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Controllability of a Neutral Stochastic Evolution Equation

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Abstract

In this paper the controllability of a stochastic impulsive second order system with infinite delay is investigated. Simple Lipschitz conditions are used to prove the controllability. This approach also removes the need to construct the controllability Gramian operator and associated limit conditions used by the authors in [20], which are practically difficult to verify and apply. An example is provided to illustrate the presented theory.
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1 Introduction

Neutral differential equations are functional differential equations in which the highest order derivative of the unknown function appear both with and without deviations. Neutral differential equations with unbounded delay appear abundantly as mathematical models in mechanics, electrical engineering, medicine, biology, ecology etc. Hence it is a widely studied topic in several papers and monographs for instance, partial neutral differential equation with unbounded delay arise in the theory of heat conduction of materials with fading memory. For instance, one may see [7],[8],[12], [15],[16], and the references cited therein. Second order neutral differential equations model variational problems in calculus of variation and in the study of vibrating

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masses attached to an electric bar. Second order systems model various dynamical systems mostly by nonlinear partial differential equations of second order in time.

Of late, much attention is paid to functional differential equation with infinite delay. We refer [7], [9],[15],[16] for details. The literature related to infinite delay mostly deals with functional differential equations in which the state belongs to a finite dimensional space. As a consequence, the study of partial functional differential equations with infinite delay is neglected. This is one of the motivations of our paper.

Impulsive differential equations are known for their utility in simulating processes and phenomena subject to short term perturbations during their evolution. Discrete perturbations are negligible to the total duration of the process. We refer[5],[10],[13],[17], [22] regarding discrete impulses.

In this paper the controllability of damped the second order stochastic impulsive neutral differential equation with nonlocal conitions, modelled in the following form is studied In this paper we study the existence of solution and control for

$$\begin{aligned} d[x'(t) &+ g(t, x_t, \int_0^t p_1(t,s,x_s)ds)] = [Ax(t) + f(t, x_t, \int_0^t p_2(t,s,x_s)ds) \\ &+ Bu(t)]dt + \int_{-\infty}^t \sigma(t,s,x_s)dw(s), \ t_k \neq t \in J := [0,T] \\ \Delta x(t_k) &= I_k^1(x_{t_k}), \quad k = \{1,2,...,m\} \\ \Delta x'(t_k) &= I_k^2(x_{t_k}), \quad k = \{1,2,...,m\} \\ x'(0) &= x_1 \in \mathbb{H} \\ x_0 &= \phi \in \mathfrak{B} \ for \ a.e. \ s \in J_0 := (-\infty, 0], \end{aligned} \tag{1.1}$$

where $0 < t_1 < t_2 < ... < t_n < T$, $n \in \mathbb{N}$; $x(.)$ is a stochastic process with values in a real separable Hilbert space $\mathbb{H}$. $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous cosine family on $\mathbb{H}$. $x_t : J_0 \to \mathbb{H}$, $x_t(\theta) = x(t+\theta)$ for $t \geq 0$, lies in the phase space $\mathfrak{B}$, defined in preliminaries section. The functions $f, g : J \times \mathfrak{B} \times \mathbb{H} \to \mathbb{H}$, $p : J \times J \times \mathfrak{B} \to \mathcal{L}_2^0$, $p_i : J \times J \times \mathfrak{B} \to \mathbb{H}$, $i = 1,2$, $I_k^1, I_k^2 : \mathfrak{B} \to \mathbb{H}$, $k = 1,...,m$, $q : \mathfrak{B}^n \to \mathfrak{B}$ are appropriate functions to be specified later. The control function $u(t)$ belongs to the space of admissible control functions $L^2(J,U)$ of a separable Hilbert space U. B is a bounded linear operator from U into H. $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$ are prefixed points and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump of the function x at time t_k with I_k, as the size of the jump. $x(t_k^+)$ and $x(t_k^-)$ denotes the right and left limits of $x(t_k)$ at $t = t_k$ respectively.

Likewise $x'(t_k^+)$ and $x'(t_k^-)$ denotes the right and left limits of $x'(t_k)$ at $t = t_k$. Let $\phi(t) \in \mathcal{L}_2(\Omega, \mathfrak{B})$ and $x_1(t)$ be $\mathbb{H}$ valued $\mathcal{F}_t$ measurable random variables independent of the Wiener process $\{w(t)\}$.

Recently, second order abstract partial neutral differential equation similar to (1.1) is extensively studied in [2],[3],[4],[18]. As a matter of fact, in these papers the authors assume severe conditions on the operator family generated by A, which imply that the underlying space X has finite dimension. Thus the equations treated in these works are really ordinary and not partial differential equations. In most of the papers the cosine family generated by the operator A is such that $C(.) \in C([0, T]; \mathcal{L}(X))$ which implies that A is bounded. Hence motivated by this fact their various applications the controllability of the stochastic partial neutral differential equation of second order with infinite delay is studied in this paper using fixed point technique. The compactness condition of the operator families generated by A and other restrictive conditions have been omitted. An example is given in the last section to illustrate the result.

