"Science Stays True Here"
Journal of Mathematics and Statistical Science (ISSN 2411-2518, USA), Vol.9 (2023), Issue 9, 19-25 | Science Signpost Publishing

SOME SPECIAL CURVES ACCORDING TO BISHOP FRAME IN EUCLIDEAN SPACE $\mathbb{E}^3$

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Abstract. In this study, we analyze the correspondence between the Bishop frame, which is an alternative way of defining a moving frame in Euclidean 3-space, of a curve in $\mathbb{E}^3$ and its natural mate, conjugate (or adjoint) mate and Mannheim partner curves. We also obtain a relation between the respective Bishop curvatures for these curves.

Keywords: Bishop frame, Bishop curvatures, Natural mate curves, Conjugate mate curves, Mannheim partner curves.

1. Introduction

In the local theory of spatial curves an interesting class are those that are related by some geometric property generating interesting facts and important problems to be addressed.

Two curves which have some special geometric properties at their corresponding points are called associated curves or curve pairs. The most famous curve pairs are Bertrand partner curves, described by Bertrand Russell in 1850 [2]; involute-evolute curves, discovered by C. Huggens [4], and Mannheim partner curves, discovered by A. Mannheim in 1878 [11]. After these papers, several studies were made about these curves in different spaces by means of the Frenet apparatus or other frames, for example: Lai [10] characterized a similar type of curves, weakened Bertrand curves and Frenet–Bertrand curves under weakened conditions, Camci et al [5] give a new approach to Bertrand curves in 3-dimensional Euclidean space, Nurkan et al [13] characterize adjoint curves of a spatial curve in Euclidean 3-space, Mendonça [12] study the Mannheim partner curves in the special linear group $SL(2,\mathbb{R})$, Yildirim and Kaya [17] study the Mannheim partner curves according to Darboux frame in the $\mathbb{E}^3$, among many others such as [6, 8, 14, 16].

The local theory of spatial curves in Euclidean space has been extensively developed by means of the Frenet frame that expresses the derivative of a given basis of $\mathbb{E}^3$ using itself, however the frame has the drawback that it is undefined at the points where the curvature is zero. In order to overcome this difficulty it was introduced by L.R. Bishop [3] in 1975 a new referential by means of fields of parallel vectors, this referential is called *Bishop frame* or, alternatively, *parallel frame* of curves. Since then, a lot of research related to this concept has been done in $\mathbb{E}^3$, for example: Körpinar et al [9] characterize parallel curves using Bishop's frame in $\mathbb{E}^3$, Yilmaz [18] obtain the versions of the Bishop frame rotating around the Frenet elements of the normal indicatrix.

In this study, natural mate, conjugate mate and Mannheim partner curves are analyzed by using Bishop frame. We relate the Bishop curvatures of these curves.

2. Preliminaries

Let $\alpha : I \to \mathbb{E}^3$ be a differentiable curve in the Euclidean space defined on an open interval I, parametrized by arc length and let $\{t = \alpha', n, b\}$ be Frenet frame satisfying

$$\begin{cases} t' = \kappa n, \\ n' = -\kappa t - \tau b, \\ b' = \tau n, \end{cases} \tag{2.1}$$

where κ e τ are differentiable functions on I called the *curvature* and the *torsion* of α, respectively, t is the tangent vector, n is the principal normal vector and b is the binormal vector of α, respectively.

