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The Neighborhood Set and the Neighbourhood Graph of the Nilpotent Cayley Graph of the Residue Class Ring $(Z_n,\oplus,\odot)$

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Abstract

The properties of the nilpotent Cayley graph $G(Z_n, N)$ associated with the set of nilpotent elements of the residue class ring $(Z_n,\oplus,\odot)$ and its vertex as well as edge domination parameters are studied by the authors. In this Paper the concepts of the neighborhood set, the neighborhood number and the neighborhood graph of the nilpotent Cayley graph $G(Z_n,N)$ are introduced and its basic properties are studied. Further it is established that the neighborhood graph $G(Z_n,N)$ is multi- partite and Hamiltonian.

Keywords: Nilpotent element, Nilpotent Cayley graph, Neighborhood graph of a graph.

AMS Subject Classification(2010): 05C25, 05C69,05C72

1. Introduction

The concept of a Cayley graph was introduced to study, whether given a group $(X,.)$, there is a graph Γ, whose automorphism group is isomorphic to the group $(X,.)$ [11]. Extensive studies have been carried out on the Cayley graphs by many graph theorists [3, 4, 15]. Given a group $(X,.)$ and a symmetric subset S of X, (a subset S of a group $(X,.)$ is called a symmetric subset, if $s^{-1}\in S$ for every $s\in S$) is the graph $G(X,S)$, whose vertex set V is X and the edge set $E = \{(x, y) / \text{ either } xy^{-1} \in S, \text{ or } yx^{-1} \in S\}$. If S does not contain e, the identity element of the group $(X,.)$, then $G(X,S)$ is a simple undirected graph. Further $G(X,S)$ is $|S|$-regular and contains $\frac{|X||S|}{2}$ edges [15]. Madhavi [15] introduced Cayley graphs associated with the arithmetical functions, namely, the Euler totient function $\varphi(n)$, the set of quadratic residues modulo a prime p and the divisor function $d(n)$, $n\geq 1$, an integer and obtained various properties of these graphs.

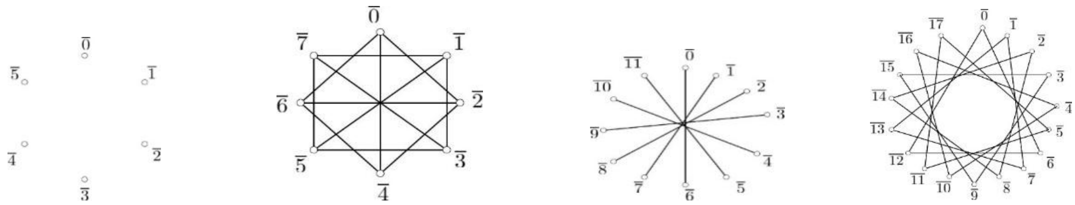
Sampath Kumar and Neerlagi [20] introduced the concept of the neighborhood number of a graph G and determined this parameter for various known graphs as well as many bounds for it. Also they studied its relationship with some well known domination parameters of G like vertex domination and edge domination. Kulli and Sigarkanthi [13] obtained Nordhaus-Gaddum type results for the neighborhood number of a graph G. Kulli and Sonar [14], Cheluvaraju [6] and Fricke *et al.*,[9] obtained some additional parameters of the neighborhood number. Kulli [12] has introduced the concept of the neighborhood graph of a graph and studied some of its properties. Further, Kulli [12] gave characterizations of the graphs (i) whose neighborhood graphs are connected, (ii) whose neighborhood graphs are r-regular and (iii) whose neighborhood graphs are eulerian. In this paper, the authors introduce a variant form of neighborhood graph of a graph and characterize the neighborhood graph of the nilpotent Cayley graph $G(Z_n,\ N)$ associated with the set of nilpotent elements in the residue class ring $(Z_n, \oplus, \odot), n \geq 1$.

In recent times Chen [7], Nikmehr and Khojasteh [19] and Dhiren Kumar Basnet *et al.*, [8] have studied the nilpotent graphs associated with a finite commutative ring R and the $n \times n$ matrix ring $M_n(R)$. In [16, 17] the authors have studied a new class of arithmetic Cayley graphs, namely, the nilpotent Cayley graphs associated with the set of nilpotent elements in the residue class ring $(Z_n, \oplus, \odot), n \geq 1$, an integer and its vertex domination. An element $\bar{a} \neq \bar{0}$, in the ring $(Z_n, \oplus, \odot)$ is called a nilpotent element, if there exists a positive integer l such that $(\bar{a})^l = \bar{0}$. It is an easy verification that the set N of all nilpotent elements in the ring $(Z_n, \oplus, \odot)$ is a symmetric subset of the group $(Z_n, \oplus)$. The Cayley graph $G(Z_n,\ N)$ associated with the group $(Z_n, \oplus)$ and its symmetric subset N, is the graph whose vertex set V is Z_n and the edge set $E = \{(x, y)/x\ , y \in Z_n$ and either $x - y \in N$, or, $y - x \in N\}$ and it is called the **nilpotent Cayley graph of the ring $(\boldsymbol{Z_n}, \oplus, \odot)$**.

