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ASYMPTOTICS IN A SMALL PARAMETER OF THE SOLUTION OF AN ELLIPTIC BOUNDARY VALUE PROBLEM IN A TWO-LAYER CYLINDER

M. MAHMUDOV[1], A. TAHIROVA[1]

[1]Azerbaijan Technical University, Baku, Azerbaijan
e-mail: maxmudov45@mail.ru, tahirova70@bk.ru

Abstract. The present paper was devoted to the study of asymptotical solution in small parameter of a boundary value problem for a second order elliptic equation with variable coefficients in a thin two-layer cylinder on the surface of which the coefficients of the equation have a discontinuity.

For studying asymptotics we used the idea of the A.Vishik and L.A.Lusternik method and the method of agreement of asymptotic expansions containing a small parameter.

1. Introduction

Many problems of mechanics and mathematical physics lead to necessity of construction of asymptotics of the solution of elliptic boundary value problem in small parameter when a small parameter enters into the geometry of the domain.

L.A.Lusternik and M.I.Vishik have developed a technique for constructing asymptotics of the solution [6] for a wide class of linear partial equations. In a more general statement these problems were studied by this method in the papers of A.L.Goldenweiser [1].

Asymptotics method for solving a boundary value problem for an elliptic equation was justified in the papers of M.G.Javadov [3].

In the papers of E.G.Ivanik [2] the solution of a dynamical problem of thermoelasticity was obtained for a two-layer (composite) cylinder.

The present paper is devoted to the study of asymptotic solution in small parameter of a boundary value problem for a second order elliptic equation with variable coefficients in a thin two-layer cylinder on whose interface the coefficients of the equation have discontinuity.

Let Q be a cylinder in n- dimensional space with the boundary Γ and $(n-1)$-dimensional surface γ divide Q into two cylinders Q^+ and Q^-. Let Q be a cylinder of height $2h$ small with respect to others measuring of the domain Q. Assume that in n-dimensional cylinder one coordinate axis, for example x_n was directed in height of the cylinder $2h$. Interface γ divides Q into two cylinders, and in γ the coefficients have discontinuity that are given by glueing conditions (sewing).

We direct the axis x_n in height $2h$ and devote the lateral surface of Q by F.

In $Q^+\bigcup Q^-\bigcup\Gamma$ we consider the problem

$$L_1u \equiv \sum_{i,j=1}^{n} a_{ij}^{(1)}(x)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^{n} a_i^1(x)\frac{\partial u}{\partial x_i} - a^1(x)u = 0 \ \ in \ \ Q^+, \tag{1}$$

$$L_2w \equiv \sum_{i,j=1}^{n} a_{ij}^{(2)}(x)\frac{\partial^2 w}{\partial x_i\partial x_j} + \sum_{i=1}^{n} a_i^{(2)}(x)\frac{\partial w}{\partial x_i} - a^2(x)w = 0 \ \ in \ \ Q^-, \tag{2}$$

$$u|_{x_n=+h} = w|_{x_n=-h} = p, \; u|_{\gamma+} = w|_{\gamma-}\,, \; \left.\frac{\partial u}{\partial x_n}\right|_{\gamma+} = \left.\frac{\partial w}{\partial x_n}\right|_{\gamma-}\,, \tag{3}$$

$$\left[h\frac{\partial u}{\partial v} + Au\right]_{\gamma+} = \left[h\frac{\partial w}{\partial v} + Aw\right]_{\gamma-} = 0, \tag{4}$$

where $a_{ij}^{(k)}(x) = a_{ji}^{(k)}(x); \sum a_{ij}^{(k)}(x)\xi_i\xi_j \geq \alpha \sum \xi_i^2$; any real $\xi_1, \xi_2, \ldots, \xi_n$,

$$a^{(k)}(x) > 0; \; k = 1,2; \; \alpha = const > 0; A > 0$$

$$\frac{\partial u}{\partial v} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\cos\left(\bar{n}, x_j\right),$$

h is a small parameter, $p(x_1, x_2, \ldots, x_{n-1})$ is a given smooth function.

The goal is to construct asymptotics of the solution of problem (1)-(4) in small parameter h.