The *Bishop frame* or *parallel transport* is an alternative approach to defining a moving frame and its constrution is based on the idea of parallel transport. For the tangent vector, normal vector and binormal vector are applicable, the Bishop frame is expressed as [3]

$$\begin{cases} T' = \kappa_1 M_1 + \kappa_2 M_2, \\ M_1' = -\kappa_1 T, \\ M_2' = -\kappa_2 T, \end{cases} \tag{2.2}$$

where we shall call the set $\{T, M_1, M_2\}$ as *Bishop trihedra* and κ_1 and κ_2 as *Bishop curvatures*. The relation with Frenet frame may be expressed as

$$\begin{cases} t = T, \\ n = \cos\theta(s) M_1 + \sin\theta(s) M_2, \\ b = -\sin\theta(s) M_1 + \cos\theta(s) M_2, \end{cases} \tag{2.3}$$

where $\theta(s) = \arctan\left(\frac{\kappa_2}{\kappa_1}\right)$, $\tau = \theta'(s)$ and $\kappa = \sqrt{\kappa_1^2 + \kappa_2^2}$. Here, Bishop curvatures are defined by

$$\kappa_1 = \kappa \cos\theta(s),$$
$$\kappa_2 = \kappa \sin\theta(s).$$

Definition 2.1. *For a unit speed curve α in $\mathbb{E}^3$ with Frenet frame $\{t, n, b\}$:*

- **i)** *An integral curve of n is called natural mate curve of α* [7].
- **ii)** *An integral curve of b is called conjugate mate curve of α* [7]*, or adjoint curve of α* [13].
- **iii)** *If the principal normal vector of α coincides with the binormal line of a curve $\beta : J \to \mathbb{E}^3$, then β is called Mannheim partner curve of α* [19].
- **iv)** *If $\beta : J \to \mathbb{E}^3$ is a natural mate or conjugate mate or Mannheim partner of a curve α, then the pair (α, β) is called natural pair or conjugate pair or Mannheim pair, respectively.*

The existence of natural and conjugate mate curves is guaranteed by existence theorem for differential equation and β, the natural mate curve of α, is given by $\beta(s) = \int n(s)\, ds$ (respectively, $\beta(s) = \int b(s)\, ds$ to conjugate mate curve of α). It can be also shown that the arc lenght parameters of the curve β can be the same of α (see [13]). Clearly, the curve β is orthogonal to α since it is tangent to the principal normal vector field n (or tangent to binormal vector field b, respectively) of α.

For the existence of Mannheim partner curves, see [15] Theorem 2.

3. Natural mate curves According Bishop frame in $\mathbb{E}^3$

In this section, we present a characterization of natural mate curves by means of Bishop frame.

Let (α, β) be natural pair with Frenet frame $\{t, n, b\}$ and $\{\bar{t}, \bar{n}, \bar{b}\}$, respectively. We write the following equations [1]

$$\begin{cases} t = -\cos\psi \bar{n} + \sin\psi \bar{b}, \\ n = \bar{t}, \\ b = \sin\psi \bar{n} + \cos\psi \bar{b}, \end{cases} \tag{3.4}$$

where $\cos\psi = \dfrac{\kappa}{\sqrt{k^2+\tau^2}}$ and $\sin\psi = -\dfrac{\tau}{\sqrt{k^2+\tau^2}}$ with κ and τ the curvature and the torsion of the curve α, respectively.

Theorem 3.1. *Let (α, β) be natural pair. The relation between Bishop frame are given by*

$$\begin{cases} T = -\cos(\psi - \bar{\theta})\ \overline{M_1} + \sin(\psi - \bar{\theta})\ \overline{M_2}, \\ M_1 = \cos\theta\ \overline{T} - \sin\theta \sin(\psi - \bar{\theta})\ \overline{M_1} - \sin\theta\cos(\psi - \bar{\theta})\ \overline{M_2}, \\ M_2 = \sin\theta\ \overline{T} + \cos\theta\sin(\psi - \bar{\theta})\ \overline{M_1} + \cos\theta\cos(\psi - \bar{\theta})\ \overline{M_2}, \end{cases} \tag{3.5}$$

where $\theta = \arctan\left(\frac{\kappa_2}{\kappa_1}\right)$, $\bar{\theta} = \arctan\left(\frac{\overline{\kappa_2}}{\overline{\kappa_1}}\right)$ and ψ is the angle between b and $\bar{b}$, with κ_1, κ_2 and $\overline{\kappa_1}, \overline{\kappa_2}$ are the Bishop curvatures of the curves α and β, respectively.

