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Analytical Study of Algebra of Operators on Algebraic Tensor Products

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Abstract

This paper presents the study of the Algebra of Operators on Algebraic Tensor Products. Here, we consider R additive group of reals with discrete topology constructing C* - algebras canonically associated with R – along with A and B are C* - algebras, it is proved in this paper that all C* - algebras have several distinct C* - norms on Algebraic tensor product A ʘ B are mutually distinct.

Keywords: Algebraic Tensor Products, C*-Algebras, W*-Algebras, C*-Tensor Norms, Normal and Binormal Norms, Hilbert Space.

Introduction

The present work is an attempt towards the generalization of work done by E.G. EFFROS (1) and Kothe (4, 5). In fact, the present area is the extension of work done by Ghosh, H.C. (2), Halub, J.R. (3), Kumar et al. (6), Srivastava et al. (7), Srivastava et al. (8), Srivastava et al. (9) and Tomiyama, J. (10). In this paper, we have studied analytically about Algebra of operators on Algebraic Tensor products.

Mathematical Treatment of the Problem:

Let, R = additive group of reals with discrete topology. There are several ways of constructing c*-algebras canonically associated with R. For example if π is the universal representation of R on the Hilbert space H, the group c*- algebra c*(R) is the c*- subalgebra of $\mathcal{L}(H)$ genrated by the set $\{\pi(g) : g \in R\}$. The left regular group c*- algebra $c_\ell^*(R)$ is the c*- subalgebra of $\mathcal{L}(\ell^2(R))$ generated by $\pi\ell(R)$, $\pi\ell$ being the left regular representation of R, when

$$\pi\ell(g)\,\xi(h) = \xi(g^{-1}(h)),\ (g, h \in R, \xi \in \ell^2(R))$$

The right regular group C*-algebra $C_r^*(R)$ is defined analogously using the right regular representation π_r of R, when

$\pi_r(g)\,\xi(h) = \xi(hg),\ (g, h \in R, \xi \in \ell^2(R))$

By its definition π contains π_ℓ and π_r, and there are natural homomorphism λ_ℓ and λ_r of c*(R) onto $c_\ell^*(R)$ and $c_r^*(R)$ resp.

(Note also that the representations π_ℓ and π_r are equivalent, so that $c_\ell^*(R)$ and $c_r^*(R)$ are in fact isomorphic).

Let A and B be c*-algebras, with algebraic tensor product $A \odot B$. In general there are several distinct (usually incomplete).

c*- norms on $A \odot B$. Two such norms are of particular interest: the maximal norm ν of Guichander and the minimal (or spatial) norm α of Turumaru.

If π_1 and π_2 are representations of A and B, respectively, on the Hilbert space H, $\{\pi_1, \pi_2\}$ is said to be a committing pair of representations of A, B if

$\pi_1(a)\,\pi_2(b) = \pi_2(b)\,\pi_1(a) \quad (a \in A, b \in B)$

The norm ν is defined by

$\nu(\sum_i a_i \otimes b_i) = \sup(\|\sum_i \pi_1(a_i)\,\pi_2(b_i)\|)$

the supremum being taken over all commuting pairs of representations of A, B. The norm α is defined as follows:

if $x \in A \odot B$, $\alpha(x)$ is the smallest non-negative real number $\mathcal{R}$ such that

$< f \otimes g, a^*x^*\,x\,a > \ \le\ \mathcal{R}^2 < f \otimes g, a^*a >$

For all $a \in A \odot B$ and all satisfies f and g of A and B respectively. If for all B$\alpha = \nu$ on $A \odot B$, A is said to be nuclear (the terminology in due to Lance, which an introduction to the theory of c*- tensor product. For a discrete group R, $c_\ell^*(R) = c^*(R)$ iff G is amenable, and this is the case iff $c_\ell^*(G)$ is nuclear.