2 Preliminaries

In this section some basic definitions and results for stochastic equations in infinite dimensions and strongly continuous cosine families of operators are recalled. For more details readers can refer [23],[11], [24]. The family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family if the following are satisfied:

(a) $C(0) = I$ (I is the identity operator in X);

(b) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$

(c) The map $t \to C(t)x$ is strongly continuous for each $x \in X$.

The one parameter family of operators $\{S(t) : t \in \mathbb{R}\}$ is the sine family associated to the strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ and it is defined as $S(t)x = \int_0^t C(s)x ds$, $x \in X$, $t \in \mathbb{R}$.

The operator A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in R}$ and $S(t)$ is the associated sine function. Let $N, \widetilde{N}$ be certain constants such that $\|C(t)\| \le N$ and $\|S(t)\| \le \widetilde{N}$ for every $t \in J = [0, T]$. In this work we use the axiomatic definition of phase space $\mathfrak{B}$, introduced by Hale and Kato [7].

$PC([0, a], X)$ is the space formed by normalized piecewise continuous function from $[0, a]$ into X. In particular it is the space PC formed by all functions $u : [0, a] \to X$ such that u is continuous at $t \ne t_i$, $u(t_i^-) = u(t_i)$ and

$u(t_i^+)$ exists for all $i = 1, 2, ..., n$. It is clear that PC endowed with the norm $\|x\|_{PC} = sup_{t\in J}\|x(t)\|$ is a Banach space. For any $x \in PC$

$$\widetilde{x}_i(t) = \begin{cases} x(t), & t \in (t_i, t_{i+1}]; \\ x(t_i^+), & t = t_i, \ i = 1, 2, ..., n. \end{cases} \tag{2.1}$$

So, $\widetilde{x} \in C([t_i, t_{i+1}], X)$.

Definition 1. *[14]:The phase space $\mathfrak{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X endowed with seminorm $\|.\|_{\mathfrak{B}}$ and satisfies the following conditions:*

(A) If $x : (-\infty, \sigma + b] \to X, b > 0$, such that $x_\theta \in \mathfrak{B}$ and $x|_{[\sigma,\sigma+b]} \in C([\sigma, \sigma + b] : X)$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold :
(i) x_t is in $\mathfrak{B}$,
(ii) $\|x(t)\| \leq H\|x_t\|_{\mathfrak{B}}$,
(iii) $\|x_t\|_{\mathfrak{B}} \leq K(t - \sigma)sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathfrak{B}}$,
where $H > 0$ is a constant $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(.)$

(B) The space $\mathfrak{B}$ is complete.

Let $(\mathbb{H}, \|.\|_{\mathbb{H}}, < ., . >_{\mathbb{H}})$ and $(\mathbb{K}, \|.\|_{\mathbb{H}}, < ., . >_{\mathbb{H}})$ denote two real separable Hilbert spaces. $\mathcal{L}(\mathbb{K}, \mathbb{H})$ denotes the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, with the usual operator norm $\|.\|$.

As the system (1.1) has instantaneous impulses,the phase space used by Balasubramaniam and Ntouyas[1] is not applicable to these systems. Therefore we use the abstract phase space $\mathfrak{B}$ defined below.
Let $l : J_0 \to (0, \infty)$ be a continuous function with $l_0 = \int_{J_0} l(t)dt < \infty$.
$\mathfrak{B} = \{\zeta : J_0 \to \mathbb{H} : (E\|\zeta(\theta)\|^2)^{1/2}$ is bounded measurable on $[-t, 0]$ $\forall t > 0$ and $\int_{J_0} l(s) \sup_{\theta\in[s,0]}(E\|\zeta(\theta)\|^2)^{1/2}ds < \infty\}$.
The phase space $\mathfrak{B}, \|.\|_{\mathfrak{B}}$ with the norm $\|\zeta\|_{\mathfrak{B}} = \int_{J_0} l(s) \sup_{\theta\in[s,0]}(E\|\zeta(\theta)\|^2)^{1/2}ds$, $\forall\zeta \in \mathfrak{B}$ is a Banach space.

Let$J_k = (t_k, t_{k+1}]$ and $J_T = (-\infty, T)$. $\mathfrak{B}_T := \{x : J_T \to \mathbb{H}$ with $x|_{(t_k,t_{k+1}]} \in C(J_k, \mathbb{H})$ and $\exists\ x(t_k^-)$ and $x(t_k^+)$ with $x(t_k^-) = x(t_k)$, $x(0) - q(x_{t_1}, x_{t_2}, ...x_{t_n}) = \phi \in \mathfrak{B}$, $k = 1, 2, ..., m\}$.

Define a seminorm $\|.\|$ on $\mathfrak{B}_T$ by $\|x\|_T = \|\phi\|_{\mathfrak{B}} + \sup(E\|x(s)\|^2)^{1/2}$, $x \in \mathfrak{B}_T$.

