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Asymptotic Frequencies of Bounded Differential Equations

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« No chaos, damn it ! »
Jackson Pollock

Abstract

We study an ordinary differential equation as an iteration with a constant fixed path when the number of iterations goes to infinite. In such conditions, very different from these of classical analysis, we greatly increase the spectrum of the solutions. First, we treat this iteration in a probabilistic framework. Curves where the probability of presence is invariant are shown. There are also critical frequencies but their interpretation remains difficult.

Keywords: Invariant measures, Perron-Frobénius, function of Plancherel - Rotach, steepest descent's method, Lorenz's attractor, critical frequencies

Introduction

Let an ordinary differential equation

$da/dt = F(a)$

where $a \in C \subset \mathbb{R}^d$ or $\mathbb{C}^d$, $t \in \mathbb{R}^+$, $F(\boldsymbol{a})$ is a polynomial application of $a \in C$ in C. The domain C is supposed bounded.

The classical problem is to find function $a(t)$ verifying this equation with an initial condition $a(t_0) = a_0$.

On another point of view, we associate a differential iteration $f(a)$ supposed belonging in the bounded set C:

$f(a) = a + \delta F(a)$

where $\delta = t/k$ is a constant fixed path. This path remains constant whereas the number of iterations goes to infinite. Iterating $f(a)$ a very long number n of times with this fixed path δ,. we will see that we obtain other solutions.

Before to examine this question, we must recall some probabilistic methods for analyze iterations. In order to simplify the presentation, we take a bounded polynomial iteration f in $\mathbb{R}^d$ and we study the invariant measure under this application. Many paragraphs of this approach are presented in our book: « Les itérations polynomiales bornées dans $\mathbb{R}^d$». Here we correct some points.

First, we recall some results and methods: the invariant Perron-Frobénius 's measure is the correct frame for this purpose and the steepest descent method allow us to approximate what happens when the number of iterations goes to infinite. In this case, the initial condition $a(t_0) = a_0$ doesn't play an important role.

Recalls about the transform of the invariant measure of Perron Frobenius

Perron-Frobenius's measure

Let f a bounded polynomial function which applies a bounded set $C \subset \mathbb{R}^d$ in itself. Let P a measure on $C \subset \mathbb{R}^d$ and $P_f = P \circ f^{-1}$ the transform of P by the function f. We say that P is invariant under f if, for all borelian set B, P verifies the Perron-Frobenius's equation (PF):

$$P_f(B) = P \circ f^{-1}(B) = P(B)$$

Under general conditions, the solution of this equation is unique. But, we will see that a PF-solution can be associated to each fixed point $f(0) = 0$ and to each point of a cycle. So, we have local solutions. All these distributions can be masked or dominated in various situations and may overlap. Our purpose is to establish an inventory of these invariant curves.

Lemma of Perron-Frobenius (known)

Let P be the invariant measure under f. X is the random vector of $\mathbb{R}^d$ with the law P . For every positive measurable function g, we have:

$$\int g \circ f(x)\, dP(x) = \int g(x)\, dP(x)$$

We use this formula and the Laplace-Fourier's transform to define **the resolving equation and the resolving gap**.

1 - Hypothesis H

f is a polynomial function and applies bounded set $C \subset \mathbb{R}^d$ in itself. It is defined at the fixed point $f(0) = 0$ or at a point of a cycle. The solution of the PF-equation needs a non-resonance condition at this point:

$$\lambda^k \neq 1, \forall k \in N^d$$

2 - Definition of the Laplace-Fourier's transform $\emptyset(y)$ and $\emptyset_f(y)$

Notations:

- *Let $\emptyset(y)$ be the Laplace-Fourier's transform of the measure P and $\emptyset_f(y)$ the Laplace-Fourier's transform of the measure P_f.*
- *x is the realization of the random vector X. $x \in C \subset \mathbb{R}^d$, $y \in \mathbb{R}^d$ or $\mathbb{C}^d$ and xy is the scalar product. As the support of P is contained in bounded C, we can use indifferently the Laplace's or the Fourier's transform which is noted:*

$$\emptyset(y) = E(e^{yX}) = \int e^{yx}\, dP(x)$$

- *The corresponding series is convergent on the bounded domain C and we write a priori for $\forall n \in N^d$:*

$$\emptyset(y) = \Sigma_n b_n y^n$$

- *We note : $\partial()/\partial a = \partial^d ()/\partial a_1, .., \partial a_\ell, .., \partial a_d$ and for $\forall n \in N^d$: $\partial^n()/\partial a^n$.*
- *The PF-equation becomes the resolving equation:*

$$\emptyset(y) = E(e^{yX}) = \emptyset_f(y) = E\left(e^{yf(X)}\right)$$

And, for all little translation $a \in C \subset R^d$ of the random vector X, the resolving equation R_a of PF becomes:

$$\theta_f(y,a) = \emptyset(y,a) - \emptyset_f(y,a) = 0$$

Where: $\emptyset(y,a) = E\left(e^{y(X+a)}\right)$ *and* $\emptyset_f(y,a) = E\left(e^{yf(X+a)}\right)$

Lemma

The resolving equation R_a of PF is: $\theta_f(y,a) = \emptyset(y,a) - \emptyset_f(y,a) = 0$ *. It means:*

$$\theta_f(y,a) = \Sigma_n b_n \partial^n (e^{ya} - e^{yf(a)})/\partial a^n = 0 \text{ for } \forall a \in C \text{ et } \forall y$$

■ Using the convergent series: $\emptyset(y) = \Sigma_n b_n y^n$, then, we obtain for all little translation $a \in C \subset R^d$ of the random vector X, the translated resolving equation

$$\emptyset(y,a) = E\left(e^{y(X+a)}\right) = \Sigma_n b_n y^n e^{ya} = \Sigma_n b_n \partial^n e^{ya}/\partial a^n$$

And: $$\emptyset_f(y,a) = E\left(e^{yf(X+a)}\right) = \Sigma_n b_n \partial^n e^{yf(a)}/\partial a^n$$ ■

Remark 1:

All solutions φ of $\theta_f(y,a) = 0$ are defined with an arbitrary constant c: $c\varphi$ is also solution. We can write $\partial^n e^{yf(a)}/\partial a^n = H_n(y,a)e^{yf(a)}$ where $H_n(y,a)$ is the polynomial of Bell of the successive derivatives of f. But, for a=0, we have a polynomial in y and we prefer to note: $H_n(y)=B_n(\mathbf{y},0)$ because of its resemblance with the Hermite's polynomial.

So, we note the ***resolving gap***: $e^n(y,a) = \partial^n(e^{ya} - e^{yf(a)})/\partial a^n = y^n(e^{ya} - H_n(y,a)\, e^{yf(a)}$

for a=0: $e^n(y) = e^n(y,a) = y^n - H_n(y)$

Equation R_a becomes R_0:

$$\theta_f(y) = \Sigma_n b_n e^n(y) = 0$$

Polynomials $e^n(y)$ are a basis if their highest degree's term doesn't be null, i.e., if the *non-resonance condition is respected:* $\lambda^n \neq 1, \forall n \in N^d$. ■

Remark 2

The Perron-Frobenius's measure is the solution $\varphi(y)$ of R_a which verifies $\theta_f(y,a) = 0, \forall y, \forall a$; so, if $\emptyset(y,a)$is any solution of R_a, Then $\varphi(y) = \partial^2\emptyset(y,a)/\partial a\partial y|_{a=0} = \partial y\emptyset(y,0)/\partial y$.