Let R be a discrete group and let be the representation of $c^*(R) \odot C^*(R)$ on $\ell^2(R)$ given by

$\lambda(\Sigma\, a_i \otimes b_i) = \Sigma\, \lambda_i(a_i)\,\lambda_r(b_i)$

It is natural to ask relative to which c*- norms η on $c^*(R) \odot c^*(R)$, λ is bounded. If $\eta = \lambda$, is already bounded with $\eta = \alpha$, the same is true if R is amenable (by the remark above). One of the main results of this section is essentially that $R = F_2$, the free group on two generators, λ is bounded when $\eta = \alpha$. A consequence of this fact is that, even in the separable case, it is not always possible to describe the ideal structure of the spatial tensor product of two c*-algebras in terms of the ideal structures of the individual algebras. It follows moreover that the spatial c*- norm is not in general preserved by quotients, (by quotients respect the maximal norm ν).

EFFROS and Lance have recently introduced two new c*- tensor norms, the normal and binormal norms. Let A and B be c*-algebras.

If A is a W*- algebra, the left normal norm ν_ℓ on $A \odot B$ is defined by

$$\nu_\ell(\Sigma\, a_i \otimes b_i) = \sup(\| \Sigma\, \pi_1(a_i)\, \pi_2(b_i) \|)$$

the supremum being taken over all commuting pairs of representations $\{\pi_1, \pi_2\}$ of A, B with π_1 normal. If B is a W*-algebra, the right normal norm ν_r is defined analogously with π_2 rather π_1 required to be normal. If A and B are both W*-algebras, the binormal norm β is given by a similar expression, the supremum being taken this time over all commuting pairs of normal representations of A, B. The notation and terminology used here follow those of for the most part.

If A and B are c*- algebras and η is a c*- norm on $A \odot B$, then η- completion of $A \odot B$ will be denoted $A \otimes_\eta B$, the α- completion will be denoted simply by $A \otimes B$. If B is a c*- subalgebra of A, a linear map $\rho : A \to B$ is a refraction, if it is a projection of norm 1, i.e. if $\|\rho\| = 1$ and $\rho(x) = x$ for $x \in B$. Finally, all c*- algebras homomorphism and representations will be assumed to be *- preserving.

Now, we discuss the five norms α , ν_l, ν_r , β and ν on $A \odot A$. Latter, we see that these all five norms are mutually distinct. Let A and B be C* - algebras with algebraic tensor product $A \odot B$. In general, there are several distinct (Usually incomplete) C* - norms on $A \odot B$. Two such norms are of particular interest: the maximal norm ν and the minimal (or spatial) norm α.

If $\prod_1$ and $\prod_2$ are representations of A and B, respectively, on the Hilbert Space H, $\{\prod_1 , \prod_2\}$ is said to be a commuting pair of representations of A, B if

$$\textstyle\prod_1(a) \prod_2(b) = \prod_2(b) \prod_1(a), \quad (a \in A,\ b \in B).$$

The norm ν is defined by

$$\nu\Big(\sum_i a_i \otimes b_i\Big) = \mathrm{Sup}\Big(\Big\|\sum_i \textstyle\prod_1(a_i) \prod_2(b_i)\Big\|\Big).$$

Main Result

Proposition: Let $A = M \oplus N$; then the five norms α , ν_l , ν_r , β and ν on $A \odot A$ are mutually distinct.

Lemma:

Let M_1 , M_2 and B be W* - algebras. Then the canonical isomorphism

$$(M_1 \oplus M_2) \odot B \cong (M_1 \odot B) \oplus (M_2 \odot B)$$

extends to an isomorphism of

$$(M_1 \oplus M_2) \otimes_\eta B \text{ onto } (M_1 \otimes_\eta B) \oplus (M_2 \otimes_\eta B),$$

When η is any of the above five norms.

Proof of Lemma:

Let e and f be the identity projections of M_1 and M_2, respectively; then, $e + f = 1$. Let $\{\prod , \prod'\}$

be a commuting pair of representations of $M_1 \oplus M_2$, B on the Hilbert Space H. $\prod (e)$ and $\prod (f)$ commute with $\prod (M_1 \oplus M_2)$ and $\prod' (B)$, so that

$$H_1 = \prod (e)H \text{ and } H_2 = \prod (f)H$$

are invariant subspaces for $\prod$ and $\prod'$.

