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Properties of (Co)Sine Clock beyond the Unit Circle Definition

L. Franklin Kemp (cosineclock@gmail.com)
(Co)Sine Clock Company, Plano Texas
Joint Mathematics Meeting Denver Colorado

Abstract

(Co)Sine Clock is a unit circle clock that tells time and trigonometry. Study of the movement of cosine and sine hands reveals their derivatives, uniform circular motion, and simple harmonic motion. Low order polynomial interpolations lead to their power series expressions which appear to defy periodicity.

(Co)Sine Clock Description

A simple math wall clock 2016 Christmas present sparked my looking into the world of math clocks narrowing to a graphical "trig" clock that used cosine and sine to tell time. I realized I could do the reverse: use time to tell trig via the unit circle. On the way I discovered the geometrical fact that a circle whose diameter is the hour hand crosses the x and y axes at cosine and sine, respectfully. I modified that gift clock with the proprietary idea of replacing the circle with a disc rotating clockwise with the hour hand but under axial slits in the clock face to yield cosine and sine hands. That clock and its refinements run through all cosine sine pairs in half a day. Figure 1 shows why the innner circle of diameter 1 cuts the x and y axes at cosine(=c) and sine(=s) by evaluating its equation at $x = 0$ and $y = 0$.

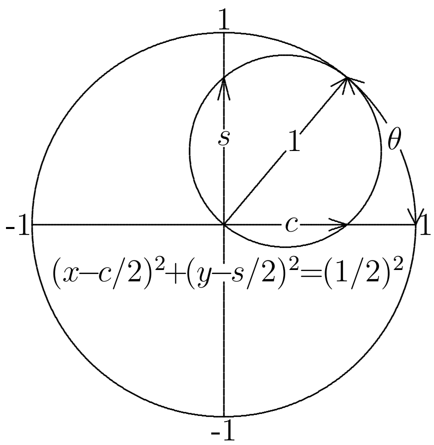

Figure 1. Inner circle rotating inside unit circle.

The unit clock face of Figure 1 doesn't show time like many clocks or watches, but (Co)Sine Clock shows 12 radian measures replacing the hour numbers. Recall that radian measure is the signed arc length along the edge of the unit circle positive counterclockwise from 3 o'clock, negative clockwise from 3 o'clock. There are two cases. Radian measure runs ccw from 2 o'clock with value .52 and ends at 3 o'clock with value 6.28 or radian measure runs ccw from 3 o'clock with value 0 and ends at 9 o'clock with value 3.14 while it also runs cw from 3 o'clock with value 0 and ends at 9 o'clock with value -3.14. This second labeling clarifies why cosine is even and sine is odd.

Trigonometric Sum Derivation

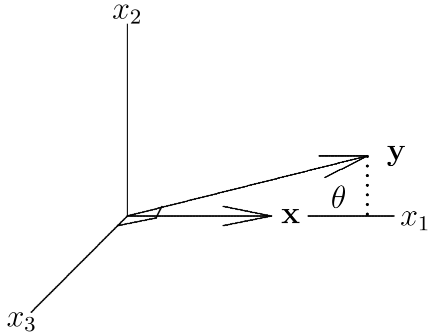

Figure 2. Interpretation of dot product Σxy.

Peter Lax (**Linear Algebra**, 2nd ed. p.78) observes that the squared distances of vectors $\mathbf{x}$ and $\mathbf{y}$ in the identity $\Sigma(x-y)^2 = \Sigma x^2 - 2\Sigma xy + \Sigma y^2$ are invariant in any Cartesian coordinate system; thus the dot product Σxy must be also. He puts the tails of $\mathbf{x}$ and $\mathbf{y}$ together and chooses a special coordinate system where the first coordinate axis runs along $\mathbf{x}$ and the second axis is in the plane spanned by $\mathbf{x}$ and $\mathbf{y}$. In this system $\mathbf{x} = (\sqrt{\Sigma x^2}, 0, ..., 0)$ and $\mathbf{y} = (\sqrt{\Sigma y^2}\cos\theta, \sqrt{\Sigma y^2}\sin\theta, 0, .., 0)$ so the dot product $\Sigma xy = \sqrt{\Sigma x^2}\sqrt{\Sigma y^2}\cos\theta$ (i.e. the length of $\mathbf{x}$ times the projection of $\mathbf{y}$ on the axis containing $\mathbf{x}$ and θ is the angle between $\mathbf{x}$ and $\mathbf{y}$).