Lemma 2.1. ([9]) *Let $x \in \mathfrak{B}_T$, then $\forall\ t \in J$, $x_t \in \mathfrak{B}$. Also,*

$$l_0(E\|x(t)\|^2)^{1/2} \leq \|x_t\|_{\mathfrak{B}} \leq \|x_0\|_{\mathfrak{B}} + l_0 \sup_{s\in[0,t]} (E\|x(s)\|^2)^{1/2}$$

Definition 2. *A $\mathcal{F}_t$ adapted cadlag stochastic process $x : J_T \to \mathbb{H}$ is called a mild solution of* (1.1) *on J_T if $x(0) - q(x_{t_1}, x_{t_2}, ..., x_{t_n}) = x_0 = \phi \in \mathfrak{B}$ and $x'(0) = x_1 \in \mathbb{H}$, satisfying $\phi, x_1, q \in \mathcal{L}_2^0(\Omega, \mathbb{H})$; the functions $C(t-s)g(s, x_s, \int_0^s p_1(s,\tau,x_\tau)d\tau)$ and $S(t-s)f(s, x_s, \int_0^s p_2(s,\tau,x_\tau)d\tau)$ are integrable on $[0,T)$ such that*

(i) $\{x_t : t \in J\}$ *is a $\mathfrak{B}-$ valued stochastic process.*

(ii) $\forall\ t \in J,\ x(t)$ *satisfies*

$$\begin{aligned} x(t) &= C(t)[\phi(0)] + S(t)[x_1 + g(0, x_0, 0)] \\ &- \int_0^t C(t-s)g(s, x_s, \int_0^s p_1(s,\tau,x_\tau)d\tau)ds \\ &+ \int_0^t S(t-s)Bu(s)ds \\ &+ \int_0^t S(t-s)f(s, x_s, \int_0^s p_2(s,\tau,x_\tau)d\tau)ds \\ &+ \int_0^t S(t-s)\int_{-\infty}^s \sigma(s,\tau,x_\tau)dw(\tau)ds \\ &+ \Sigma_{0<t_k<t}C(t-t_k)I_k^1(x_{t_k}) + \Sigma_{0<t_k<t}S(t-t_k)I_k^2(x_{t_k}) \end{aligned} \tag{2.2}$$

, *for a.e.* $t \in [t_j, t_{j+1}],\ j = 0, 1, 2, ..., m$

(iii) $\Delta x(t_k) = I_k^1(x_{t_k}),\ \Delta x'(t_k) = I_k^2(x_{t_k}), \quad k = \{1, 2, ..., m\}$

Definition 3. *System* (1.1) *is called controllable on the interval J_T, if $\forall$ initial stochastic process $\phi \in \mathfrak{B}$ defined in J_0, $x'(0) = x_1 \in \mathbb{H}$ and $x_f \in \mathbb{H}$; $\exists$ a stochastic control $u \in L^2(J, U)$ which is adapted to filtration $\{\mathcal{F}_t\}_{t\in J}$ such that the solution $x(.)$ of the system* (1.1) *satisfies $x(T) = x_f$ where x_f and T are preassigned terminal state and time.*

The following hypotheses are made to establish the main results.

(H1) $\exists\ M_C,\ M_S,\ M_{p_1}$ such that $\forall\ t, s, \in J$, and $x, y \in \mathfrak{B}$, $\|C(t)\|^2 \leq M_C$, $\|S(t)\|^2 \leq M_S$; $E\|\int_0^t [p_1(t,s,x) - p_1(t,s,y)]ds\|^2 \leq M_{p_1}\|x-y\|_{\mathfrak{B}}^2$.

(H2) $g : J \times \mathfrak{B} \times \mathbb{H} \to \mathbb{H}$ is continuous and $\exists\ M_g > 0$ such that $\forall\ t \in J,\ x, y, \in \mathfrak{B},\ z_1, z_2 \in \mathcal{L}_2(\Omega, \mathbb{H})$

$$E\|g(t,x,z_1) - g(t,y,z_2)\|^2 \leq M_g(\|x-y\|_{\mathfrak{B}}^2 + E\|z_1 - z_2\|^2).$$

(H3) $\forall\ (t,s) \in J \times J$, $p_2 : J \times J \times \mathfrak{B} \to \mathbb{H}$ is continuous and $\exists\ M_{p_2} > 0$ such that $\forall\ t, s \in J,\ x, y, \in \mathfrak{B}$

$$E\| \int_0^t [p_2(t,s,x) - p_2(t,s,y)]ds\|^2 \leq M_{p_2}\|x-y\|^2_{\mathfrak{B}}.$$

(H4) $f : J \times \mathfrak{B} \times \mathbb{H} \to \mathbb{H}$ is continuous and $\exists\ M_f > 0$ such that $\forall\ t \in J,\ x, y, \in \mathfrak{B},\ z_1, z_2 \in \mathcal{L}_2(\Omega, \mathbb{H})$

$$E\|f(t,x,z_1) - f(t,y,z_2)\|^2 \leq M_f(\|x-y\|^2_{\mathfrak{B}} + E\|z_1 - z_2\|^2).$$