Let $\prod_i = \prod/H_i$, $\prod'_i = \prod'/H_i$ $(i = 1 , 2)$;

then $\{\prod_1 , \prod'_1\}$ and $\{\prod_2 , \prod'_2\}$

are commuting pairs of representations of $M_1 \oplus M_2$, B on H_1 and H_2 respectively.

Moreover, $\prod$ is normal if and only if $\prod_1$ and $\prod_2$ are ; and for

$$\sum x_i \otimes b_i \in M_1 \odot B, \sum y_j \otimes c_j \in M_2 \odot B,$$

$$\| \sum\prod(x_i) \prod'(b_i) + \sum\prod(y_j) \prod'(c_j)\|$$

$$= \max (\|\sum\prod_1(x_i) \prod'_1(b_i)\|, \|\sum\prod_2(y_j) \prod'_2(c_j)\|)$$

The Lemma follows easily from this relation and the definitions of the various norms.

Proof of Proposition:

In view of the lemma it is sufficient to check that any two of the norms α , ν_l , ν_r , β and ν differ on at least one of the algebraic tensor products $M \odot M$, $M \odot N$,

$N \circledcirc M$ and $N \circledcirc N$,

(i) on $M \circledcirc N$, $\alpha = \nu_l = \beta$.

In the notation of the homomorphisms

$$x \rightarrow \Phi(x), (x \in M)$$

and $$y \rightarrow R(\tilde{y}), (y \in N)$$

constitute a commuting pair of representations of M, N on H(N), the second representation being normal. Thus the homomorphism.

$\sum x^i \otimes y^i \rightarrow \sum \Phi(x^i) R(\tilde{y}^i)$; $M \circledcirc N \rightarrow L(H(N))$ is continuous relative to the norm ν_r on $M \circledcirc N$ and also if it is not continuous relative to α, so that $\alpha \neq \nu_r \leq \nu$ on $M \circledcirc N$.

(ii) In exactly the same way

$$\alpha = \nu_r = \beta \neq \nu_l \leq \nu \text{ on } N \circledcirc M.$$

(iii) The representation

$$\sum x_i \otimes y_i \rightarrow x_i R(\tilde{y}_i) \text{ of } N \circledcirc N \text{ on } H(N)$$

is clearly continuous relative to the norm β on $N \circledcirc N$.

Again by the other relevant proposition, this representation is not continuous relative to α.

Thus, $\alpha \neq \beta$ on $N \circledcirc N$. Thus the proposition is now completed. Hence the result.

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References

[1]. E.G. Effros and E.C. Lance . (1970): Tensor Products of Operator Algebras.

[2]. Ghosh, H.C., Yadav, S.K., and Srivastava U.K., (2021): International Journal of Recent Development in Engineering and Technology, Vol. 10(9), pp. 01-03

[3]. Halub, J.R. (1970): Tensor Product Mapping Math Ann., Vol. 188, pp 01-12.

[4]. Kothe , G. (1969): Topological Vector Spaces , Springer-Verlag, I, New York.

[5]. Kothe, G. (1979): Topological Vector Spaces , Springer-Verlag, II, New York.

[6]. Kumar Nirmal, Choudhary, D., Talukdar, A.U., Srivastava, U.K., (2016): International Journal of Emerging Technology and Advanced Engineering Vol. 6(11), pp 170-173.

[7]. Srivastava , U.K.,Kumar, S., and Singh, T.N. (2011): J.P.A.S (Mathematical Science), Vol. 17, pp 188-191.

[8]. Srivastava, U.K., kumar, N., Kumar, S., and Singh, T.N. (2012): Proc. Of Math. Soc., B.H.U. Vol. 28, pp 25-28.

[9]. Srivastava, U.K., Roy, L.K.; Prasad, B; Pathak, C.D., and Ray, S. (2020): Journal of Mathematics and statistical science, U.S.A. Vol. 06 Issue (05), pp. 182-186.

[10]. Tomiyama, J. (1971): Tensor Products and Projection of Norms. One in VON-NEUMANN Algebras.