In [16], it is proved that, if $n = \prod_{i=1}^{r} {p_i}^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes, $\alpha_i \geq 1$ and $1 \leq i \leq r$, are integers and $m = p_1 p_2 p_3 \ldots p_r$, then

i. the graph $G(Z_n, N)$ is $(\prod_{i=1}^{r} {p_i}^{\alpha_i - 1} - 1) -$ regular and contains $\frac{n}{2}(\prod_{i=1}^{r} {p_i}^{\alpha_i - 1} - 1)$ edges,
ii. the graph $G(Z_m, N)$ contains only vertices and
iii. the graph $G(Z_n,\ N)$ is a union of m disjoint connected components of $G(Z_n,\ N)$, each of which is a complete subgraph of $G(Z_n,\ N)$.

The graphs of $G(Z_6, N), G(Z_8, N), G(Z_{12}, N)$ and $G(Z_{18}, N)$ are given below:



$\boldsymbol{G(Z_6, N)}$ $\boldsymbol{G(Z_8, N)}$ $\boldsymbol{G(Z_{12}, N)}$ $\boldsymbol{G(Z_{18}, N)}$

Figure 1.1

The terminology and notations that are used in this paper can be found in [5] for graph theory, [10] for algebra and [2] for number theory.

2. The Neighborhood Set and the Neighborhood Number of the Nilpotent Cayley Graph $G(Z_n, N)$

Let G be a graph with the vertex set V and the edge set E. For any subset S of V, the **induced** subgraph $\langle S \rangle$ is the graph, whose vertex set is S and the edge set consists of all those edges in E that have both endpoints in S. For any vertex $v \in V$ the **neighborhood** $N(v)$ of v is the set $N(v) = \{u \in V: (u, v) \in E\}$, and the **closed neighborhood** $N[v]$ of v is the set $N[v] = N(v) \cup \{v\}$. A subset S of the vertex set V of G is called a **neighborhood set** of G, if $G = \cup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph of G induced by $N[v]$. A **minimum neighborhood set** of a graph G, is the neighborhood set of G with minimum cardinality. The cardinality of a minimum neighborhood set of the graph G is called the **neighborhood number** of G and it is denoted by $n_0(G)$.

Example 2.1: Consider the graph G, whose vertex set V and edge set E, are respectively given by

$$V = \{a, b, c, d, e, f, g\} \text{ and } E = \{(a, b)(b, c)(b, g)(c, d)(d, e)(d, f)(e, f)(f, g)\}.$$

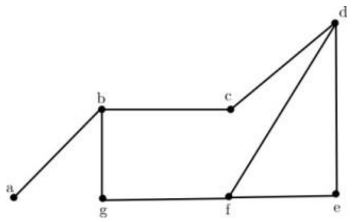


Figure 2.1

The closed neighborhoods of the vertices are $N[a] = \{a, b\}$, $N\,[b] = \{a, b, c, g\}$, $N\,[c] = \{b, c, d\}$, $N\,[d] = \{c, e, d, f\}$, $N\,[e] = \{d, e, f\}$, $N[f] = \{d, e, f, g\}$, $N\,[g] = \{b, f, g\}$. The induced subgraphs of these closed neighborhoods of the vertices are given in the Table 2.1.

Induced subgraph	Vertex set	Edge set
$\langle N(a) \rangle$	$\{a, b\}$,	$\{(a, b)\}$
$\langle N(b) \rangle$	$\{a, b, c, g\}$	$\{(a, b)(b, c)(b, g)\}$
$\langle N(c) \rangle$	$\{b, c, d\}$	$\{(b, c)(c, d)\}$
$\langle N(d) \rangle$	$\{c, e, d, f\}$	$\{(c, d)(e, d)(e, f)(d, f)\}$
$\langle N(e) \rangle$	$\{d, e, f\}$	$\{(d, e)(d, f)(e, f)\}$
$\langle N(f) \rangle$	$\{d, e, f, g\}$	$\{(d, e)(d, f)(e, f)(f, g)\}$
$\langle N(g) \rangle$	$\{b, f, g\}$	$\{(b, g)(f, g)\}$

Table 2.1

One can observe that $G = \langle N(b) \rangle \cup \langle N(c) \rangle \cup \langle N(f) \rangle$. So the vertex set $\{b, c, f\}$ is a neighborhood set of G. Similarly one can see that the vertex set $\{a, d, g\}$ is another neighborhood set of G. Each of the above neighborhood sets of G contains three vertices. It can also be seen that no other subset of the vertex set containing less than three vertices of G, is not a neighborhood set of G. So each one of the vertex sets, $\{b, c, f\}$ and $\{a, d, g\}$ is a minimum neighborhood set of G and the neighborhood number $n_0(G)$ is 3.

Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes, $\alpha_i \geq 1, 1 \leq i \leq r$ are integers, such that $\alpha_i > 1$, for at least one i, and let $m = p_1 p_2 \dots p_r$. Consider the subsets $C_0, C_1, C_2, \dots, C_{m-1}$ of the vertex set of $G(Z_n, N)$, where

$$C_k = \left\{\bar{k}, \overline{m+k}, \overline{2m+k}, \dots, \overline{\imath m+k}, \dots, \overline{\left(\frac{n}{m}-1\right)m+k}\right\}, 1 \leq k \leq m-1.$$

Lemma 2.2: For $0 \leq k \leq m-1$, $C_k = N[\bar{k}]$, the closed neighborhood of $\bar{k}$ and the subgraph $\langle N[\bar{k}] \rangle$ induced by the closed neighborhood $N[\bar{k}]$ is the complete component $\langle C_k \rangle$ of the graph $G(Z_n, N)$.

Proof: For any $\overline{\imath m+k} \in C_k$, $1 \leq i \leq \frac{n}{m}-1$, $1 \leq k \leq m-1$ we have $\overline{\imath m+k} - \bar{k} = \overline{\imath m}$. Since $\bar{m}$ is a nilpotent element of the ring $(Z_n, \oplus, \odot)$ and $i \leq \frac{n}{m} - 1 < \frac{n}{m}$, it follows that $\overline{\imath m}$ is also a nilpotent element of the ring $(Z_n, \oplus, \odot)$, so that each $\overline{\imath m+k}$ is adjacent to $\bar{k}$. That is, $C_k = N[\bar{k}]$, the closed neighborhood of $\bar{k}$. Hence $\langle N[\bar{k}] \rangle = \langle C_k \rangle$. By the Lemma 2.3.5 [16, p42], the subgraph $\langle C_k \rangle$ induced by C_k is a complete subgraph of the graph $G(Z_n, N)$, so that $\langle N[\bar{k}] \rangle$ is a complete subgraph of the graph $G(Z_n, N)$. ■

Theorem 2.3: The subset $S = \{\bar{0}, \bar{1}, \dots, \overline{m-1}\}$ of the vertex set of the nilpotent Cayley graph $G(Z_n, N)$ is a minimum neighborhood set of the graph $G(Z_n, N)$.

Proof: By the Theorem 2.3.7 [16, p43], the nilpotent Cayley graph $G(Z_n, N)$ is a disjoint union of the complete subgraphs $\langle C_0 \rangle, \langle C_1 \rangle, \langle C_2 \rangle, \dots, \langle C_{m-1} \rangle$, where

$$C_k = \left\{\bar{k}, \overline{m+k}, \overline{2m+k}, \dots, \overline{\imath m+k}, \dots, \overline{\left(\frac{n}{m}-1\right)m+k}\right\}, \text{ for } 0 \leq k \leq m-1.$$

But, by the Lemma 2.2**,** $\langle N[\bar{k}] \rangle = \langle C_k \rangle$, for $0 \leq k \leq m-1$. So,

$$G(Z_n, N) = \cup_0^{m-1} \langle C_k \rangle = \cup_{v \in S} \langle N[v] \rangle.$$

This shows that the subset $S = \{\bar{0}, \bar{1}, \dots, \overline{m-1}\}$ of the vertex set of the nilpotent Cayley graph $G(Z_n, N)$ is a neighborhood set of the graph $G(Z_n, N)$.

Let T be the set got by deleting a vertex $\bar{\imath}$ from the set S, that is, $T = S - \{\bar{\imath}\}$, for some $i, 1 \leq i \leq m-1$. Then the subgraph $\cup_{v \in T} \langle N[v] \rangle$ does not contain the induced subgraph $\langle N[\bar{\imath}] \rangle$, of the graph $G(Z_n, N)$, since $\bar{\imath} \notin T$. So, $G(Z_n, N) \neq \cup_{v \in T} \langle N[v] \rangle$. This shows that T is not a neighborhood set of the graph $G(Z_n, N)$, so that S is a minimum neighborhood set of G ■

The following corollary is immediate from the Theorem 2.3.

Corollary 2.4: The neighborhood number $n_0(G(Z_n, N))$ of the graph $G(Z_n, N)$ is m.

Remark 2.5: One can easily see that for $0 \leq 1 \leq m-1$, each of the following subsets

$$S_i = \{\bar{\imath}, \overline{\imath+1}, \overline{\imath+2}, \dots, \overline{\imath+(m-1)}\},$$

of the vertex set of the graph $G(Z_n, N)$ is also a neighborhood set of the graph $G(Z_n, N)$. It is clear that $S_0 = S$.

Example 2.6: Consider the graph $G(Z_{18}, N)$. By the Theorem 2.3, the subset $S = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ of the vertex set of the graph $G(Z_{18}, N)$ is a minimum neighborhood set of the graph $G(Z_{18}, N)$ and its neighborhood number $n_0(G(Z_{18}, N))$ is 6. The vertices in this minimum neighborhood set of the graph $G(Z_{18}, N)$ are denoted by boldface dots, in the graph $G(Z_{18}, N)$ is given below.