Proof. From equations (2.3), the relation of the curve β between the Frenet frame and the Bishop frame is

$$\begin{bmatrix} \bar{t} \\ \bar{n} \\ \bar{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\bar{\theta} & \sin\bar{\theta} \\ 0 & -\sin\bar{\theta} & \cos\bar{\theta} \end{bmatrix} \begin{bmatrix} \overline{T} \\ \overline{M_1} \\ \overline{M_2} \end{bmatrix}, \tag{3.6}$$

and the relation of the curve α between the Frenet frame and the Bishop frame can be expressed by

$$\begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{3.7}$$

From equations (3.4), (3.6) and (3.7), we get

$$\begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -\cos\psi & \sin\psi \\ 1 & 0 & 0 \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\bar{\theta} & \sin\bar{\theta} \\ 0 & -\sin\bar{\theta} & \cos\bar{\theta} \end{bmatrix} \begin{bmatrix} \overline{T} \\ \overline{M_1} \\ \overline{M_2} \end{bmatrix}.$$

The result follows. □

The next result relates the Bishop curvatures of the curve α to those of the curve β.

Theorem 3.2. *Let (α, β) be natural pair. We get the following equations*

$$\kappa_1\cos\theta + \kappa_2\sin\theta = \overline{\kappa_1}\cos(\psi - \bar{\theta}) - \overline{\kappa_2}\sin(\psi - \bar{\theta}), \tag{3.8}$$

$$-\kappa_1\sin\theta + \kappa_2\cos\theta = \psi' - \bar{\theta}'. \tag{3.9}$$

Proof. From Theorem 3.1 we have

$$T = -\cos(\psi - \bar{\theta})\overline{M_1} + \sin(\psi - \bar{\theta})\overline{M_2}. \tag{3.10}$$

Taking the derivative of the equation (3.10) and use the Bishop frame, we obtain

$$\begin{aligned} \kappa_1 M_1 + \kappa_2 M_2 &= [\overline{\kappa_1}\cos(\psi - \bar{\theta}) - \overline{\kappa_2}\sin(\psi - \bar{\theta})]\overline{T} \\ &+ (\psi' - \bar{\theta}')[\sin(\psi - \bar{\theta})\overline{M_1} + \cos(\psi - \bar{\theta})\overline{M_2}]. \end{aligned} \tag{3.11}$$

Multiply both sides of the equation (3.11) by $\overline{T}$, we obtain

$$\kappa_1\langle M_1, \overline{T}\rangle + \kappa_2\langle M_2, \overline{T}\rangle = \overline{\kappa_1}\cos(\psi - \bar{\theta}) - \overline{\kappa_2}\sin(\psi - \bar{\theta}). \tag{3.12}$$

On ther other hand, multiplying both sides of the equations (3.5) by $\overline{T}$, we obtain

$$\langle M_1, \overline{T}\rangle = \cos\theta \quad \text{and} \quad \langle M_2, \overline{T}\rangle = \sin\theta.$$

By using these equations in the equation (3.12), we obtain (3.8).

Multiply both sides of the equation (3.11) by $\overline{M_1}$, we obtain

$$\kappa_1\langle M_1, \overline{M_1}\rangle + \kappa_2\langle M_2, \overline{M_1}\rangle = (\psi' - \bar{\theta}')\sin(\psi - \bar{\theta}). \tag{3.13}$$

On ther other hand, multiplying both sides of the equations (3.5) by $\overline{M_1}$, we obtain

$$\langle M_1, \overline{M_1}\rangle = -\sin\theta\sin(\psi - \bar{\theta}) \quad \text{and} \quad \langle M_2, \overline{M_1}\rangle = \cos\theta\sin(\psi - \bar{\theta}).$$

By using these equations in the equation (3.13), we obtain (3.9). □

The following Corollary are direct consequence of Theorem 3.2.