(H5) $I_k^1,\ I_k^2 \in C(\mathfrak{B}, \mathbb{H}),\ \ k = 1,2,...,m$ and $\exists\ M_{I_k^1},\ \overline{M}_{I_k^1},\ M_{I_k^2},\ \overline{M}_{I_k^2}$ such that $\forall\ x, y \in \mathfrak{B}$ $E\|I_k^1(x)\|^2 \leq M_{I_k^1}$, $E\|I_k^2(x)\|^2 \leq M_{I_k^2}$, $E\|I_k^1(x) - I_k^1(y)\|^2 \leq \overline{M}_{I_k^1}\|x-y\|^2_{\mathfrak{B}}$, $E\|I_k^2(x) - I_k^2(y)\|^2 \leq \overline{M}_{I_k^2}\|x-y\|^2_{\mathfrak{B}}$

(H6) $\forall \phi \in \mathfrak{B}$, $h(t) = \lim_{c\to\infty} \int_{-c}^0 \sigma(t,s,\phi)dw(s)$ $\exists$ and is continuous. Also $\exists$ $M_h > 0$ such that $E\|h(t)\|^2 \leq M_h$.

(H7) $\sigma : J \times J \times \mathfrak{B} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ is continuous and $\exists\ M_\sigma > 0,\ \overline{M}_\sigma > 0$ such that $\forall s, t \in J$ and $x, y, \in \mathfrak{B}$ $E\|\sigma(t,s,x)\|^2_{\mathcal{L}_2^0} \leq M_\sigma$ $E\|\sigma(t,s,x) - \sigma(t,s,y)\|^2_{\mathcal{L}_2^0} \leq \overline{M}_\sigma\|x-y\|^2_{\mathfrak{B}}$.

(H8) $W : L^2(J, U) \to L^2(\Omega, \mathbb{H})$ defined as

$$Wu = \int_J S(T-s)Bu(s)ds$$

is a linear operator with induced inverse W^{-1} taking values in $L^2(J,U)/KerW$ [6]. $\exists$ positive constants M_B and M_W such that $\|B\|^2 \leq M_B$ and $\|W^{-1}\|^2 \leq M_W$.

3 Main Results

The controllability of the distributed impulsive stochastic delay system (1.1) in Hilbert spaces is investigated in this section.

Theorem 3.1. *Let* $(H1) - (H9)$ *hold. If* $\lambda < 1$ *and* $\Lambda < 1$ *then the system* (1.1) *is controllable on* J_T*, where*

$$\lambda = 28(1 + 8T^2 M_B M_S M_W)[2l_0^2 T^2\{M_C M_g(1 + 2M_{p_1}) + M_S M_f(1 + 2M_{p_2})\} \quad (3.1)$$

and

$$\begin{aligned}
\Lambda &= \{(12l_0^2 + 60T^2 M_B M_S M_W) \\
&\times [T^2 M_C M_g(1+M_{p_1}) + T^2 M_S M_f(1+M_{p_2}) \\
&+ T^3 M_S \bar{M}_\sigma Tr(Q) \\
&+ mM_C \sum_{k=1}^{m} \bar{M}_{I_k^1} + mM_S \sum_{k=1}^{m} \bar{M}_{I_k^2}]\} \qquad (3.2)
\end{aligned}$$

Proof. By hypothesis $(H9)$, for any state function $x(.)$, the control is defined as

$$\begin{aligned}
u_x^T(t) &= W^{-1}\{x_f - C(T)[\phi(0)] - S(T)[x_1 + g(0, x_0, 0)] \\
&+ \int_0^T C(T-s) g(s, x_s, \int_0^s p_1(s, \tau, x_\tau) d\tau) ds \\
&- \int_0^T S(T-s) f(s, x_s, \int_0^s p_2(s, \tau, x_\tau) d\tau) ds \\
&- \int_0^T S(T-s)[h(s) + \int_0^s \sigma(s, \tau, x_\tau) dw(\tau)] ds \\
&- \Sigma_{0<t_k<t} C(T-t_k) I_k^1(x_{t_k}) - \Sigma_{0<t_k<t} S(T-t_k) I_k^2(x_{t_k})\} \qquad (3.3)
\end{aligned}$$

Let $\Upsilon : \mathfrak{B}_T \to \mathfrak{B}_T$ defined by

$$\begin{aligned}
\Upsilon x(t) &= \phi(t)(t), \ t \in J_0 \\
\Upsilon x(t) &= C(t)[\phi(0)] + S(t)[x_1 + g(0, x_0, 0)] \\
&- \int_0^t C(t-s) g(s, x_s, \int_0^s p_1(s, \tau, x_\tau) d\tau) ds \\
&+ \int_0^t S(t-s) B u_x^T(s) ds \\
&+ \int_0^t S(t-s) f(s, x_s, \int_0^s p_2(s, \tau, x_\tau) d\tau) ds \\
&+ \int_0^t S(t-s)[h(s) + \int_0^s \sigma(s, \tau, x_\tau) dw(\tau)] ds \\
&+ \Sigma_{0<t_k<t} C(t-t_k) I_k^1(x_{t_k}) + \Sigma_{0<t_k<t} S(t-t_k) I_k^2(x_{t_k})
\end{aligned}$$