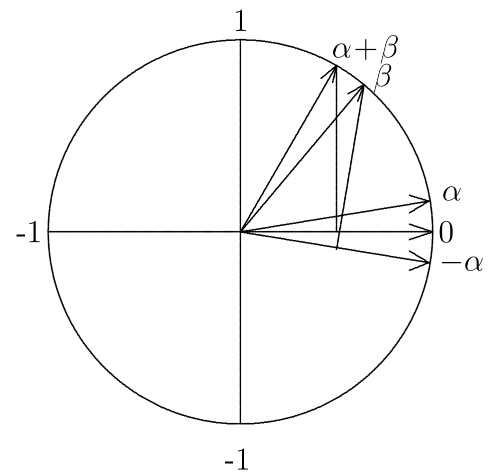

Figure 3. Projection of unit arrow at angle $\alpha+\beta$.

$c_{\alpha+\beta}$ is the projection of the unit arrow at angle $\alpha+\beta$ on the unit arrow at (1,0). This projection is invariant under rotation; so rotating arrows cw by α, we arrive at the projection of the unit arrow at $\beta = (c_\beta, s_\beta)$ on unit arrow at $-\alpha = (c_{-\alpha}, s_{-\alpha})$; hence by Lax $c_{\alpha+\beta}$ is the inner product of unit arrow at β on unit arrow at $-\alpha$ giving $c_{-\alpha}c_\beta + s_{-\alpha}s_\beta = c_\alpha c_\beta - s_\alpha s_\beta$ which yields the trig sum formula for cosine.

Now $(\pm s_{\alpha+\beta})^2 = 1 - c^2_{\alpha+\beta} = 1 - (c_\alpha c_\beta - s_\alpha s_\beta)^2 = (c_\alpha s_\beta + s_\alpha c_\beta)^2$. Hence $\pm s_{\alpha+\beta} = c_\alpha s_\beta + s_\alpha c_\beta$, but the minus case fails if $\alpha = 0$ so we get the trig sum formula for sine.

All the arrows are unit vectors, but the arrow at $\alpha+\beta$ is **not** the **sum** of the arrow at α and the arrow at β, but more akin to a **product** of the two arrows in order to remain a unit arrow. For example, we might write $(c_\alpha, s_\alpha)(c_\beta, s_\beta) \equiv (c_{\alpha+\beta}, s_{\alpha+\beta}) = (c_\alpha c_\beta - s_\alpha s_\beta, c_\alpha s_\beta + s_\alpha c_\beta)$. Letting $a = c_\alpha, b = s_\alpha, c = c_\beta$, and $d = s_\beta$, we find $(a, b)(c, d) = (ac - bd, ad + bc)$ which is equivalent to the product of complex numbers $a+bi$ and $c+di$ residing on the unit circle. It is now easy to arrive at the polar representation (r, θ) of any point in the plane as the radial distance $r = \sqrt{x^2 + y^2}$ from the origin to the point times a unit arrow; i.e $(x, y) = \sqrt{x^2 + y^2}(c_\theta, s_\theta)$ where $\theta = \arctan\frac{y}{x}$.

Derivatives of Cosine and Sine

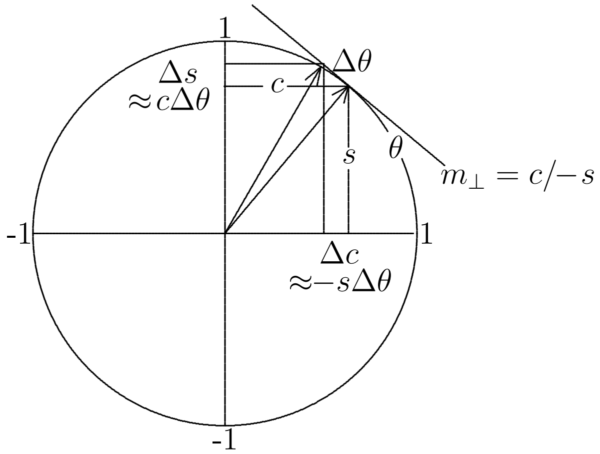

Figure 4. Tangent line ⊥ to unit arrow at angle θ.