Corollary 3.1. *Let (α, β) be natural pair. Then the Bishop curvatures satisfy the equations*

$$\begin{aligned} \kappa_1 &= [\overline{\kappa_1}\cos(\psi - \bar{\theta}) - \overline{\kappa_2}\sin(\psi - \bar{\theta})]\cos\theta - (\psi' - \bar{\theta}')\sin\theta, \\ \kappa_2 &= [\overline{\kappa_1}\cos(\psi - \bar{\theta}) - \overline{\kappa_2}\sin(\psi - \bar{\theta})]\sin\theta + (\psi' - \bar{\theta}')\cos\theta. \end{aligned}$$

Remark 3.1. *Taking the derivative of others equations of (3.5), we obtain the same equations to κ_1 and κ_2 give in Corollary 3.1.*

4. Conjugate Mate Curves According Bishop Frame in $\mathbb{E}^3$

In this section, we present a characterization of conjugate mate curves by means of Bishop frame.

Let $\alpha, \beta : I \to \mathbb{E}^3$ be conjugate pair with Frenet frame $\{t, n, b\}$ and $\{t^*, n^*, b^*\}$, respectively. We write the following equations [13]

$$\begin{cases} t = b^*, \\ n = -n^*, \\ b = t^*. \end{cases} \tag{4.14}$$

Theorem 4.3. *Let (α, β) be conjugate pair. The relation between Bishop frame are given by*

$$\begin{cases} T = -\sin\theta^*\ M_1^* + \cos\theta^*\ M_2^*, \\ M_1 = -\sin\theta\ T^* - \cos\theta\cos\theta^*\ M_1^* - \cos\theta\sin\theta^*\ M_2^*, \\ M_2 = \cos\theta\ T^* - \sin\theta\cos\theta^*\ M_1^* - \sin\theta\sin\theta^*\ M_2^*, \end{cases} \tag{4.15}$$

where $\theta = \arctan\left(\frac{\kappa_2}{\kappa_1}\right)$ *and* $\theta^* = \arctan\left(\frac{\kappa_2^*}{\kappa_1^*}\right)$ *with* κ_1, κ_2 *and* κ_1^*, κ_2^* *are the Bishop curvatures of the curves* α *and* β*, respectively.*

Proof. The proof is similar to the proof of Theorem 3.1. □

The next result relates the Bishop curvatures of the curve α to those of the curve β.

Theorem 4.4. *Let* (α, β) *be conjugate pair. We get following equations*

$$-\kappa_1 \sin\theta + \kappa_2 \cos\theta = \kappa_1^* \sin\theta^* - \kappa_2^* \cos\theta^*, \tag{4.16}$$

$$-\kappa_1 \cos\theta - \kappa_2 \sin\theta = -(\theta^*)'. \tag{4.17}$$

Proof. The proof is similar to the proof of Theorem 3.2. □

The following Corollary are direct consequence of Theorem 4.4.

Corollary 4.2. *Let* (α, β) *be conjugate pair. Then the Bishop curvatures satisfy the equations*

$$\kappa_1 = [-\kappa_1^* \sin\theta^* + \kappa_2^* \cos\theta^*] \sin\theta + (\theta^*)' \cos\theta,$$

$$\kappa_2 = [\kappa_1^* \sin\theta^* - \kappa_2^* \cos\theta^*] \cos\theta + (\theta^*)' \sin\theta.$$

5. Mannheim Partner Curves According Bishop Frame in $\mathbb{E}^3$

In this section, we present a characterization of Mannheim partner curves by means of Bishop frame.