3 – Effect of an iteration on the resolving equation $\theta_f(y)$ and on $e^n(y,0)$

It is interesting to understand the effect of an iteration on the resolving equation $\theta_f(y,a)$. When we iterate f, we replace successively and respectively: $a_1,\ldots a_\ell,\ldots, a_d$ by $f_1(a),\ldots, f_\ell(a),\ldots, f_d(a)$, for $\ell = 1,2,\ldots,d$. Let $a = (\overline{a_\ell}, a_\ell)$ where $\overline{a_\ell}$ is the set of all the coordinates of a distinct of a_ℓ. We study the unidimensional transformation $a_\ell \longmapsto f_\ell(a)$ leaving the other components $\overline{a_\ell}$ of a unchanged and we note:

$$D = \theta_f\left(y, \overline{a_\ell}, f_\ell(a)\right) = \emptyset\left(y, \overline{a_\ell}, f_\ell(a)\right) - \emptyset_f\left(y, \overline{a_\ell}, f_\ell(a)\right),$$

where $f_\ell(a)$ take the place of a_ℓ in $\theta_f(y,a)$.

Proposition

Iteration $a_\ell \longmapsto f_\ell(a)$ *acts as a derivation on* $\theta_f(y,0)$ *and on* $e^n(y,0)$= 0 in the sense that :

$$\theta_f(y,0) \longmapsto \partial\theta_f(y,0)/\partial a_\ell|_{a=0} \text{ and } e^n(y,0) \longmapsto e^{n+1_\ell}(y,0).$$

■ We study the impact of $a_\ell \longmapsto f_\ell(a)$ on $D = \theta_f\left(y, \overline{a_\ell}, f_\ell(a)\right)$.

As $\theta_f(y, \overline{a_\ell}, a_\ell) = 0$:

$$D = \theta_f(y, \overline{a_\ell}, f_\ell(a)) \text{-} \theta_f(y, \overline{a_\ell}, a_\ell)$$

So: $D = (f_\ell(a) - a_\ell)(\partial\theta_f(y, \overline{a_\ell}, a_\ell + r((f_\ell(a) - a_\ell)))/\partial a_\ell)$

When $a \to 0$ $\quad f_\ell(a) - a_\ell \sim a_\ell(\lambda_\ell - 1)$.

$$D \sim a_\ell(\lambda_\ell - 1)(\partial\theta_f(y,0)/\partial a_\ell)$$

For similar raisons, if $e^n(y, a) = 0$, then: $e^n(y, f_\ell(a)) \sim a_\ell(\lambda_\ell - 1)\partial e^n(y,0))/\partial a_\ell$.

More, if we iterate f , that means $a \mapsto f(a)$ in $e^n(y, a)$, we obtain:

$$D \sim \Sigma_{\ell=1}^{\ell=d} a_\ell(\lambda_\ell - 1)(\partial\theta_f(y,0)/\partial a_\ell)$$

If this quantity is null for all a, we must have d equations independent null.

We have the same result for $e^n(y,0)$. ■

In other words, when f is iterated *n* times (the *d*-tuple (i.e. $f_1(a),\dots f_\ell(a),\dots, f_d(a)$) is iterated *n* times), this operation is equivalent to derivate *n* times $\theta_f(y, a)$ with respect to each component $(a_1, a_\ell, \dots, a_d)$ at $a = 0$. Now, as an iteration is corresponding to a derivation, unless otherwise expressly stated, all the indices of derivation are equal.

4- Solution de R_0 (recalls)

We choose a sufficiently large index $n \in N^d$, with $n = n_1 = \dots = n_\ell, \dots = n_d$

Lemma

For a fixed $b_n \neq 0$, under non-resonance conditions, we can find a unique series $b*_m$, $\forall m \leq n \in N^d$, depending only on the coefficients h_{mk} of the polynomials $H_m(y)$, such as:

$$\theta*_n(y) = \Sigma_{m\leq n} b*_m e^m(y) = 0$$

■ The solution of this equation is obtained as the following:

- We choose a sufficiently large index $n \in N^d$, such as:

1- $b_n \neq 0$

2- and $\theta_{nf}(y) = \Sigma_{m\leq n} b_m e^m(y)$ verifies uniformly $|\theta_{nf}(y) - \theta_f(y)| < \epsilon$

- As $\theta_f(y) = 0$, we search an approximation $\theta*_{nf}(y) = \theta*_n(y) = 0$, and estimators $b*_m$ such as, for $b*_n = b_n \neq 0$ arbitrarily fixed, we have:

$$\theta*_n(y) = \Sigma_{m\leq n} b*_m e^m(y) = 0$$

We note the polynomials $\emptyset*_m(y) = 1 + \Sigma_{0<m\leq n} b*_m y^m$ and $\emptyset_f *_n(y) = 1 + \Sigma_{0<m\leq n} H_m(y)$

So : $\theta*_n(y) = \emptyset*_n(y) - \emptyset_f *_n(y)$

- As $\theta*_n(y) = 0$:

1- It is easy to verify for all y verifying $\theta*_n(y) = 0$:

$$\theta*_{n-1}(y) = \Sigma_{m<n} b*_m e^m(y) = \theta*_n(y) - b_n e^n(y).$$

$$\theta*_{n-1}(y) = -b_n e^n(y)$$

But, all the coefficients of $e^m(y) = y^m - H_m(y)$ are known: $H_m(y) = \Sigma_{0<k\leq m} h_{mk} y^k$, so :

$$\theta*_{n-1}(y) = \Sigma_{m<n} b*_m (y^m - \Sigma_{0<k\leq m} h_{mk} y^k) = -b_n(y^n - \Sigma_{0<k\leq n} h_{nk} y^k$$

2 – We obtain a finite triangular system of linear equations which can be solved step by step, and we can identify in a unique way all the unknown coefficients $b*_m$ with b_n and the coefficients h_{mk} of $H_m(y) = \Sigma_{k\leq m} h_{mk} y^k$ with $m \leq n \in N^d$.

3- This solution is unique for $b*_n = b_n \neq 0$ arbitrarily fixed, near to the solution of $\theta_f(y) = 0$, as the $b*_m$ converge to the b_m. So, we can construct the polynomials $\emptyset*_n(y) - 1$ and $\emptyset_f *_n(y) - 1$.■

We will see in the next paragraph, that, under hypothesis H', all the zeros of $H_n(y)$ are distinct with multiplicity 1(all the *d* equations $\partial e^n(y,a)/\partial a_\ell$ =0 gives *n+1* points.

Theorem

Under H' and under the non-resonance condition $\emptyset *_n (y)$ *and* $e^n(y)$*are null simultaneously.*

If all the zeros of $H_n(y)$ are distinct with multiplicity 1

$$\emptyset *_n (y) = 1 + ce^n(y)$$

Where the distribution of the real zeros of the polynomials $e^n(y)$ *gets the distribution of the Perron-Frobenius's measure when n→ ∞.*

If $\lambda^n \gg 1$, *real zeros of the polynomials* $H_n(y)$ *gets the distribution of the Perron-Frobenius's measure.*

■ By construction, polynomials $\emptyset *_n (y) - 1$ and $\emptyset_f *_n (y) - 1$ have proportional coefficients. They cancel simultaneously, and also for every linear non-resonant combinations.