Clearly $\Upsilon x(T) = x_f$.
Let $\widetilde{\phi}$ be defined as $\widetilde{\phi}(t) = \begin{cases} \phi(t)(t), & t \in J_0; \\ C(t)[\phi(0)], & t \in J. \end{cases}$

$\forall \phi \in \mathfrak{B}$. So, $\widetilde{\phi} \in \mathfrak{B}_T$. Let $x(t) = z(t) + \widetilde{\phi},\ t \in J_T$. Then x satisfies (4.1) iff z satisfies $z_0 = 0,\ x'(0) = x_1 = z'(0) = z_1$ where

$$\begin{aligned} z(t) &= S(t)[z_1 + g(0, \widetilde{\phi}_0, 0)] + \int_0^t S(t-s)Bu^T_{z+\widetilde{\phi}}(s)ds \\ &- \int_0^t C(t-s)g(s, z_s + \widetilde{\phi}_s, \int_0^s p_1(s,\tau, z_\tau + \widetilde{\phi}_\tau)d\tau)ds \\ &+ \int_0^t S(t-s)f(s, z_s + \widetilde{\phi}_s, \int_0^s p_2(s,\tau, z_\tau + \widetilde{\phi}_\tau)d\tau)ds \\ &+ \int_0^t S(t-s)[h(s) + \int_0^s \sigma(s,\tau, z_\tau + \widetilde{\phi}_\tau)dw(\tau)]ds \\ &+ \Sigma_{0<t_k<t}C(t-t_k)I^1_k(z_{t_k} + \widetilde{\phi}_{t_k}) + \Sigma_{0<t_k<t}S(t-t_k)I^2_k(z_{t_k} + \widetilde{\phi}_{t_k}), \end{aligned}$$

for $a.e.$ $t \in [t_j, t_{j+1}],\ j = 0, 1, 2, ..., m$. Here $u^T_{z+\widetilde{\phi}}(t)$ is obtained from (3.3) by replacing $x_t = z_t + \widetilde{\phi}_t$. Let $\mathfrak{B}^0_T = \{y \in \mathfrak{B}_T : y_0 = 0 \in \mathfrak{B}\}$. $\forall\ y \in \mathfrak{B}^0_T$ the norm is defined as $\|y\|_T = \|y_0\|_{\mathfrak{B}} + \sup(E\|y(s)\|^2)^{1/2} = \sup_{s\in J}(E\|y(s)\|^2)^{1/2}$, for $(\mathfrak{B}^0_T, \|.\|_T)$ to be a Banach space. Let for some $r > 0$,

$$B_r = \{y \in \mathfrak{B}^0_T : \|y\|^2_T \le r\}.$$

Then $B_r \subset \mathfrak{B}^0_T$ is uniformly bounded and by lemma (2.1)

$$\begin{aligned} \|z_t + \widetilde{\phi}_t\|^2_{\mathfrak{B}} &= 2(\|z_t\|^2_{\mathfrak{B}} + \|\widetilde{\phi}_t\|^2_{\mathfrak{B}}) \\ &\le 4(l_0^2 \sup_{s\in[0,t]} E\|z(s)\|^2 + \|z_0\|^2_{\mathfrak{B}} + l_0^2 \sup_{s\in[0,t]} E\|\widetilde{\phi}(s)\|^2 + \|\widetilde{\phi}_0\|^2_{\mathfrak{B}}) \\ &\le 4l_0^2(r + M_C[E\|\phi(0)\|^2]) + 4\|\widetilde{\phi}\|^2_{\mathfrak{B}} \\ &:= r^*. \end{aligned} \tag{3.4}$$

Let the map $\overline{\Upsilon} : \mathfrak{B}^0_T \to \mathfrak{B}^0_T$ be defined by $\overline{\Upsilon}z(t) = 0,\ \forall\ t \in J_0$ and

$$\begin{aligned} \overline{\Upsilon}z(t) &= S(t)[z_1 + g(0, \widetilde{\phi}_0, 0)] + \int_0^t S(t-s)Bu^T_{z+\widetilde{\phi}}(s)ds \\ &- \int_0^t C(t-s)g(s, z_s + \widetilde{\phi}_s, \int_0^s p_1(s,\tau, z_\tau + \widetilde{\phi}_\tau)d\tau)ds \\ &+ \int_0^t S(t-s)f(s, z_s + \widetilde{\phi}_s, \int_0^s p_2(s,\tau, z_\tau + \widetilde{\phi}_\tau)d\tau)ds \\ &+ \int_0^t S(t-s)[h(s) + \int_0^s \sigma(s,\tau, z_\tau + \widetilde{\phi}_\tau)dw(\tau)]ds \\ &+ \Sigma_{0<t_k<t}C(t-t_k)I^1_k(z_{t_k} + \widetilde{\phi}_{t_k}) + \Sigma_{0<t_k<t}S(t-t_k)I^2_k(z_{t_k} + \widetilde{\phi}_{t_k}), \end{aligned}$$

$for\ a.e.\ t \in [t_j, t_{j+1}],\ j = 0, 1, 2, ..., m$. Clearly, the operator Υ has a fixed point which is equivalent to show $\overline{\Upsilon}$ has a fixed point. Also from the hypotheses $\overline{\Upsilon}$ is continuous as all the functions involved in the operator are continuous.