Let $\alpha : I \to \mathbb{E}^3$ and $\beta : J \to \mathbb{E}^3$ be Mannheim pair parameterized by arc length s and $\tilde{s}$ with Frenet frame $\{t, n, b\}$ and $\{\tilde{t}, \tilde{n}, \tilde{b}\}$, respectively. We write the following equations [15]

$$\begin{cases} t = \cos\gamma \tilde{t} + \sin\gamma \tilde{n}, \\ n = \tilde{b}, \\ b = \sin\gamma \tilde{t} - \cos\gamma \tilde{n}, \end{cases} \tag{5.18}$$

where γ be the angle function between t and $\tilde{t}$.

Theorem 5.5. *Let* (α, β) *be Mannheim pair. The relation between Bishop frame are given by*

$$\begin{cases} T &= \cos\gamma\ \tilde{T} + \sin\gamma \cos\tilde{\theta}\ \tilde{M}_1 + \sin\gamma \sin\tilde{\theta}\ \tilde{M}_2, \\ M_1 &= -\sin\theta \sin\gamma\ \tilde{T} + (-\cos\theta \sin\tilde{\theta} + \sin\theta \cos\tilde{\theta} \cos\gamma)\ \tilde{M}_1 \\ &+ (\cos\theta \cos\tilde{\theta} + \sin\theta \sin\tilde{\theta} \cos\gamma)\ \tilde{M}_2, \\ M_2 &= \cos\theta \sin\gamma\ \tilde{T} + (-\sin\theta \sin\tilde{\theta} - \cos\theta \cos\tilde{\theta} \cos\gamma)\ \tilde{M}_1 \\ &+ (\sin\theta \cos\tilde{\theta} - \cos\theta \sin\tilde{\theta} \cos\gamma)\ \tilde{M}_2, \end{cases} \tag{5.19}$$

where γ *be the angle between* t *and* $\tilde{t}$, $\theta = \arctan\left(\frac{\kappa_2}{\kappa_1}\right)$ *and* $\tilde{\theta} = \arctan\left(\frac{\tilde{\kappa}_2}{\tilde{\kappa}_1}\right)$ *with* κ_1, κ_2 *and* $\tilde{\kappa}_1, \tilde{\kappa}_2$ *are the Bishop curvatures of the curves* α *and* β*, respectively.*

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 5.6. *Let Mannheim pair (α, β) be specified. We get following equations*

$$\kappa_1 \sin\theta - \kappa_2 \cos\theta = \frac{d\tilde{s}}{ds}\left(\frac{d\gamma}{d\tilde{s}} + \tilde{\kappa}_1 \cos\tilde{\theta} + \tilde{\kappa}_2 \sin\tilde{\theta}\right), \tag{5.20}$$

$$\kappa_1 \cos\theta + \kappa_2 \sin\theta = \frac{d\tilde{s}}{ds}\left(\frac{d\tilde{\theta}}{d\tilde{s}} \sin\gamma - \tilde{\kappa}_1 \sin\tilde{\theta}\cos\gamma + \tilde{\kappa}_2 \cos\tilde{\theta}\cos\gamma\right). \tag{5.21}$$

Proof. The proof is similar to the proof of Theorem 3.2. □

Corollary 5.3. *Let Mannheim pair (α, β) be specified. Then the Bishop curvatures satisfy the equations*

$$\begin{aligned} \kappa_1 &= \frac{d\tilde{s}}{ds}\Big[\tilde{\kappa}_1(\sin\theta\cos\tilde{\theta} - \cos\theta\sin\tilde{\theta}\cos\gamma) + \tilde{\kappa}_2(\sin\theta\sin\tilde{\theta} + \cos\theta\cos\tilde{\theta}\cos\gamma) \\ &+ \sin\theta\frac{d\gamma}{d\tilde{s}} + \cos\theta\sin\gamma\frac{d\tilde{\theta}}{d\tilde{s}}\Big], \\ \kappa_2 &= \frac{d\tilde{s}}{ds}\Big[\tilde{\kappa}_1(-\cos\theta\cos\tilde{\theta} - \sin\theta\sin\tilde{\theta}\cos\gamma) + \tilde{\kappa}_2(-\cos\theta\sin\tilde{\theta} + \sin\theta\cos\tilde{\theta}\cos\gamma) \\ &- \cos\theta\frac{d\gamma}{d\tilde{s}} + \sin\theta\sin\gamma\frac{d\tilde{\theta}}{d\tilde{s}}\Big]. \end{aligned}$$

References

[1] R. A. Abdel-Baky - M. F. Naghi, *Sweeping surfaces with Natural mate curve of a spatial curve in Euclidean 3-Space*, WSEAS Transactions on Mathematics 19 (2020), 581-588.