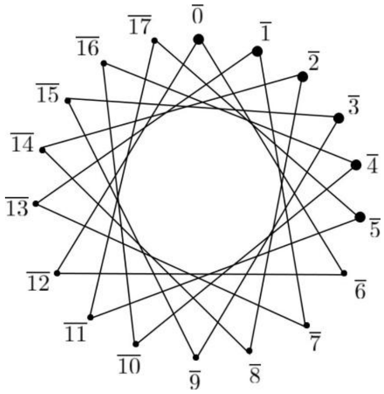


Figure 2.2 $\boldsymbol{G(Z_{18}\,, N)}$

As mentioned in the Remark 2.5, each of the following subsets of vertices of $G(Z_n, N)$ is also a minimum neighborhood set of the graph $G(Z_n, N)$.

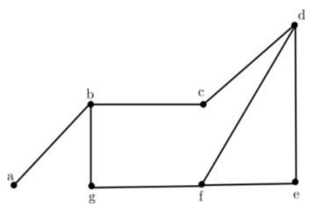
$S_0 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$, $S_1 = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$, $S_2 = \{\overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$,

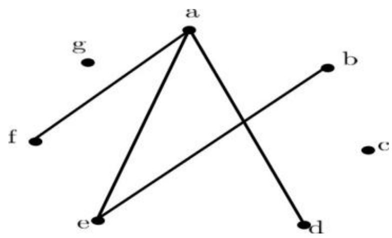
$S_3 = \{\overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}\}$, $S_4 = \{\overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9}\}$, $S_5 = \{\overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9}, \overline{10}\}$.

3. The Neighborhood Graph of the Nilpotent Cayley Graph $\boldsymbol{G(Z_n, N)}$

Let G be a graph with the vertex set V and the edge set E. The graph $N[G]$, whose vertex set $V_N = V$ and the edge set E_N is given by $E_N = \{(u, v): u, v \in V_N \text{ and } N[u] \cap N[v] = \emptyset\}$, is called the **neighborhood graph** of G.

Example 3.1: Consider the graph G given in the Example 2.1, the neighborhood graph $N[G]$ has the vertex set $V_n = \{a, b, c, d, e, f, g\}$ and edge set $E_n = \{(a, d)(a, e)(a, f)(b, e)\}$. The graph G and its neighborhood graph $N[G]$ are given below.





Graph G

The Neighborhood graph $\boldsymbol{N[G]}$

Figure 3.1

Theorem 3.2: If $n = p_1 p_2 p_3 \dots p_r$ where $p_1 < p_2 < \cdots < p_r$ are primes, then the neighborhood graph $N[G(Z_n, N)]$ of the nilpotent Cayley graph $G(Z_n, N)$ is a complete graph isomorphic to K_n, that is, $N[G(Z_n, N)] \cong K_n$.

Proof: If $n = p_1 p_2 p_3 \dots p_r$ where $p_1 < p_2 < \cdots < p_r$ are primes, then by the Lemma 2.2.11 of [16], the nilpotent Cayley graph $G(Z_n, N)$ contains only vertices $\overline{0}, \overline{1}, \dots, \overline{n-1}$ and no edges. So the neighborhoods of $\overline{0}, \overline{1}, \dots, \overline{n-1}$ are respectively given by $N(\overline{0}) = \phi$, $N(\overline{1}) = \phi$, $N(\overline{2}) = \phi$, ... $N(\overline{n-1}) = \phi$, so that the closed neighborhoods of $\overline{0}, \overline{1}, \dots, \overline{n-1}$, are given by

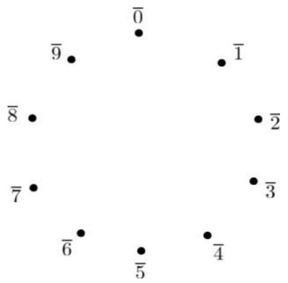
$N[\overline{0}] = \{\overline{0}\}$, $N[\overline{1}] = \{\overline{1}\}$, $N[\overline{2}] = \{\overline{2}\}$, ... , $N[\overline{n-1}] = \{\overline{n-1}\}$.

Since $N[\bar{\imath}] \cap N[\overline{\jmath+1}] = \{\bar{\imath}\} \cap \{\overline{\jmath+1}\} = \phi$, for $0 \le i, j \le n-1,\ i \ne j$, it follows that there is an edge between every pair of distinct the vertices $\bar{\imath}$ and $\overline{\jmath+1}$, of the neighborhood graph $N[G(Z_n, N)]$. Hence the neighborhood graph $N[G(Z_n, N)]$ is a complete graph of n vertices, so that

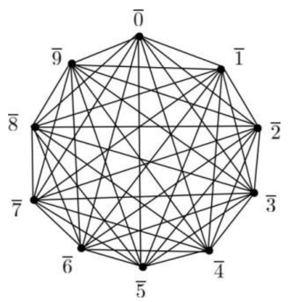
$$N[G(Z_n, N)] \cong K_n. \qquad \blacksquare$$

Example 3.3: For the graph, $G(Z_{10}, N)$ whose vertex set $V = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}\}$ and the edge set E is empty. The closed neighborhoods of the vertices of the graph $G(Z_{10}, N)$ are $N[\bar{0}] = \{\bar{0}\}$,$N[\bar{1}] = \{\bar{1}\}$,$N[\bar{2}] = \{\bar{2}\}$,$N[\bar{3}] = \{\bar{3}\}$,$N[\bar{4}] = \{\bar{4}\}$,$N[\bar{5}] = \{\bar{5}\}$, $N[\bar{6}] = \{\bar{6}\}$, $N[\bar{7}] = \{\bar{7}\}$, $N[\bar{8}] = \{\bar{8}\}$ and $N[\bar{9}] = \{\bar{9}\}$. Since $N[\bar{\imath}] \cap N[\overline{\jmath+1}] = \phi$, for $0 \le i, j \le 9, i \ne j$, it follows that there is an edge between the vertices $\bar{\imath}$ and $\overline{\jmath+1}$, for $0 \le i \le 9$. Hence the neighborhood graph $N[G(Z_n, N)]$ is a complete graph of 10 vertices, that is $N[G(Z_{10}, N)] \cong K_{10}$.

