 21, 33, 47 | 60) and its nucleolus and Shapley value are given by the allocation (6.66, 20.66, 32.66).

10.4 Provision of an indivisible public good (Dehez, 2013)

A group of n cities consider the provision of an indivisible public good, for instance a waste treatment plant. A known cost is associated to each town, $c_i \geq 0$ for city i. The question is to decide where to locate the treatment plant and what compensations to award to the city that host it. Without loss of generality, cities are ordered in terms of their cost: $0 < c_1 \leq c_2 \leq \ldots \leq c_n$. If a coalition forms, it will naturally rely on the player with the lowest cost. The associated cost-sharing game (N,v) is then defined by:

$$v(S) = Min_{i \in S}\; c_i \text{ for all } S \subset N, S \neq \varnothing.$$

This game is *subadditive* game and *essential*. Its core is non-empty: it always contains the no-compensation allocation $(c_1, 0, \ldots, 0)$. It is *concave* if $c_2 = c_n$: the marginal contributions of player 1 are all equal to $c_1 - c_2$ while the marginal contributions of the other players are all equal to zero.

Notice that the function v can be written as $v(S) = c_n - w(S)$ where $w(S) = Max_{i \in S}(c_n - c_i)$ defines an *airport game* on N with cost parameters $(0, c_n - c_{n-1}, \ldots, c_n - c_1)$.[9]

The upper-vector is defined by $b_1(N,v) = c_1 - c_2$ and $b_i(N,v) = 0$ for all $i \neq 1$. In terms of the gap function, we then have:

$$g_v(N) = \sum_{i \in N} b_i(N,v) - v(N) = -c_2 \leq 0$$

and

[9] Airport games were introduced by Littlechild and Owen (1973).

$$g_v(N)-g_v(S)=-c_2-\left(\sum_{i\in S}b_i(N,v)-v(S)\right)=\begin{cases}0 & \text{if } 1\in S\\ -c_2+Min_{i\in S}c_i\geq 0 & \text{if } 1\notin S\end{cases}$$

The opposite inequalities (3) are verified and the game (N,v) is 1-concave. The core and its vertices are given by:

$$AC(N,v)=\{x\in\mathbb{R}^n \mid x(N)=c_1,\ x_1\geq c_1-c_2 \text{ and } x_i\geq 0 \text{ for all } i=2,...,n\},$$

$$\theta^1=(nc_1-(n-1)c_2,c_1-c_2,...,c_1-c_2),\ \theta^2=(0,c_2,0...,0),...,\theta^n=(0,0...,c_2),$$

from which we deduce the nucleolus

$$NUC_i(N,v)=\begin{cases}\dfrac{c_2}{n}-(c_2-c_1)=c_1-\dfrac{n-1}{n}c_2 & \text{for } i=1\\ \dfrac{c_2}{n} & \text{for all } i\neq 1.\end{cases}$$

The provider supports his cost and receives a compensation from the other players who contribute an equal amount. We observe that only the two lowest costs enter into account. This is not so for the Shapley value that may differ significantly from the nucleolus. It can be derived from the Shapley value of the associated airport game, hence the following triangular formula:

$$SV_i(N,v)=\sum_{j=i}^{n}\frac{c_j-c_{j+1}}{j}\quad (i=1,...,n)$$

where $c_{n+1}=0.$ The Shapley value and the nucleolus coincide in the particular case where $c_2=c_n$. The game is then indeed concave and Proposition 15 applies.

Example 11 The 3-player game associated to the cost vector $c=(3,7,25)$ is defined by $v=(3,7,25\,|\,3,3,7\,|\,3).$ Its nucleolus and Shapley value are given by:

$$NUC(N,v)=(-1.66,\ 2.33,\ 2.33),$$
$$SV(N,v)=(-4.66,-0.66,8.34).$$

Notice that the Shapley value compensates the second player. This is not appropriate in this context: there is no reason to compensate other players than the provider. Actually, core allocations do not compensate any player other than the provider.

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