As $\lambda^n \neq 1$, the highest degree's term of the polynomial $\theta *_n (y)$ is*:* $b_n(\lambda^n - 1)y^n \neq 0$ for $\forall y$. Then, $\theta *_n (y) = 0$ needs $e^n(y) = 0$, because all the other gaps $e^m(y)$ have a lower degree.

The polynomials $e^n(y)$, $\theta *_n (y)$, $\emptyset *_n (y) - 1$ and $\emptyset_f *_n (y) - 1$ are simultaneously null. This result is generally sufficient for our purpose, but it can be interesting to have a formula. As they have the same highest degree's term y^n, we apply the Hilbert's theorem of the zeros with the fact that all the roots of $H_n(y)$ have a multiplicity one. So, they are all proportional and $\theta *_n (y) = c\ e^n(y)$ =$\emptyset *_n (y) - 1$. Then:

$$\emptyset *_n (y) - 1 = ce^n(y)$$

But, if : $\lambda^n \gg 1$, then : $y^n\lambda^n \gg y^n$:

$$\emptyset *_n(y) = 1 - c'H_n(y).\ ■$$

Remarque 3 *(see general reference)*

We deduce, under general conditions, that, if *q(y)* is the density of real zeros of $H_n(y)$ when *n*→ ∞, then the invariant density p(x) of the Perron-Frobenius's measure is:

$$p(x) = (-x)\partial q(x)/\partial x$$

Recalls about the steepest descent applied to $H_n(y)$

Now, we get an estimation of the asymptotic distribution of the real zeros of $H_n(y)$. For this, we use the steepest descent's method as Plancherel and Rotach use. We recall that the polynomial:

$H_n(y) = e^{-yf(a)}\partial^n e^{yf(a)}/\partial a^n|_{a=0} = \partial^n e^{yf(a)}/\partial a^n|_{a=0}$

can be represented by the Cauchy's integral:

$$H_{n-1}(y) = K\oint_\Gamma \frac{e^{yf(a)}}{a^n}da$$

where Γ is a closed polydisk around the fixed point *0* of *f*, $a \in \mathbb{C}^d$, *K* can be taken as some finite non-null function, for all *n* = (n_1,..., n_ℓ,…, n_d).

1- The steepest descent's method

In order to apply the steepest descent's method to $H_{n-1}(y)$ when *n*→ ∞, we must write the integrand of:

$$\oint_\Gamma e^{\mu\gamma(a)}da$$

$\mu\gamma(a)$, where μ → ∞ and the function $\gamma(a)$ doesn't depend on μ.
Now, the integrand, that we call the Plancherel-Rotach's function, is written here:

$$yf(a) - n \ln a = \Sigma_\ell \ (y_\ell a f_\ell(a) - n_\ell \ln a_\ell),$$

We meet two difficulties: the y_ℓ can vary arbitrarily in such manner that makes the method impossible; we must find a common μ → ∞ such as $\gamma(a)$ doesn't depend on μ.
- So, in our case, we choose:

$$\mu = n_1 + ... + n_\ell ... + n_d \text{), and we fix: } z_\ell = n_\ell/\mu.$$

We put: $y_\ell = n_\ell s_\ell = \mu z_\ell s_\ell$ and $n_\ell \ln a_\ell = \mu z_\ell \ln a_\ell$, the Plancherel-Rotach's function is:

$$\mu\gamma(a) = \mu\Sigma_\ell \ (z_\ell s_\ell f_\ell(a) - z_\ell \ln a_\ell)$$

with:

$$\gamma(a) = \Sigma_\ell \ (z_\ell s_\ell f_\ell(a) - z_\ell \ln a_\ell)$$

which doesn't depend on μ because the y_ℓ, thus s_ℓ, can be taken arbitrarily.
- But, according to the previous paragraph 4, we take in general all the n_ℓ equal: n_ℓ=*n*, and we have μ=*nd* with $z_\ell = 1/d$. So, we can simplify the notation and we take μ= *n* instead of *nd* and $z_\ell = 1$ to get a simplified formula:

$$n\gamma(a) = n \ \Sigma_\ell(s_\ell f_\ell(a) - \ln a_\ell)$$

The wealth and the variety of the situations is so large that we see here only the most common and simple. We consider:

$$H_{n-1}(y) = K \oint_\Gamma \ \exp \ (n \ \gamma(a)) da$$

The method consists to search the critical points a_C of $\gamma(a)$: $\partial\gamma(a)/\partial a = 0$. We recall the conditions:

Hypothesis H': *The general position*
The critical point a_c maximizing $e^{\,n\,\gamma(a)}$ is supposed be in general position if:
- The critical point is isolated from the other critical points and at a finite distance.
- A sufficient condition to get this maximum is that the hessian matrix (which is hermitian) of $\gamma(a)$ is definite negative at a_c .
- $\Re e(\mu\gamma\ (a))$ → - ∞ when μ → ∞ to make sure the convergence of the Laplace's integral. In the bounded set $C \subset \mathbb{R}^d$, it is not important. The critical point is called point of Morse.

Here, the critical point a_c is supposed a Morse's point defined by the equation:

$$\frac{y\partial f(a)}{\partial a} - \frac{n}{a} = 0$$

Which is written explicitly with *n* equations with real coefficients:

$$\partial\gamma(a)/(\partial a_\ell) = y\partial f(a)/(\partial a_\ell) - n_\ell/a_\ell$$

$$= \Sigma_i z_i(s_i \partial f_i(a)/\partial a_\ell) - z_\ell/a_\ell = 0 \qquad \ell = 1, 2, ..., d$$

And, if n_ℓ=*n*, $\forall\ell$:

$$= \Sigma_i(s_i \partial f_i(a)/\partial a_\ell) - 1/a_\ell = 0$$

Comments
As we have many problems to solve, here we don't be interested by all questions about the deformation of the contour of the integral, topics treated in detail by many authors. It is also unnecessary to speak about the depth of the critical point.
If there are many critical points, the approximation is the sum of the contributions neglecting those when the real part is negligible.
Finally, even if the critical point maximizing $e^{\,n\,\gamma(a)}$ is unique, the Plancherel-Rotach's function may have many determinations in relation with the complex log a.

Besides the inherent difficulties of the method, the critical point depends on the arbitrary vector y or s.

But here, we are only especially interested by the real values of y which give $H_{n-1}(y) = 0$ because they get the invariant measure. As the real part of the contributions cannot vanish, except perhaps if y becomes infinite, only imaginary part can cancel $H_{n-1}(y)$.

If f is a polynomial with real coefficients, every complex critical point a_C has a conjugate $\overline{a_C}$ solution. So integrands of $e^{n\gamma(a_C)}$ et $e^{n\gamma(\overline{a_C})}$ are imaginary conjugated and the sum of their contributions is null only if the imaginary part of $\gamma(a)$ is $k\pi$.

$$Im(\gamma(a)) = Im(sf(a)) - \Sigma_\ell \theta_\ell = k\pi/n$$

where $\Sigma_\ell \theta_\ell$ is the sum of the arguments of the d components of the complex a_C.
We observe that, for fixed n, we have n distinct equations with $k = 1, 2, ..., n$.