Let $z,\ \overline{z} \in \mathfrak{B}_T^0$. Then using equation (3.3), the hypotheses and lemma (2.1) it can be derived that

$$\begin{aligned} E &\ \|u^T_{z+\widetilde{\phi}}(t)\|^2 \\ \leq &\ 8M_W\{E\|y_1\|^2 + M_C(E\|\phi(0)\|^2) + 2M_S[E\|x_1\|^2 + 2(M_g\|\widetilde{\phi}\|^2_{\mathfrak{B}} + C_2)] \\ + &\ 2T^2M_C[M_g([1+2M_{p_1}]r^* + 2C_1) + C_2] + 2T^2M_S[M_f([1+2M_{p_2}]r^* + 2C_3) + C_4] \\ + &\ 2T^2M_S(M_h + TTr(Q)M_\sigma) + mM_C\sum_{k=1}^{m} M_{I_k^1} + mM_S\sum_{k=1}^{m} M_{I_k^2}\} := bl, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} E &\ \|u^T_{z+\widetilde{\phi}}(t) - u^T_{\bar{z}+\widetilde{\phi}}(t)\|^2 \\ \leq &\ 10l_0^2M_W\{T^2M_CM_g(1+M_{p_1}) + T^2M_SM_f(1+M_{p_2}) \\ + &\ T^3M_S\bar{M}_\sigma Tr(Q) + mM_C\sum_{k=1}^{m}\bar{M}_{I_k^1} + mM_S\sum_{k=1}^{m}\bar{M}_{I_k^2}\}\sup_{s\in J} E\|z(t) - \widetilde{z}(t)\|^2 \end{aligned} \tag{3.6}$$

$C_1 := T\sup_{(t,s)\in J\times J} p_1^2(t,s,0)$, $\quad C_2 := \sup_{t\in J}\|g(t,0,0)\|^2$,
$C_3 := T\sup_{(t,s)\in J\times J} p_2^2(t,s,0)$, $\quad C_4 := \sup_{t\in J}\|f(t,0,0)\|^2$. Now it is to be proved that $\bar{\Upsilon}(B_r) \subset B_r^*$ Suppose on the contrary $\forall r > 0$, $\exists$ a function $z^r(.) \in B_r$ but $\bar{\Upsilon}(z^r)$ does not belong to B_r, i.e. $\|\bar{\Upsilon}(z^r)(t)\|^2 > r$ for some $t \in J$. Although by the hypotheses

$$\begin{aligned} r \ \leq &\ E\|\overline{\Upsilon}(z^r)(t)\|^2 \\ \leq &\ 7M_W\{2M_S[E\|x_1\|^2 + 2(M_g\|\widetilde{\phi}\|^2_{\mathfrak{B}} + C_2)] \\ + &\ 2T^2M_C[M_g([1+2M_{p_1}]r^* + 2C_1) + C_2] + 2T^2M_S[M_f([1+2M_{p_2}]r^* + 2C_3) + C_4] \\ + &\ T^2M_SM_Bbl \\ + &\ 2T^2M_S(M_h + TTr(Q)M_\sigma) + mM_C\sum_{k=1}^{m} M_{I_k^1} + mM_S\sum_{k=1}^{m} M_{I_k^2}\}, \\ \leq &\ M^{**} + 7(1 + 8T^2M_BM_SM_W)[2T^2(M_CM_g(1+2M_{p_1}) + M_SM_f(1+2M_{p_2})) \\ + &\ 3M_SN_G/l_0^2 + T^2M_CN_G/l_0^2]r^* \end{aligned} \tag{3.7}$$

where

$$
\begin{aligned}
& M^{**} \\
:= \ & 56(1 + 8T^2 M_B M_S M_W)[E\|y_1\|^2 + M_C(E\|\phi(0)\|^2)] \\
+ \ & 7(1 + 8T^2 M_B M_S M_W) \times [2M_S[E\|x_1\|^2 + 2(M_g\|\widetilde{\phi}\|^2_{\mathfrak{B}} + C_2)] \\
+ \ & 2T^2 M_C(2M_g C_1 + C_2) + 2T^2 M_S(2M_f C_3 + C_4) \\
+ \ & 2T^2 M_S(M_h + TTr(Q)M_\sigma) + mM_C \sum_{k=1}^{m} M_{I_k^1} + mM_S \sum_{k=1}^{m} M_{I_k^2}] \quad (3.8)
\end{aligned}
$$

Now divide both sides of (3.9) by r, using (3.4) and taking limit as $r \to \infty$ we get $1 \leq \lambda$ which contradicts the assumption (3.1). Hence $\exists\ r > 0$ such that $\overline{\Upsilon}(B_r) \subset B_r$.