We introduce new variables $x_n = th$ then problem (1)-(4) will be written as

$$L_1 u \equiv \frac{1}{h^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{h}M_1^{(1)}u + M_2^{(1)}u = 0, \;\; in \;\; Q^+, \tag{5}$$

$$u|_{t=+1} = p; \; u|_{\gamma+} = w|_{\gamma-}\,; \; \frac{1}{h}\left.\frac{\partial u}{\partial t}\right|_{t=+0} = \left.\frac{\partial w}{\partial x_n}\right|_{\gamma-}\,, \tag{6}$$

$$\left[\frac{\partial u}{\partial v} + u\right]_{F1+} = 0, \tag{7}$$

$$L_2 w \equiv \frac{1}{h^2}\frac{\partial^2 w}{\partial t^2} + \frac{1}{h}M_1^{(2)}w + M_2^{(2)}w = 0, \;\; in \;\; \bar{D}, \tag{8}$$

$$w|_{t=-1} = p; \quad w|_{\gamma-} = u|_{\gamma+}\,; \; \frac{1}{h}\left.\frac{\partial w}{\partial t}\right|_{t=-0} = -\left.\frac{\partial u}{\partial x_n}\right|_{\gamma+}\,, \tag{9}$$

where $(x_1, x_2, \ldots, x_{n-1}, x_n) = (r, th)$ and $\mu_i^{(1)}, \mu_2^{(1)}, \mu_i^{(2)}, \mu_2^{(2)}$, are the known differential expressions. Sufficient smoothness of the coefficients (5)-(8) are assumed. We expand them by Taylor formula in powers th and making some simplification we get problems whose asymptotical expansion of the solution will be sought in the form

$$u = \frac{u_{-1}}{h} + u_0 + hu_1 + h^2 u_2 + \cdots, \tag{10}$$

$$w = \frac{w_{-1}}{h} + w_0 + hw_1 + h^2 w_2 + \cdots. \tag{11}$$

Having put expansion (10)-(11) in the obtained problems and equating the coefficients at the same powers of h, in the cylinder Q^+ and Q^- we get problems for the function $u_{-1}, u_0, u_1, \ldots, w_{-1}$ $w_0, \; w_1, \ldots$ determine all the functions $u_i(x,t)$and $w_i(x,t), \; i = -1, 0, 1, \ldots$. The found functions are generally speaking do not satisfy all boundary conditions, i.e. do not satisfy the boundary conditions on the lateral surface.

Therefore to the found solution we should add a boundary layer type function so that the obtained sum satisfy all boundary conditions. In order to construct these functions in rather small vicinity of F_1 and F_2 we introduce the local coordinates $\rho = \tau h$, where ρ is the distance in normal to F, while $y = (y_1, y_2, \ldots, y_{n-2}, t)$ are the bases of the normal. Having written equation (5),(8) boundary conditions with a new variable and making a change of variable $\rho = \tau h$ in the obtained expression we will look for the solution in the form

$$u^{(1)} = \frac{u_{-1}^{(1)}}{h} + u_0^{(1)} + hu_1^{(1)} + \cdots , \tag{12}$$

$$w^{(1)} = \frac{w_{-1}^{(1)}}{h} + w_0^{(1)} + hw_1^{(1)} + \cdots . \tag{13}$$

Substituting expression $u^{(1)}$ and $w^{(1)}$ from (12) and (13) in the problem that are obtained after change of $\rho = \tau h$ note that the function u and $u^{(1)}$ in Q^+ at each step are connected between themselves by the boundary conditions.

As the equations satisfied by the functions $u_k^{(1)}$ and $w_k^{(1)}$ are the equations with partial derivatives, then the finding of a boundary layer type solution of these problems is difficult. Therefore, we try to find their approximate solutions provided that the admissible error does not exceed h^{n+1} in the corresponding norm.

For solving problem (1)-(4) we get the asymptotic representation

$$u = \sum_{i=-1}^{n+2} h^i u_i + \sum_{j=-1}^{n+2} h^j u_j^{(1)} + zh^{n+1} \ \ in \ \ Q^+ \tag{14}$$

and

$$w = \sum_{i=-1}^{n+2} h^i w_i + \sum_{j=-1}^{n+2} h^j w_j^{(1)} + zh^{n+1}, \ \ in \ \ Q^-, \tag{15}$$

where $h^{n+1}z$ is a residual term.

We have the following theorem.

Theorem. Let $p\left(x_1, x_2, \ldots, x_{n-1}\right)$ be a rather smooth function. Then for solving problem (1)-(4) we have asymptotic representation of the form (14), (15) where the residual term tends to zero as have with velocity $h \to 0$ in the h^{n+1} metrics of the space $L_2\left(Q\right)$.

Keywords: Parameter, domain, method, cylinder, surface, condition, expansion, boundary layer, metrics, space.

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