2 - Encoding the solution

But, what happens when the number n_ℓ of derivations of the ℓ-th coordinate a_ℓ of a_c becomes $n_\ell + m$.

Theorem

*Under the previous hypothesis, if the critical point a_C is in general position, real zeros of H_{n-1} (**y**) are reciprocals images of the p complex coordinates of a_C :* with $k_\ell \in (1,..., n_\ell$;$\forall\ \ell$) *encoding these zeros by uniform independent random variables κ_ℓ on (0,1).*

$$Im(s_\ell f_\ell(a)) - \theta_\ell = \pi\kappa_\ell \qquad\qquad \ell = 1, 2, \dots, d$$

■ If the number $n_\ell = n$ of derivations of the ℓ-th coordinate a_ℓ of a_c becomes $n + m$, we cannot simplify the notation because $\mu = nd$ becomes $\mu = nd + m$.

- All the equations defining the coordinates of a_C remain unchanged when $n_i = n$ becomes $n+m$, except the one of the ℓ-th component of a_C which becomes $\mu\ \partial\gamma(a)/\partial a_\ell - m/a_\ell = 0$.

We suppose a_c known at time μ and we keep its value at time μ+m. The equations defining the coordinates of a_C are linear in y:

$$y\partial f(a)/\partial a = n/a$$

If $\partial f(a)/\partial a$ is invertible, at μ; $\quad y = (\partial f(a)/\partial a)^{-1}(n/a)$

And at μ+m: $\quad y = (\partial f(a)/\partial a)^{-1}((n/a) + (m/a_\ell))$

Where (m/a_ℓ) is the vector of coordinates all null except the ℓ -th valued m/a_ℓ.

Is y_ℓ is the ℓ-th coordinate of y at μ, $y_\ell + (m/n)y_\ell$ will be the ℓ -th coordinate of y at μ+m.

We observe that, if $n \to \infty$, a_C being bounded and m being fixed, $m/na_\ell \to 0$, then, the equations defining a_C in function of y remain asymptotically unchanged. We put:

$s_i = y_i/\text{n}$, so $z_i = n/\mu$ $\qquad$ if $i \neq \ell$

and $\qquad s_\ell = y_\ell/(n+m)$, so: $z_\ell = (n+m)/\mu$ $\quad$ if $i = \ell$

Then:

- We distinguish the Plancherel-Rotach's function at time μ from the one at time $\mu + m$, γ is noted γ_μ at time μ and $\gamma_{\mu+m}$ at time $\mu + m$:

at time μ: $\qquad \mu\gamma_\mu(a) = yf(a) - n\ln a = \mu\Sigma_i\ (z_\ell s_i f_i(a) - z_\ell \ln a_\ell)$

becomes at time $\mu + m$: $(\mu + m)\gamma_{\mu+m}(a) = \mu\gamma\mu(a) + ms_\ell f_\ell(a) - m\ln a_\ell$

- its imaginary part: $\mathrm{Im}\left(\mu\gamma_\mu(a)\right) - k_0\pi$

becomes at time $\mu + m$: $\mathrm{Im}\left(\mu\gamma_\mu(a) + ms_\ell f_\ell(a)\right) - m\theta_\ell - k_\ell\pi$

with: k_0 = 1, 2, ..,n , if $i \neq \ell$, but k_ℓ =1,2,.., n+m.

The imaginary part of the Plancherel-Rotach's function must be null at n and at $n + m$:

$Im\left(\mu\gamma_\mu(a)\right) = k\pi$ and $\mathrm{Im}\left(\mu\gamma_\mu(a) + ms_\ell f_\ell(a)\right) - m\theta_\ell - k_\ell'\pi = 0$

Then, by subtraction, we have for all $\forall k_\ell' = 1, 2, .., m$ and $\forall \ell = 1, 2, ..., d$:

$\mathrm{Im}\left(s_\ell f_\ell(a)\right) - \theta_\ell = k_\ell'\pi/m$

For any fixed m. For instance, we can take $n=e^m$and $m\to\infty$.

In other words, for all $n\to\infty$, we get a uniform coding with $k_\ell'/m\to\kappa_\ell$ of the coordinates of the zeros. For each coordinate of a_c, the distribution asymptotic is uniform κ_ℓ on (0,1). The κ_ℓ are random independent and uniform variables. ■

Example 1

- If $f(x)$ is unidimensional, then, at the critical point x, the density of zeros between (s, $s+ds$) is:

$$q(s)ds = \text{Prob } (1 \text{ zero between } s, s+ds) = |Imf(x(s))|ds/\pi$$

- Let the iteration $f(x) = x(b - cx/2)$. Then: $n\gamma(x) = n(sf(x) - \ln x)$. the critical point is:

$$x(s) = b/2c \pm i\sqrt{\frac{1}{sc} - b^2/4c^2}$$

According to the remarks 1 and 2, and the previous theorem, the density of the invariant PF-measure is:

$$q(s) = |Imf(x(s))|/\pi = b|Imx(s)|/2\pi = b\sqrt{4c/s - b^2}/4\pi c$$

and the invariant measure P has a density $p(x)$ generalizing the low of Ulam-Von Neuman.

$$p(x) = -xdq(x)/dx = 1/(\pi\sqrt{\frac{4cx}{b^2} - x^2}).$$

If we write: $xb/\sqrt{2c} = \beta$, then β follows a low β(1/2,1/2).

Remark 4

The distribution of the zeros of $H_n(y)$ is the reciprocal image of uniform by $|Imf(x(s))|/\pi$. So, if this function is injective, we recognize a generalization of a known situation that the zeros of $H_n(y)$ are all distinct and distinct of those of $H_{n-1}(y)$.

Some very curious situations can happen when the distribution at a fixed point is dominating an another. Or when the hessian degenerates.

Analysis of an ordinary differential equation

1 - A differential equation as an iteration (Recalls)

We consider ordinary differential equation:

da/dt= $F(a)$

where $a \in C \subset \mathbb{R}^d or\ \mathbb{C}^d$, $t \in \mathbb{R}^+$, $F(\boldsymbol{a})$ is a polynomial application of $a \in C$ in C. The domain C is supposed bounded. We associate the differential iteration $f(a)$ supposed belonging in the bounded domain C:

$f(a) = a + \delta\, F(a)$

where $\delta = t/k$ is the path. The problem is to find a function $a(t)$ verifying this equation with an initial condition $a(t_0) = a_0$.

With the invariant measure of Perron-Frobénius, we obtain limit cycles:

So, the resolving equation is for each fixed point:

$$\theta_f(y,a) = \emptyset(y,a) - \emptyset_f(y,a) = 0$$

$$\emptyset(y) = E(e^{yX}) = \emptyset_f(y) = E\left(e^{yf(X)}\right)$$

With $f(x) = x + \delta\, F(x)$; and: $\quad E(e^{yX}) = \emptyset_f(y) = E\left(e^{yX+\delta F(X)}\right)$

So: $\quad E\left(e^{yX}(1 - e^{\delta y F(X)})\right) \sim E(\delta y F(X) e^{yX}) = 0$

This implies generally $F(X) = 0$ and determines the fixed points of the differential equation. We verify the same behaviour for the cycles (which are dependent on δ).