[2] J. Bertrand, *Memoire sur la théorie des courbes à double courbure*, Journal de Mathématiques Pureset Appliquées 15 (1850), 332–350.

[3] L. R. Bishop, *There is More Than One Way to Frame a Curve*, Amer. Math. Monthly 82 (1975), 246-251.

[4] C. Boyer. *A history of Mathematics*, New York, Wiley, 1968.

[5] Ç. Camci - A. Uçum - K. İlarslan, *A new approach to Bertrand curves in Euclidean 3-space*, J. Geom. 111 (2020), https://doi.org/10.1007/s00022-020-00560-5.

[6] A.Y. Ceylan - A.A. Ergin, *Mannheim partner curves in Cartan-Vranceanu 3-space*, Filomat 30 (2016), 1089-1095.

[7] S. Deshmukh S - B. Y. Chen - A. Alghanemi, *Natural mate of Frenet curves in Euclidean 3-space*, Turkish Journal of Mathematics 42 (2018), 2826-2840.

[8] A. Has - B. Yilmaz, *Various Characterizations for Quaternionic Mannheim Curves in Three-Dimensional Euclidean Space*. Journal of Universal Mathematics 4 (2021), 62-72.

[9] T. Körpinar - V. Asil V - M. T. Sariaydin - M. İncesu, *A Characterization for Bishop Equations of Parallel Curves according to Bishop Frame in* $\mathbb{E}^3$, Bol. Soc. Paran. Mat. 33 (2015), 33-39.

[10] H. F. Lai, *Weakened Bertrand curves*, Tohoku Math. Journal 19 (1967), 141-155.

[11] A. Mannheim, *De l'emploi de la courbe représentative de la surface des normales principales d'une courbe gauche pour la démonstration de propriétés relatives à cette courbure*, C.R. Comptes Rendus des Séances de l'Académie des Sciences, Paris 86 (1878), 1254-1256.

[12] T. Mendonça, *Mannheim partner curves in the special linear group*, Int. J. Phys. Sci. 18 (2023), 53-58.

[13] S. K. Nurkan - *İ*. A. Güven - M. K. Karacan, *Characterizations of Adjoint Curves in Euclidean 3-space*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences 89 (2018), 155-161.

[14] O. Z. Okuyucu - B. D. Yazıcı, *Generalized Bertrand and Mannheim Curves in 3D Lie Groups*, Fundamental Journal of Mathematics and Applications 5 (2022), 201-209.
[15] K. Orbay - E. Kasap, *On Mannheim partner curves in* $\mathbb{E}^3$, International Journal of Physical Sciences 4 (2009), 261-264.
[16] S. Şenyurt - Y. Altun - C. Cavahir. *Smarandache curves according to Sabban frame for Darboux vector of Mannheim partner curve*, American Intitute of Physics, (2017), http://dx.doi.org/10.1063/1.4981672.
[17] A. Yildirim - F. Kaya, *Mannheim Partner Curves According to Darboux Frame in the Euclidean 3-space* $\mathbb{E}^3$, Mathematical Sciences and Applications E-Notes 8 (2020), 54-59.
[18] B. Yilmaz, *Some curve pairs according to types of Bishop frame*, International Journal of Mathematics Combinatorics 2 (2021), 84-91.
[19] F. Wang - H. Liu, *Mannheim partner curves in 3-space*, Journal of Geometry 88 (2008), 120-126.

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