The graph $G(Z_{10}, N)$ and the neighborhood graph $N[G(Z_{10}, N)]$ are given below:



Graph $G(Z_{10}, N)$



The neighborhood graph $N[G(Z_{10}, N)] \cong K_{10}$

Figure 3.2

Theorem 3.4: If $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes, $\alpha_i \ge 1, 1 \le i \le r$ are integers, such that $\alpha_i > 1$ for at least one i and if $m = p_1 p_2 \dots p_r$, then the neighborhood graph $N[G(Z_n, N)]$ of the nilpotent Cayley graph $G(Z_n, N)$ is a **complete m −partite graph** and

$$N[G] \cong K_{\frac{n}{m}, \frac{n}{m}, \cdots, \frac{n}{m}}(m \text{ times}).$$

Proof: Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes, $\alpha_i \ge 1, 1 \le i \le r$ are integers, such that $\alpha_i > 1$ for at least one i, and let $m = p_1 p_2 \dots p_r$. In the Theorem 3.5[17], it is established that $G(Z_n, N)$ is a disjoint union of the subsets $C_0, C_1, C_2, \dots, C_{m-1}$ of the vertex set of $G(Z_n, N)$, where

$$C_k = \left\{\bar{k}, \overline{m+k}, \overline{2m+k}, \dots, \overline{\imath m+k}, \dots, \overline{\left(\frac{n}{m}-1\right)m+k}\right\}, 1 \le k \le m-1.$$

By the Lemma 2.2, C_k is the closed neighborhood $N[\bar{k}]$ of $\bar{k}$, for $1 \le k \le m-1$. So, for$\overline{\imath m+k} \in C_k$, $\overline{\jmath m+l} \in C_l$, $0 \le i, j \le \frac{n}{m}$, we have $N[\overline{\imath m+k}] = N[\bar{k}] = C_k$, and $N[\overline{\jmath m+l}] = N[\bar{l}] = C_l$. Since $C_k \cap C_l = \phi$, for$0 \le k, l \le m-1$, $k \ne l$, we have $N[\overline{\imath m+k}] \cap N[\overline{\jmath m+l}] = \phi$. **So there is an edge between the vertices $\overline{\imath m+k}$ and $\overline{\jmath m+l}$ for $0 \le k, l \le m-1$, $k \ne l$ and for $0 \le i, j \le \frac{n}{m}$.**

Further for any two distinct vertices $\overline{\imath m+k}$ and $\overline{\jmath m+k}$ in C_k, $N[\overline{\imath m+k}] = N[\overline{\jmath m+k}] = N[\bar{k}]$, so that, $N[\overline{\imath m+k}] \cap N[\overline{\jmath m+k}] = N[\bar{k}] \ne \phi$.

This shows that there is no edge between any two distinct vertices $\overline{\imath m+k}$ and $\overline{\jmath m+k}$ of C_k.

That is, the vertex set $V_n(= V)$ of the neighborhood graph $N[G(Z_n, N)]$ is partitioned into m disjoint subsets namely, $C_0, C_1, C_2, \dots, C_k, \dots, C_{m-1}$, such that there is no edge between any two vertices in C_k, for

$1 \leq k \leq m-1$, and there is an edge between any vertex of C_k and any vertex of C_l , for $0 \leq k, l \leq m-1$, $k \neq l$. From this it follows that the neighborhood graph $N[G(Z_n, N)]$ of $G(Z_n, N)$ is a **complete $\boldsymbol{m}$ −partite graph with the $\boldsymbol{m}$ − partition** $\{\boldsymbol{C_0}, \boldsymbol{C_1}, \boldsymbol{C_2}, \ldots, \boldsymbol{C_{m-1}}\}$ of the vertex set V_n of $N[G(Z_n, N)]$, That is $N[G(Z_n, N)] \cong K_{\frac{n}{m}, \frac{n}{m}, \ldots, \frac{n}{m}(m \text{ times})}$. ■

Corollary 3.5: If $n = 2^2 p_2 p_3 \ldots p_r$, where $2 < p_2 < \cdots < p_r$ are primes, then the neighborhood graph $N[G(Z_n, N)]$ of the nilpotent Cayley graph $G(Z_n, N)$ is a **complete $\frac{\boldsymbol{n}}{\boldsymbol{2}}$ − partite graph**. That is, $N[G(Z_n, N)] \cong K_{2,2,\ldots,2(\frac{n}{2} \text{ times})}$.