Then it is proved that $\overline{\Upsilon} : \mathfrak{B}^0_T \to \mathfrak{B}^0_T$ is a contraction mapping. Suppose $z,\ \bar{z}\ \in \mathfrak{B}^0_T$, then

$$
\begin{aligned}
& E\|\overline{\Upsilon}(z)(t) - \overline{\Upsilon}\bar{z}(t)\|^2 \\
\leq \ & 12 l_0^2 M_W\{T^2 M_C M_g(1 + M_{p_1}) + T^2 M_S M_f(1 + M_{p_2}) \\
+ \ & T^3 M_S \bar{M}_\sigma Tr(Q) \\
+ \ & mM_C \sum_{k=1}^{m} \bar{M}_{I_k^1} + mM_S \sum_{k=1}^{m} \bar{M}_{I_k^2}\} \sup_{s\in J} E\|z(t) - \widetilde{z}(t)\|^2 \\
+ \ & 6T^2 M_S M_B E\|u^T_{z+\widetilde{\phi}}(t) - u^T_{\bar{z}+\widetilde{\phi}}(t)\|^2 \\
\leq \ & \{(12 l_0^2 + 6 \times 10 T^2 M_B M_S M_W) \\
\times \ & [T^2 M_C M_g(1 + M_{p_1}) + T^2 M_S M_f(1 + M_{p_2}) \\
+ \ & T^3 M_S \bar{M}_\sigma Tr(Q) \\
+ \ & mM_C \sum_{k=1}^{m} \bar{M}_{I_k^1} + mM_S \sum_{k=1}^{m} \bar{M}_{I_k^2}]\} \sup_{s\in J} E\|z(t) - \widetilde{z}(t)\|^2
\end{aligned} \quad (3.9)
$$

Now by taking supremum over t and using (3.2), it is found that

$$\|\overline{\Upsilon}(z)(t) - \overline{\Upsilon}\bar{z}(t)\|^2_T \leq \Lambda\|z - \bar{z}\|^2_T.$$

Hence $\overline{\Upsilon}$ is a contraction mapping on $\mathfrak{B}^T_0$. Thus by Banach fixed point theorem, $\exists\ x(.) \in \mathfrak{B}^T_0$ such that $\overline{\Upsilon}(x)(t) = x(t)$ with $x(T) = \overline{\Upsilon}(x)(T) = x_f$. Therefore the system (1.1) is controllable on J_T. □

4 Example

In this section a partial differential equation applying the abstract results of this paper is discussed.

Example 1 : Consider the second order neutral differential equation with instantaneous impulses

$$\begin{aligned}
\frac{\partial}{\partial t}[\frac{\partial}{\partial t}x(t,\varepsilon) &+ \int_{-\infty}^{t}\xi_1(t,\varepsilon,s-t)\delta_1(x(s,\varepsilon))ds + \int_0^t\int_{-\infty}^{s}p_1(s-\tau)\delta_2(x(\tau,\varepsilon))d\tau ds] \\
= [\frac{\partial^2}{\partial\varepsilon^2}x(t,\varepsilon) &+ \int_{-\infty}^{t}\xi_2(t,\varepsilon,s-t)\varrho_1(x(s,\varepsilon))ds + \int_0^t\int_{-\infty}^{s}p_2(s-\tau)\varrho_2(x(\tau,\varepsilon))d\tau ds \\
+b(\varepsilon)u(t) &+ \int_{-\infty}^{t}c(s-t)x(t,\varepsilon)d\omega(s), \\
&+ t_k \neq t \in J,\ \varepsilon \in [0,\pi], \\
x(t,0) &= x(t,\pi) = 0,\ t \in J, \\
x(t,\varepsilon) &= q(t,\varepsilon),\ t \in J_0, 0 \le \varepsilon \le \pi, \\
\frac{\partial}{\partial t}x(0,\varepsilon) &= x_1(\varepsilon),\ 0 \le \varepsilon \le \pi, \\
\Delta x(t_i)(\varepsilon) &= \int_{-\infty}^{t_i} n_i^1(t_i-s)x(s,\varepsilon)ds,\ i = 1,...,m \\
\Delta x'(t_i)(\varepsilon) &= \int_{-\infty}^{t_i} n_i^2(t_i-s)x(s,\varepsilon)ds,\ i = 1,...,m,\ \varepsilon \in [0,\pi] \qquad (4.1)
\end{aligned}$$

where $x_1 \in X = L^2([0,\pi]),\ X = K = U = L^2([0,\pi])$. $\omega(t)$ is a standard one dimensional Wiener process in X, on a stochastic basis $(\Omega,\mathcal{F},P),\ q \in \mathfrak{B}$ $\mathfrak{B} = PC_0 \times L^2(\rho,X),\ A \subset D(A) \subset X \to X$ is the map defined by $A = \frac{\partial^2}{\partial\sigma^2}$ with domain $D(A) = H^2([0,\pi]) \cap H_0^1([0,\pi])$. where

$$H_0^1([0,\pi]) = \{w \in L^2([0,\pi]) : \frac{\partial w}{\partial z} \in L^2([0,\pi]),\ w(0) = w(\pi) = 0\}$$

$$H^2([0,\pi]) = \{w \in L^2([0,\pi]) : \frac{\partial w}{\partial z},\ \frac{\partial^2 w}{\partial z^2} \in L^2([0,\pi])\}$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in R}$ on X. Also, A has a discrete spectrum, and the following properties hold

(C1) $A\phi = -\sum_{n=1}^{\infty}\lambda_n^2 < \phi, z_n > z_n$ where $\phi \in D(A),\ \lambda_n,\ z_n,\ n \in \mathfrak{N}$ are eigenvalues and eigenvectors of A.