The classical method gives the solution $a(t)$ by iterating k times $f(a)$ from a starting point a_0 with the path $\delta = t/k$:

$$a(t) = f^{(k)}(a_0)$$

then, $k \to \infty$. This solution is theoretically:

$$a(t) = a(0) + \int_0^t F\left(a(u)\right) du$$

2- Invariant measure of the differential equation

We examine now another situation taking $\delta = t/k$ very small, but fixed for every number of iterations; on iterate now f a number of times n very great before k: $n \gg k$, and tending to infinite.

Justification

We can view $da / dt = F(a)$ as the realization of the family of equations $da / dt = cF(a)$ with $c = 1$, or as the study of the equation: $da / d(t) = da / d(ct') = F(a)$ with $ct' = t$ at the neighbourhood of the infinite, when c is very great. In term of iteration, it is like if we subdivide the path.

Lemma

The invariant measure of the differential iteration f(a)= a+δF(a) is defined by real zeros of the derivative with respect to $\forall\, t \le \delta$ *of* $H_n(y) = \partial^n e^{yf(a)}/\partial a^n|_{a=0}$:

$$\partial(\partial^n(e^{yf(a)})/\partial a^n)\partial\delta|_{a=0} = 0$$

■ the resolving equation of the differential iteration becomes:

$$\theta_f(y,a) = \Sigma_n b_n \partial^n(e^{ya} - e^{yf(a)})/\partial a^n = 0$$

for $\forall a \in C$ and $\forall y$ where $f(a) = a + \delta\, F(a)$, the resolving gaps must be null:

$$e^n(y,a) = \partial^n(e^{ya}(1 - e^{\delta y F(a)}))/\partial a^n|_{a=0} = 0$$

but: $\quad 1 - e^{\delta y F(a)} = -\int_0^\delta (de^{tyF(a)}/dt)\, dt$

As δ doesn't depend on a, we can invert integration and derivations:

$$\partial e^n(y,a) = -\partial^n(e^{ya} \int_0^\delta e^{tyF(a)}\, \mathrm{d}t)/\partial a^n = 0$$

$$e^n(y,a) = -\int_0^\delta (d(\frac{\partial^n(e^{ya+tyF(a)})}{\partial a^n})/dt)\, \mathrm{d}t = \int_0^\delta (\frac{\partial^n\left(yF(a)e^{ya+tyF(a)}\right)}{\partial a^n})\, \mathrm{d}t = 0$$

If $\frac{\partial^n\left(yF(a)e^{ya+tyF(a)}\right)}{\partial a^n} = 0$ is true for $\forall\, t \le \delta$, the resolving equation will be verified for $\forall a$.

We see now that the resolving gap can be null, not only if the critical point a has complex coordinates, but also if a get $yF(a) = 0$. ■

Proposition

When the number of iterations $n \to \infty$ and if the la hessian of yF is definite negative, the approximation with the steepest descent's method *defines $s = y/n$ in function of the critical point a:*

$$s + ts\partial F(a)/\partial a - 1/a = 0$$

where $1/a = (1/a_\ell, \ell = 1, 2, ..., d)$*.*

If s_0 is a particular solution and if $\vec{s}$ is an eigenvector of $-\partial F(a)/\partial a$ for the eigenvalue $1/t$, the general solution is:

$$s = s_a + \vec{s}$$

The eigenvalue $1/t$ can be interpreted as a critical asymptotic frequency.

■ We search an approximation of $e^n(y,0)$=$H_n(y)$= $\partial^n yF(a)e^{yf(a)}/\partial a^n|_{a=0}$ for $\delta = t$ arbitrary with the steepest descent's method to: $f(a)= a + t\, F(a)$.

The hessian of yF has to be definite negative:

$$H_{n-1}(y)=K \oint_\Gamma \frac{yF(a)e^{yf(a)}}{a^n}\, da$$

y$F(a)$ is not exponential, so the Plancherel-Rotach's function is unchanged:

$n\gamma(a) = yf(a) - n \ln a = ya + tyF(a) - n \ln a$

Let $s = y/n$, then: $yf(a) - n \ln a = n(sa + tsF(a) - \ln a)$

The critical point is defined by:

$$s + ts\partial F(a)/\partial a - 1/a = 0$$

where: $1/a = (1/a_\ell, \ell = 1, 2, ..., d)$.

If t =0, the critical point a_c is real.

We note that the probabilistic distribution is defined by the imaginary part of the complex critical point a_c. Theoretically, $t \leq \delta$, but, as the imaginary part of $\gamma(a)$ gets $H_{n-1}(y)$=0, we have to choose the real part of $\gamma(a)$ maximal as the steepest descent's method needs to have the best approximation of $H_{n-1}(y)$. Then, t can be arbitrary to this purpose. As a_c is complex, we note a the real part of the critical point.

Contrary to the previous chapter, we don't write the real part of critical point a as a function of s, but s as a function of a. For fixed a, we recognize a linear affine equation of s depending on the parameter t. We have to find a particular solution s_a :

$$s_a + ts_a\partial F(a)/\partial a - 1/a = 0$$

Formally: $s_a = (Id + t\partial F(a)/\partial a)^{-1} 1/a$

The equation of s_a is now an elementary Fredholm's equation and has a unique solution for all $t \neq -1/\lambda_a$ where λ_a is eigenvalue of $\partial F(a)/\partial a$ at the fixed critical point a,

Let the general solution be $s = s_a + \vec{s}$, where s_a is a particular solution of the general equation:

As: $s_a + \vec{s} + t(s_a + \vec{s})\partial F(a)/\partial a - 1/a = 0$

then : $\vec{s} = -t\vec{s}\partial F(a)/\partial a$

$\vec{s}$ is eigenvector of $\partial F(a)/\partial a$ for the eigenvalue λ_a and giving t=-1/λ_a maximal positive ; $\vec{s}$ is defined with a multiplicative constant arbitrary. The general solution is: s =s_a +$\vec{s}$ and shows a discontinuity at the eigenvalues λ_a.■

Remark 5*: calculation of* s_a

s_a is obtained with $(Id + t\partial F(a)/\partial a)^{-1}$ for all t≠-1/$\lambda_a$which doesn't belong to the spectrum of -$\partial F(a)/\partial a$ with the series development of *t*.

3 - Encoding the solution

As previously, if the number n_ℓ of derivations of the ℓ-th coordinate a_ℓ of *a* becomes n_ℓ *+1*, we cannot use the simplified formula because μ=*nd+1*. We find a condition on the ℓ-th coordinate a_ℓ of *a*.