The rate of change of cosine (sine) with respect to θ is the limit of the change in the run(rise) of the tangent line with the change in θ at angle theta on the unit circle. The tangent line's slope is the negative reciprocal of $\tan(\theta) = s/c$; i.e $c/-s$. For a change $\Delta\theta > 0$ along the tangent line, the run is $-s\Delta\theta$ and the rise is $c\Delta\theta$. But the run(rise) is the change of cosine(sine); so $c' = -s$ and $s' = c$ upon dividing out $\Delta\theta$. From this it follows that $c'' + c = 0$ and $s'' + s = 0$ because $c'' = (c')' = (-s)' = -c$ and $s'' = (s')' = (-c)' = -s$ or following the first $\Delta\theta$ with another and comparing their directions shows a change toward the center of the unit circle like acceleration of uniform circular motion. This can be viewed as another right angle ccw rotation of the tangent line yielding the negative reciprocal $-s/-c$ opposing the initial arrow to θ.

As the eye tracks ccw around the circle, it is easy to see the change in sine with respect to arc length at $\theta = 0$ is 1, at $\theta = \pi/2$ is 0, at $\theta = \pi$ is -1, and finally at $\theta = 3\pi/2$ is 0 suggesting the change in sine is cosine, at least at these points, and, furthermore, the change is bounded between -1 and 1. A similar observation can be made for cosine at these points suggesting its change is negative sine.

The back and forth motion of the cosine hand of (Co)Sine Clock mimics the simple harmonic motion of a pendulum bob along its arc while the up and down motion of its sine hand mimics the simple harmonic motion of a suspended mass; thus, (Co)Sine Clock exhibits simple harmonic motion besides obvious uniform circular motion. This is best observed with the computer version of the (Co)Sine Clock because there the second hand is the angle indicator so the period of the motion is one minute.

Polynomial Interpolation of $\sin\theta \approx P(\theta)$

power	$\leq \theta^9$	$\leq \theta^{13}$	$< \theta^\infty$	factorial
θ^1	1.00000000000E-0	1.00000000000E-0	1.00000000000E-0	1/1!
$-\theta^3$	1.66666655526E-1	1.66666666664E-1	1.66666666667E-1	1/3!
θ^5	8.33325988879E-3	8.33333332502E-3	8.33333333333E-3	1/5!
$-\theta^7$	1.98264911000E-4	1.98412689651E-4	1.98412698413E-4	1/7!
θ^9	2.64631167811E-6	2.75567874191E-6	2.75573192240E-6	1/9!
$-\theta^{11}$		2.50177017329E-8	2.50521083854E-8	1/11!
θ^{13}		1.54876873666E-10	1.60590438368E-10	1/13!

Table 1: Coefficients of Powers of θ for $\sin(\theta)$

The table shows coefficients of two interpolations at 0,30,45,60,90 degrees plus 15 and 75 showing approach to those of the convergent power series of $\sin\theta$. The two interpolations are within 10^{-7} and 10^{-9} of sine, respectively. Their number of operations are 4 nested parentheses, 5 adds, 6 multiplies, and 6 nested parentheses, 7 adds, 8 multiplies, respectively.

The abstract of the paper **Essential Trigonometry Without Geometry Analysis** presented at the 99th Annual Meeting of the Texas Section of the MAA, Tarleton State University, Stephenville TX March 28-30, 2019 by Bryant Wyatt, John Gresham, and Jesse Crawford reads "In this study we develop the theorems and calculus of traditional trigonometry without reference to geometric definitions or constructions." I find in Eli Maor's **Trigonometric Delights** pp. 192-197 that Edmund Landau took this approach in his mercilessly rigorous 1930 book **Foundations of Analysis**. Both manipulate the power series solution to the differential equation $x'' + x = 0, x(0) = 0, x'(0) = 1$. One of the manipulations shows the power series has period 2π. It's not possible to numerically sum an infinite series, so one must use a finite process like a finite polynomial. For example, summing the first seven terms of the MaClaurin power series of $\sin\theta$, we find the seven partial sums at $x = \pi$: 3.14159265359, -2.026122012646, .524043913417,-,07522060615904, .006925270707, -.00445160238, .00021142567 evaluated on a TI 89 calculator. Can you imagine summing the power series for $\theta = 50\pi$? Yet it yields 0. This is reminiscent of an asymptotic series where for fixed n, x can get large and the sum converges, but here with x large the series is useless seeming to defy convergence. Fortunately, periodicity saves the situation in this case requiring only few power terms for approximation because all values of $\mid \sin\theta \mid$ occur for $0 \leq \theta \leq \pi/2$.