Proof: Let $n = 2^2 p_2 p_3 \ldots p_r$, where $2 < p_2 < \cdots < p_r$ are primes. Then $n = 2m$, where $m = 2p_2 \ldots p_r$. By the Theorem 2.2.12 of [16, p36], the graph $G(Z_n, N)$ is a bipartite graph with bipartition (U, V) where $U = \{\bar{0}, \bar{1}, \ldots, \overline{m-1}\}$ and $v = \{\bar{m}, \overline{m+1}, \ldots, \overline{n-1}\}$. The closed neighborhoods of the vertices are given by $N[\bar{\imath}] = \{\bar{\imath}, \overline{m+\imath}\} = C_i$, $0 \leq i \leq m-1$.

So by Theorem 3.4, the neighborhood graph $N[G(Z_n, N)]$ of the graph $G(Z_n, N)$ is a complete $\frac{n}{2}$ −partite graph, since $m = \frac{n}{2}$, with $\frac{n}{2}$ −partition $\{C_0, C_1, \ldots C_{m-1}\}$, where $C_i = \{\bar{\imath}, \overline{m+\imath}\}$. That is,

$$N[G(Z_n, N)] \cong K_{2,2,\ldots,2\left(\frac{n}{2} \text{ times}\right)}.$$ ■

Remark 3.6: If $n = 2^2 p_2 p_3 \ldots p_r$, where $2 < p_2 < \cdots < p_r$ are primes then by the Theorem 2.2.12 of [16, pg36], the graph $G(Z_n, N)$ is a bipartite graph, and by the Corollary 3.5, $N[G(Z_n, N)]$ is a $\frac{n}{2}$-partite graph. **This is an instance of a graph and its neighborhood graph are both partite graphs.**

Example 3.7: Consider the graph $G(Z_{18}, N)$, whose vertex set

$$V = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{10}, \overline{11}, \overline{12}, \overline{13}, \overline{14}, \overline{15}, \overline{16}, \overline{17}\}$$

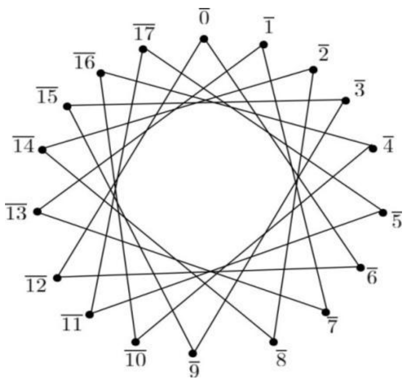
and the edge set

$$E = \{(\bar{0}, \bar{6})(\bar{0}, \overline{12})(\bar{1}, \bar{7})(\bar{1}, \overline{13})(\bar{2}, \bar{8})(\bar{2}, \overline{14})(\bar{3}, \bar{9})(\bar{3}, \overline{15})(\bar{4}, \overline{10})(\bar{4}, \overline{16})$$
$$(\bar{5}, \overline{11})(\bar{5}, \overline{17})(\bar{6}, \overline{12})(\bar{7}, \overline{13})(\bar{8}, \overline{14})(\bar{9}, \overline{15})(\overline{10}, \overline{16})(\overline{11}, \overline{17})\}.$$

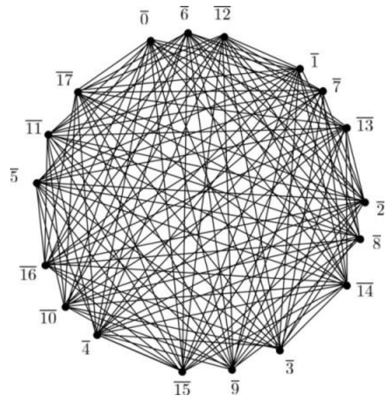
Here $18 = 2.3^2$ and $m = 6$. The closed neighborhoods of the vertices of the graph $G(Z_{18}, N)$ are as follows.

$N[\bar{0}] = \{\bar{0}, \bar{6}, \overline{12}\}$, $N[\bar{1}] = \{\bar{1}, \bar{7}, \overline{13}\}$, $N[\bar{2}] = \{\bar{2}, \bar{8}, \overline{14}\}$, $N[\bar{3}] = \{\bar{3}, \bar{9}, \overline{15}\}$,
$N[\bar{4}] = \{\bar{4}, \overline{10}, \overline{16}\}$, $N[\bar{5}] = \{\bar{5}, \overline{11}, \overline{17}\}$, $N[\bar{6}] = \{\bar{0}, \bar{6}, \overline{12}\}$, $N[\bar{7}] = \{\bar{1}, \bar{7}, \overline{13}\}$
$N[\bar{8}] = \{\bar{2}, \bar{8}, \overline{14}\}$, $N[\bar{9}] = \{\bar{3}, \bar{9}, \overline{15}\}$, $N[\overline{10}] = \{\bar{4}, \overline{10}, \overline{16}\}$, $N[\overline{11}] = \{\bar{5}, \overline{11}, \overline{17}\}$, $N[\overline{12}] = \{\bar{0}, \bar{6}, \overline{12}\}$, $N[\overline{13}] = \{\bar{1}, \bar{7}, \overline{13}\}$ $N[\overline{14}] = \{\bar{2}, \bar{8}, \overline{14}\}$, $N[\overline{15}] = \{\bar{3}, \bar{9}, \overline{15}\}$,
$N[\overline{16}] = \{\bar{4}, \overline{10}, \overline{16}\}$, $N[\overline{17}] = \{\bar{5}, \overline{11}, \overline{17}\}$.