(C2) $C(t)\phi = \sum_{n=1}^{\infty} cos(\lambda_n t) < \phi, z_n > z_n$ and $S(t)\phi = \sum_{n=1}^{\infty}\frac{sin(\lambda_n t)}{n} < \phi, z_n > z_n$, for $\phi \in X$.

Clearly $\forall\ x \in X,\ t \in \mathbb{R},\ C(.)x, S(.)x$ are periodic functions and $\|C(t)\| \le 1,\ \|S(t)x\| \le 1$. Hence $(H1)$ is true. Let $l(s) = e^{2s},\ s \le 0$, so $l_0 = \int_{J_0} l(s)ds =$

1/2. Define $\|\psi\|_{\mathfrak{B}} = \int_{J_0} l(s) \sup(E\|\psi(\theta)\|^2)^{1/2} ds, \quad \forall \psi \in \mathfrak{B}$. Thus for $(t, \psi) \in J \times \mathfrak{B}$, $\psi(\theta)x = \psi(\theta, x)$, $(\theta, x) \in J_0 \times [0, \pi]$. Denote $x(t, \varepsilon)$ as $x(t)(\varepsilon)$ The result of this paper is applied to the system (4.1), by assuming the following

1. Suppose $B \in \mathcal{L}(\mathbb{R}, X)$, be defined by $Bu(\varepsilon) = b(\varepsilon)u, \ 0 \leq \varepsilon \leq \pi, \ u \in \mathbb{R}, \ b(\varepsilon) \in L^2([0, \pi])$.

2. The linear operator $W : L^2(J, U) \to X$ be defined as $Wu = \int_J S(T - s)b(\varepsilon)u(s)ds$ is a bounded linear operator but need not be injective. Let $Ker\ W = \{u \in L^2(J, U) : Wu = 0\}$ denote null space of W and $[Ker\ W]^{\perp}$ denote its orthogonal complement in $L^2(J, U)$. Consider the restriction of W to $W^* : [Ker\ W]^{\perp} \to Range(W)$, which is necessarily one-to-one. By inverse mapping theorem $(W^*)^{-1}$ is bounded as $[Ker\ W]^{\perp}$ and $Range(W)$ are Banach spaces. As W^{-1} is bounded and takes values in $L^2(J, U)/Ker\ W$, the hypothesis $(H8)$ is satisfied.

3. $n_i^1, n_i^2 \in C(\mathbb{R}, \mathbb{R})$ such that for $= 1, 2, ..., m$ $\overline{M}_{I_i^1} = \int_{J_0} l(s)n_i^1(s)ds \leq \infty$, $\overline{M}_{I_i^2} = \int_{J_0} l(s)n_i^2(s)ds \leq \infty$.

Now define the functions $g, f : J \times \mathfrak{B} \times X \to X$, $\varepsilon : J \times J \times \mathfrak{B} \to \mathcal{L}_2^0$, and $I_i^1, I_i^2 : \mathfrak{B} \to X$, $i = 1, 2, ..., m$ as

$$g(t, \psi, \eta_1\psi)(\varepsilon) = \int_{J_0} \xi_1(t, \varepsilon, \theta)\delta_1(\psi(\theta)(\varepsilon))d\theta + \eta_1\psi(\varepsilon),$$

$$f(t, \psi, \eta_2\psi)(\varepsilon) = \int_{J_0} \xi_2(t, \varepsilon, \theta)\varrho_1(\psi(\theta)(\varepsilon))d\theta + \eta_2\psi(\varepsilon),$$

$$\sigma(t, s, \psi)(\varepsilon) = (\int_{J_0} c(\theta)\psi(\theta)(\varepsilon)d\theta$$

$$I_k^1(t, \psi)(\varepsilon) = \int_{J_0} n_i^1(-s)\psi(\theta)(\varepsilon)ds, \quad k = i = 1, ..., m$$

$$I_k^2(t, \psi)(\varepsilon) = \int_{J_0} n_i^2(-s)\psi(\theta)(\varepsilon)ds, \quad k = i = 1, ..., m$$

Here $\eta_1\psi(\varepsilon) = \int_0^t \int_{J_0} p_1(s-\theta)\delta_2(\psi(\theta))(\varepsilon))d\theta ds, \eta_1\psi(\varepsilon) = \int_0^t \int_{J_0} p_1(s-\theta)\delta_2(\psi(\theta))(\varepsilon))d\theta ds,$ Thus the system (4.1) can be writtten in the abstract form as system (1.1). Then imposing suitable conditions on the above functions as per hypotheses $(H1) - (H8)$ and using theorem (3.1) we get that the system (4.1) is controllable on J_T.

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