Theorem

For each fixed point of F, we have d asymptotic curves verifying the Perron-Frobenius's equation and solutions of the the differential *equation da/dt= F(a). These curves are defined by 2d equations under the condition that the hessian of F be definite negative :*

- a is critical point of the Plancherel-Rotach's function:

$$s + ts\partial F(a)/\partial a = 1/a$$

Where the vector $1/a = (1/a_\ell\ , \ell = 1, 2, ...,d)$ *and the vector s is d-dimensional.*

If the critical point a is known and if s_a *is particular solution of* $s_a + ts_a\partial F(a)/\partial a - 1/a = 0$. *s is solution of this linear affine equation:*

$$s = s_a + \vec{s},$$

where $\vec{s}$ *is eigenvector of the matrix* $\partial F(a)/\partial a$ *at the critical point a for the eigenvalue* λ_a and for *t*=-1/$\lambda_a > 0$. *The eigenvalue* $1/t$ *can be seen as critical asymptotic frequency.*

■ The theorem takes the previous proposition, and adds the coding of the coordinates. Demonstration of this encoding is the same that the one of the coding before but with *f(a)*= $a + t$ *F*(a) where we take two times μ=*nd* and μ=*nd*+1_ℓ with one further derivation of the coordinate a_ℓ :

at μ: $\quad y + ty\partial F(a)/\partial a = n/a$

at μ +m for a_ℓ: $\quad y_\ell + ty\partial F(a)/\partial a_\ell = n/a_\ell + (1/a_\ell)$

with $(1/a_\ell) = (0, \dots, 1/a_\ell, 0 \dots 0)$. Only the particular solution changes with $y_\ell = (n+1)/a_\ell$, instead of $y_\ell = n/a_\ell$, solution of the homogenous equation is unchanged, that means, $\vec{s}$ doesn't change. The equations of the critical point give with summation at μ+1_ℓ=nd+1_ℓ :

$$ya = (\mu + 1) - tya\partial F(a)/\partial a - (1/n)ta_\ell y\partial F(a)/\partial a_\ell$$

Substituting ya by this expression in the Plancherel-Rotach's function, we have at $\mu + 1_\ell$:

$$(\mu + 1)\gamma_{\mu+1}(a) =$$

$$= (\mu + 1) + tyF(a) - \frac{tya\partial F(a)}{\partial a} - n \ln\ a + (1/n)(ty_\ell F_\ell(a) - a_\ell ty\partial F(a)/\partial a_\ell) - \ln\ a_\ell$$

$$= \mu\partial(\gamma_\mu(a) + 1 + (1/n)(ty_\ell F_\ell(a) - a_\ell ty\partial F(a)/\partial a_\ell) - \ln\ a_\ell$$

The resolving gap: $\quad \partial(\partial^n(e^{yf(a)})/\partial a^n)\partial t|_{a=0,}$

must be null at μ and at μ+1_ℓ : $\quad \partial(\partial^1\partial^n(e^{yf(a)})/\partial a^n\ \partial a_\ell)\partial t|_{a=0} = 0$

After approximation of the Cauchy's representation, we obtain from the difference, if $t \neq 0$:

$$y_\ell F_\ell(a) - a_\ell y \partial F(a)/\partial a_\ell = 0 \qquad \ell = 1, 2, \ldots, d$$

Putting: $s = s_a + \vec{s}$: $\qquad (s_{a\ell} + \vec{s}_\ell) F_\ell(a) - a_\ell (s_a + \vec{s}) \partial F(a)/\partial a_\ell = 0$

We recall that we have *2d* equations with *2d+1* unknown variable. ■

Remark 6:

If we can take δ so small such as: $0 \leq t \leq \delta < \inf(\lambda_a, \forall a \in C)$, solution is unique and gives the classical solution; if not $\inf(\lambda_a) = 0$, we must have a relation de compatibility.

Finally, we don't forget that this study is valid in the neighbourhood of each fixed point of *F*. Some curious situations happen when the distribution around a fixed point dominates an another of the same fixed point (phenomenon of Stokes) or at the frontier of fixed point. Or when the hessian degenerate.

Some questions remain pending:

Are the curves solutions be closed?

In physics, Can we reach the critical limit frequencies for certain phenomenon? `

Remark 7:

In this paper, we have insisted about the real coordinates of the critical point a. But, the complex coordinates of a give the probability of presence of the solution of differential equation.

Remark 8:

We note that the resolving equation of the differential iteration is:

$$E\left(e^{yX}(1 - e^{\delta y F(X)})\right) = -\mathrm{E}\left(\int_0^\delta \left(\frac{de^{tyF(X)}}{dt}\right) \mathrm{d}t\right) = 0$$

As an iteration acts as a derivation, we obtain after n derivations:

$$\mathrm{E}\left(\int_0^\delta \frac{\partial^n}{\partial X^n}\left(\frac{de^{tyF(X)}}{dt}\right) \mathrm{d}t\right) = \mathrm{E}\left(-\int_0^\delta \left(\frac{\partial^n\left(yF(X)e^{yX+tyF(X)}\right)}{\partial X^n}\right) \mathrm{d}t\right) = 0$$

That is coherent with our calculations.

Case where the hessian is degenerate: equation of Lorenz

Generally, the hessian is not definite negative. The Lorenz's equation is an example particularly important because it can be broken down into three independent iterations which have a remarkable feature: a partial linearity; an iteration with a negative hessian which induces a probabilistic solution and another with a positive hessian. It is an ideal example to clarify the previous results.

However, as there is an interpenetration of the distributions related to each fixed point, the connection between the various results remains delicate. The probabilistic presentation seems to be the least bad: it gives the probability of presence except at the places where the domination changes; in this case, we go from a basin of iteration to an another.

- *Presentation of the differential iteration at its fixed points.* These equations are written in our notations:

$d\boldsymbol{a} / dt = F(\boldsymbol{a})$ where $\boldsymbol{a} = (a, b, c)$:

$$da/dt = \sigma(b-a)$$
$$db/dt = \rho a - b - ac$$
$$dc/dt = -\beta c + ab$$

the differential iteration $\boldsymbol{a}_1 = f(\boldsymbol{a})$ associated with a given path $\delta = t / n$ is the following:

$$a_1 = a + \delta\sigma(b - a)$$
$$b_1 = b + \delta(\rho a - b - ac)$$
$$c_1 = c + \delta(-\beta c + ab)$$

We recall the known results concerning the fixed points:

The fixed points are zeros of $F(\boldsymbol{a})=0$. If $\rho > 1$ and $\alpha = \sqrt{\beta(\rho - 1)}$, it exists three fixed points:

The point $0 = (0,0,0)$, and two others symmetric to the axis of c:

$$\alpha_+ = (\alpha, \alpha, \alpha^2/\beta) \text{ et } \alpha_- = (-\alpha, -\alpha, \alpha^2/\beta).$$

At 0, the eigenvalue's equation λ of the linear part is:

$$(\beta + \lambda)\,[(\sigma + \lambda)(1 + \lambda) - \sigma\rho] = 0,$$

But, at α_+ or at α_-:

$$\lambda(\beta + \lambda)(1 + \sigma + \lambda) - \alpha^2(2\sigma + \lambda) = 0,$$

Coefficients β, σ, ρ are such as these three repellent fixed points, that means we have to study the distributions around each fixed point. We don't speak here about attractive cycles, resonances, and some particular values of the parameters, etc. It remains many things to clarify.

The iteration applies a compact set C in itself for $\delta > t > 0$ (the phenomenon occurring between a cold sphere at -50° and hot sphere, the earth, at +15° as the terrestrial atmosphere is modelled).

This iteration is quadratic, but has a linearity in a.