The neighborhood graph $N[G(Z_{18}, N)]$ is a 6 -partite graph with the 6 -partition $\{U_0, U_1, U_2, U_3, U_4, U_5\}$, where $U_0 = \{\bar{0}, \bar{6}, \overline{12}\}$, $U_1 = \{\bar{1}, \bar{7}, \overline{13}\}$, $U_2 = \{\bar{2}, \bar{8}, \overline{14}\}$, $U_3 = \{\bar{3}, \bar{9}, \overline{15}\}$, $U_4 = \{\bar{4}, \overline{10}, \overline{16}\}$, $U_5 = \{\bar{5}, \overline{11}, \overline{17}\}$ of the vertex set of $N[G(Z_{18}, N)]$.



Graph $G(Z_{18}, N)$



$N(G(Z_{18}, N)) \cong k_{3,3,3,3,3,3}$

Figure 3.3

Example 3.8: Conside the graph $G(Z_{12}, N)$, whose vertex set V and the edge set $E$are given by

$V = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9}, \overline{10}, \overline{11}\}$

$E = \{(\overline{0}, \overline{6})(\overline{1}, \overline{7})(\overline{2}, \overline{8})(\overline{3}, \overline{9})(\overline{4}, \overline{10})(\overline{5}, \overline{11})\}$.

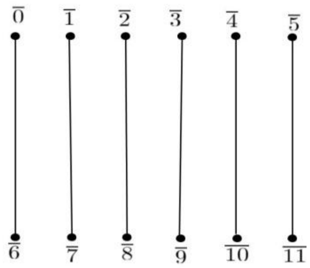
The closed neighborhoods of the vertices of the graph $G(Z_{12}, N)$ are as follows

$N[\overline{0}] = \{\overline{0}, \overline{6}\}$, $N[\overline{1}] = \{\overline{1}, \overline{7}\}$, $N[\overline{2}] = \{\overline{2}, \overline{8}\}$, $N[\overline{3}] = \{\overline{3}, \overline{9}\}$,

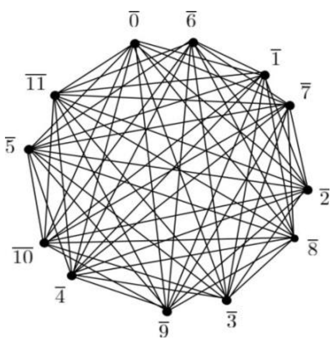
$N[\overline{4}] = \{\overline{4}, \overline{10}\}$, $N[\overline{5}] = \{\overline{5}, \overline{11}\}$, $N[\overline{6}] = \{\overline{0}, \overline{6}\}$, $N[\overline{7}] = \{\overline{1}, \overline{7}\}$,

$N[\overline{8}] = \{\overline{2}, \overline{8}\}$, $N[\overline{9}] = \{\overline{3}, \overline{9}\}$, $N[\overline{10}] = \{\overline{4}, \overline{10}\}$, $N[\overline{11}] = \{\overline{5}, \overline{11}\}$.

The neighborhood graph $N[G(Z_{12}, N)]$ is a 6 -partite graph with the 6 -partition $\{U_0, U_1, U_2, U_3, U_4, U_5\}$,where$U_0 = \{\overline{0}, \overline{6}\}$, $U_1 = \{\overline{1}, \overline{7}\}$, $U_2 = \{\overline{2}, \overline{8}\}$, $U_3 = \{\overline{3}, \overline{9}\}$, $U_4 = \{\overline{4}, \overline{10}\}$, $U_5 = \{\overline{5}, \overline{11}\}$ of the vertex set of $N[G(Z_{12}, N)]$. The graph $G(Z_{12}, N)$ and its neighborhood graph $N[G(Z_{12}, N)]$ are given below:



$G(Z_{12}, N)$



$N[G(Z_{12}, N)] \cong K_{2,2,2,2,2,2}$

Figure 3.4

4. Some Properties of the Neighborhood Graph $N[G(Z_n, N)]$

In the following it is shown that the neighborhood graph $N[G(Z_n, N)]$ is regular and Hamiltonian.

Theorem 4.1: If $n = \prod_{i=1}^{r} {p_i}^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes, $\alpha_i \geq 1, 1 \leq i \leq r$ are integers, such that $\alpha_i > 1$ for at least one i and if $m = p_1 p_2 \ldots p_r$, the neighborhood graph $N[G(Z_n, N)]$ is $(n - \frac{n}{m})$ −regular and the number of edges in $N[G(Z_n, N)]$ is $\frac{n^2}{2m}(m-1)$.

Proof: In the Theorem 3.4, it is proved that the neighborhood graph $N[G(Z_n, N)]$ is a m −partite graph with the m -partition $\{C_0, C_1, C_2, \ldots, C_{m-1}\}$ So the vertex set of $N[G(Z_n, N)]$ is the disjoint union of $C_0, C_1, C_2, \ldots, C_{m-1}$ and very $C_k, 0 \leq k \leq m-1$, contains $\frac{n}{m}$ vertices further vertex $\overline{\imath m + k}, 0 \leq i \leq \frac{n}{m}$ in C_k is adjacent to every vertex in each of $C_0, C_1, C_2, \ldots, C_{k-1}, C_{k+1}, \ldots\ C_{m-1}$ and the vertex $\overline{\imath m + k}$, $0 \leq i \leq \frac{n}{m}$ in C_k is not adjacent to every other vertex in C_k. So the degree of every vertex in C_k is $(n - \frac{n}{m})$. Thus the degree of every vertex in the neighborhood graph $N[G(Z_n, N)]$ is $(n - \frac{n}{m})$ and the graph $N[G(Z_n, N)]$ is $\left(n - \frac{n}{m}\right)$ −regular.

The graph $N[G(Z_n, N)]$ contains n vertices and the degree of each vertex in the graph $N[G(Z_n, N)]$ is $\left(n - \frac{n}{m}\right)$. So the number of edges in $N[G(Z_n, N)]$ is $n\left(n - \frac{n}{m}\right)$. In the above enumeration every edge is counted by each of the end vertices, so that the number of distinct edges in $N[G(Z_n, N)]$ is $\frac{n}{2}\left(n - \frac{n}{m}\right)$, which is equal to $\frac{n^2}{2m}(m-1)$. ■

Theorem 4.2: If $n = \prod_{i=1}^{r} {p_i}^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_r$, are primes, $\alpha_i \geq 1, 1 \leq i \leq r$ are integers, such that $\alpha_i > 1$, for at least one i and if $m = p_1 p_2 \ldots p_r$, then the neighborhood graph $N[G(Z_n, N)]$ is Hamiltonian.

Proof: By the Theorem 3.4, the neighborhood graph $N[G(Z_n, N)]$ is a m −partite graph with the m-partition $\{C_0, C_1, C_2, \ldots, C_{m-1}\}$ of the vertex set of $N[G(Z_n, N)]$, where

$$C_k = \left\{\bar{k}, \overline{m + k}, \overline{2m + k}, \ldots, \overline{\imath m + k}, \ldots, \overline{\left(\frac{n}{m} - 1\right)m + k}\right\}, 1 \leq k \leq m - 1.$$

Since, the vertex $\overline{\imath m + k} \in C_k$, and the vertex $\overline{\jmath m + l} \in C_l$, for $0 \leq k, l \leq m-1,\ k \neq l,\quad 0 \leq i, j \leq \frac{n}{m}$, are adjacent, the vertex set

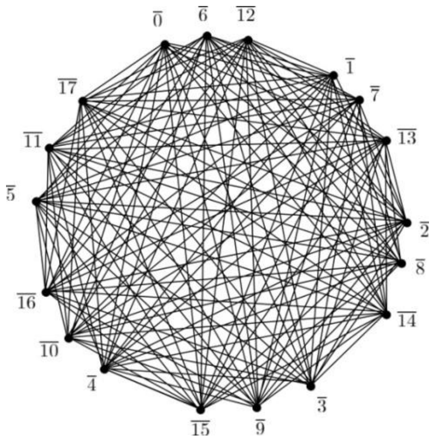
$$H = (\bar{0}, \bar{1}, \bar{2}, \bar{3}, \ldots, \bar{\imath}, \ldots, \overline{n-1}, \bar{0}),$$

is a cycle of length n. Clearly H is a Hamilton cycle in the neighborhood graph $N[G(Z_n, N)]$ and hence it is Hamiltonian. ■

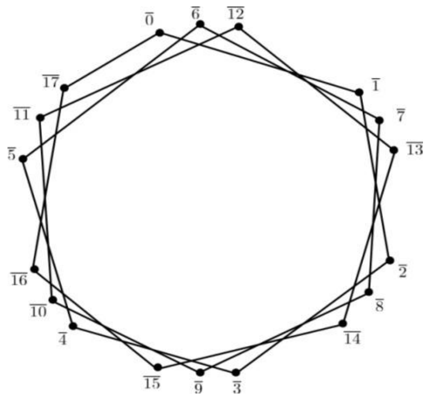
Example 4.3: Consider the neighborhood graph $N[G(Z_{18}, N)]$ of the nilpotent Cayley graph $G(Z_{18}, N)$. By the Theorem 4.2, the cycle

$$H = (\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{10}, \overline{11}, \overline{12}, \overline{13}, \overline{14}, \overline{15}, \overline{16}, \overline{17}, \bar{0}),$$

is a Hamilton cycle in $N[G(Z_{18}, N)]$. The graph $N[G(Z_{18}, N)]$ and its Hamilton cycle H are exhibited below.



$G(Z_{18}, N)$



Hamilton cycle in $N[G(Z_{18}, N)]$

Figure 4.1

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