Analysis of the hessian

Projecting $f(\boldsymbol{a})$ onto an axis $\boldsymbol{y} = (x, y, z)$, we write:

$$yf(\boldsymbol{a}) = L(\boldsymbol{a}) + \delta Q(\boldsymbol{a})$$

where $L(\boldsymbol{a})$ is linear for $\boldsymbol{a}$: $L(\boldsymbol{a}) = x(a + \delta\sigma(b - a)) + y(b + \delta(\rho a - b)) + zc(1 - \delta\beta)$

$$L(\boldsymbol{a}) = aL_1 + bL_2 + cL_3$$

with : $L_1 = x(1 - \delta\sigma) + \delta\rho y$

$$L_2 = \delta\sigma x + y(1 - \delta)$$
$$L_3 = z(1 - \delta\beta)$$

and $Q(\boldsymbol{a})$ is quadratic: $\delta Q(\boldsymbol{a}) = \delta(zb - yc)a$

The hessian is degenerated and not definite negative. We cannot apply the previous results. On the other hand, we can always use the lemma which requires: $\partial(\partial^n(e^{yf(a)})/\partial a^n)\partial\delta|_{a=0} = 0$.

A- Before to study this equation, we examine the quadratic application and its matrix $Q(\boldsymbol{a})$:

$$Q = \begin{bmatrix} 0 & z & -y \\ z & 0 & 0 \\ -y & 0 & 0 \end{bmatrix}$$

If $\mu = \sqrt{y^2 + z^2}$ is the positive eigenvalue of the characteristic equation of Q : $\mu(\mu^2 - y^2 - z^2) = 0$

The matrix of the eigenvectors T is orthogonal and constant for all $\boldsymbol{a}$.

$$T = \frac{1}{\mu\sqrt{2}} \begin{bmatrix} 0 & \mu & \mu \\ y\sqrt{2} & -z & z \\ z\sqrt{2} & y & -y \end{bmatrix}$$

Corresponding to the diagonal matrix of the eigenvectors.

$$\Lambda=\begin{bmatrix}0 & 0 & 0\\ 0 & -\mu & 0\\ 0 & 0 & \mu\end{bmatrix}$$

- *Changing of basis*

We calculate directly with the Hermite's polynomials.

The application $\boldsymbol{u} = T\boldsymbol{a}$ with $\boldsymbol{u} = (u, v, w)$ is orthogonal and transforms $yf(\boldsymbol{a})$ in $G(\boldsymbol{u}) = yf(T'\boldsymbol{u})$:

- $Q(\boldsymbol{a})$ in: $Q(\boldsymbol{u})=\delta\mu(w^2 - v^2)$
- $\boldsymbol{L}(\boldsymbol{a})$ in: $\boldsymbol{L}T'\boldsymbol{u}$

Where T' means the transposed of the orthogonal matrix T, which is also its inverse: $T'=T^{-1}$.

Now, in the new basis $\boldsymbol{u}$, the function $\boldsymbol{y}f(\boldsymbol{a})$ is factorized: $yf(T'\boldsymbol{u}) = G(\boldsymbol{u})$ into three independent functions:

$$G(\boldsymbol{u})= g_1(u)+ g_2(v)+ g_3(w)$$

with:

$$g_1(u) = l_1 u$$
$$g_2(v)= l_2 v-\delta\mu\, v^2$$
$$g_3(w)= l_3 w +\delta\, \mu\, w^2$$

We get 3 independent iterations:

- the first is linear;
- the second is a random iteration;
- the third remains positive, except if $l_3 = 0$.

To calculate l_1, l_2 et l_3, we form $L(\boldsymbol{a})=a(x- \delta\sigma x + \delta\rho y) + b(\delta\sigma x+ y(1- \delta))+zc(1-\delta\beta)$

With: $L_1= x(1- \delta\sigma) + \delta\rho y$; $L_2= \delta\sigma x+ y(1- \delta)$; $L_3= z(1-\delta\beta)$

Then: $\boldsymbol{l}u=(l_1, l_2, l_3)\boldsymbol{u} =LT'\boldsymbol{u}=(L_1, L_2, L_3)\,\frac{1}{\mu\sqrt{2}}\begin{bmatrix}0 & y\sqrt{2} & z\sqrt{2}\\ \mu & -z & y\\ \mu & z & -y\end{bmatrix}$

$$l_1= (\delta\sigma x+ y(1- \delta)+z(1-\delta\beta))/\sqrt{2}$$
$$l_2 = (x- \delta\sigma x + \delta\rho y)y/\mu - (\delta\sigma x+ y(1- \delta))\text{-}z(1-\delta\beta))z/\mu\sqrt{2}$$
$$l_3 = (x- \delta\sigma x + \delta\rho y)z/\mu + (\delta\sigma x+ y(1- \delta))\text{-}z(1-\delta\beta))y/\mu\sqrt{2}$$

B- Let the resolving gap $e^n = \partial(\partial^n(e^{yf(\boldsymbol{a})})/\partial\boldsymbol{a}^n)\partial\delta|_{a=0} = 0$

For $\forall\ t \leq \delta$. Putting $\boldsymbol{a} = T'\boldsymbol{u}$, we have:

$$e^n = T^n\partial(\partial^n(e^{yf(T'\boldsymbol{u})})/\partial\boldsymbol{u}^n)\partial\delta|_{\boldsymbol{u}=0} = 0$$

$$\partial^n(e^{yf(T'\boldsymbol{u})})/\partial\boldsymbol{u}^n = \partial^n(e^{g_1(u)})/\partial u^n\,.\,\partial^n(e^{g_2(v)})/\partial v^n.\ \partial^n(e^{g_3(w)})/\partial w^n$$

This gives: $\partial^n(e^{g_1(u)})/\partial u^n= {l_1}^n e^{g_1(u)}$;

$$\partial^n(e^{g_2(v)})/\partial v^n = H_n(g_2(v))e^{g_2(v)} ;$$
$$\partial^n(e^{g_3(w)})/\partial w^n = H_n(g_3(w))e^{g_3(w)}$$

And: $e^n = \partial\, {l_1}^n H_n(g_2(v))H_n(g_3(w))(e^{yf(\boldsymbol{T'u})})\partial\delta|_{u=0} = 0$

The main results

With the same calculations of encodings and interchanging the derivations, we have:

$\partial\, {l_1}^n/\partial\delta$=0; $\partial\, H_n(g_2(v))/\partial\delta$=0; $\partial\, H_n(g_3(w))/\partial\delta$=0

We study separately the three expressions:

- First: $\partial\, {l_1}^n/\partial\delta$=$n(\partial l_1/\partial\delta){l_1}^{n-1}$ =0: Either $\partial l_1/\partial\delta= \sigma x - y\text{-}z\beta$ =0, or: $l_1 \sim (y + z)/\sqrt{2} = 0$

- Second: the polynomial $H_n(g_3(w))$ when $w = 0$ is a Hermite's polynomial $H_n(x)$ where x is $x=il_3/(\sqrt{2\delta\mu}$. this polynomial $i^n H_n(il_3/(\sqrt{2\delta\mu})$ is always positive whatever n. In a general way :
$\partial H_n(x)/\partial\delta=nH_{n-1}(x)\ \partial x/\partial\delta=0$. So: $d(l_3/\sqrt{2\delta\mu})/d\delta = 0$, and $l_3\sim (xz\sqrt{2} + (y-z)y)/\mu\sqrt{2}=0$
- Third: in the case of $H_n(g_2(w))$, in addition to the solution $l_2 = 0$, we have to find the possible invariant distribution of $H_n(l_2/\sqrt{2\delta\mu}) = 0$.
Let the integrand of $n\gamma(w) = g_2(w) - n\ln w$
When $\delta \to 0$, $l_2\sim (x\,y\sqrt{2} + (y-z)\,z)/\sqrt{2}\mu$ with $\mu = \sqrt{y^2 + z^2}$.
By normalization of the coordinates $\boldsymbol{x} = (x, y, z) = \delta n\boldsymbol{s} = (\delta nr, \delta ns, \delta nt)$, we obtain:

$$l_2 \sim n\delta(rs\sqrt{2} + (s-t)t\,)/\,2(s^2+t^2)^{\frac{1}{2}} = n\delta l_2(\boldsymbol{s})$$
$$\delta\mu = n\delta^2(s^2+t^2)^{1/2} = n\delta^2\,\mu(\boldsymbol{s})$$

$n\gamma(v) = n(\delta l_2(\boldsymbol{s})v - \mu(\boldsymbol{s})(\delta v)^2 - \ln\delta v + \ln\delta)$
Putting $\delta v = \mathrm{v}$, we have: $\quad n\gamma(\mathrm{v}) = n(l_2(\boldsymbol{s})\mathrm{v} - \mu(\boldsymbol{s})\mathrm{v}^2 - \ln\mathrm{v})$
We search the critical point: $\quad d\gamma(\mathrm{v})/d\mathrm{v} = l_2(\boldsymbol{s}) - 2\mu(\boldsymbol{s})\mathrm{v} - 1/\mathrm{v} = 0$
The imaginary roots are: $\quad \mathrm{v}(\boldsymbol{s}) = l_2(\boldsymbol{s})/4\mu(\boldsymbol{s}) \pm i\sqrt{1/2\mu(\boldsymbol{s}) - l_2(\boldsymbol{s})^2/16\mu(\boldsymbol{s})^2}$
Under the condition: $l_2(\boldsymbol{s})^2 < 8\mu(\boldsymbol{s})$:

$$l_3\sim(xz\sqrt{2} + (y-z)y)/\mu\sqrt{2}=0$$

Implies: $\quad l_2(\boldsymbol{s}) = -(s-t)^2\,/\,\sqrt{2}(s^2+t^2)^{1/2}$
The condition becomes; $(s-t)^4\,/\,(s^2+t^2)^{3/2} < 16$
$l_1 = 0$ implies $s + t = 0$, then: $s < 8$
In any case, we observe that the conditions $l_3 = l_1 = 0$ allow us to express r et t depending on s and we can write that the density of zeros of s is now:

$$q(s)ds = \text{Prob (1 } zero\ between\ s,\ s+\mathrm{d}s) = |Imf(\mathrm{v}(s))|\mathrm{d}s/\pi$$
$$q(s)ds = l_2(\boldsymbol{s})\sqrt{8\mu(\boldsymbol{s}) - l_2(\boldsymbol{s})^2}/8\pi\mu(\boldsymbol{s}) = \mathrm{d}\kappa$$

κ follows a uniform low on (0,1) with: $s + t = 0$ (or $\sigma x - y\text{-}z\beta = 0$) and: $x\,y\sqrt{2} + (y-z)\,z+0$
We also remark that the normalization doesn't affect the coefficients of the orthogonal matrix:
$T(x, y, z) = T(\delta nr, n\delta s, n\delta\, t) = T(r, s, t)$

Proposition
The solution around the fixed point 0 consists of the intersection of
the familly of random surfaces defined by: $l_2/2\sqrt{\mu} \longmapsto$ *low* $\beta(1/2,1/2)$ *with*
The surfaces $\sigma x - y\text{-}z\beta = 0$ *et* $(-\sigma x + \rho y)z + (\sigma x - y + z\beta)y/\sqrt{2} = 0$.

We now verify similar results the two other fixed points $\boldsymbol{\alpha}_+$ et $\boldsymbol{\alpha}_-$.

C - Calculation for the two other fixed points
We search the distributions around the two other fixed points. To pass from the fixed point *0* to the fixed point α_+ or α_- , it is sufficient to put in the differential iteration instead of $\boldsymbol{a} = (a,b,c)$
$\boldsymbol{a'}+\boldsymbol{\alpha}_+ = (a'+\alpha, b'+\alpha, c'+\alpha^2/\beta)$ and $\boldsymbol{a'}+\boldsymbol{\alpha}_- = (a'-\alpha, b'-\alpha, c'+\alpha^2/\beta)$:
So, for $\boldsymbol{a'}+\boldsymbol{\alpha}_+$: $\quad \boldsymbol{a}_1 = f(\boldsymbol{a})$ where $\boldsymbol{a}_1 = (a_1, b_1\ c_1)$ becomes $\boldsymbol{a}_1 = \boldsymbol{a'}_1 + \alpha_+ = f(\boldsymbol{a}) = f(\boldsymbol{a'}+\alpha_+)$;
then : $\quad \boldsymbol{a'}_1 = \boldsymbol{a'} + \delta F(\boldsymbol{a'}+\alpha_+)$
And $\boldsymbol{a}_1 = f(\boldsymbol{a})$: $\quad a_1 = a + \delta\sigma(b-a)$

b_1=b+ $\delta(\rho a$– b-ac)

c_1=c+ δ(-βc + ab)

Becomes for $\boldsymbol{a}+\alpha_+$ (we remove apostrophe of ***a'*** to make the notation less cluttered):

a'_1=a + $\delta\sigma$(b-a) = a_1

b'_1=b + $\delta(\rho a$– b -ac) +δ(- αc-$a\alpha^2/\beta$)= $b_1 + \delta$(- αc-$a\alpha^2/\beta$)

c'_1=c + δ(-βc+ab) + $\delta\alpha$ (a+b)= c_1+ $\delta\alpha$ (a+b)

The projection of $f(\boldsymbol{a})$ on an axis $\boldsymbol{y}$ = (x, y, z) can be written:

$\boldsymbol{y}f(\boldsymbol{a'}) = xa_1+yb_1 + \delta y(- \alpha c\text{-}a\alpha^2/\beta)+zc_1+ z\delta\alpha(a+b)$

$\boldsymbol{y}f(\boldsymbol{a'}) = \boldsymbol{y}f(\boldsymbol{a})+ \delta(\text{a } (z\alpha\text{-}y\alpha^2/\beta) +z\alpha b\text{-}\ y\alpha c)$

and $Q(\boldsymbol{a})$ is invariant: $\boldsymbol{y}f(\boldsymbol{a'}) = L'(\boldsymbol{a}) + \delta Q(\boldsymbol{a})$

$L(\boldsymbol{a})$ is linear for $\boldsymbol{a}$: $L'(\boldsymbol{a})= L(\boldsymbol{a})+ \delta(a\ (z\alpha\text{-}y\alpha^2/\beta) +z\alpha b\text{-}\ y\alpha c)$

$L'(\boldsymbol{a})=aL'_1 + bL'_2+cL'_3$

with: $L'_1= L_1+ \delta(z\alpha\text{-}y\alpha^2/\beta)$

$L'_2= L_2 + \delta z\alpha$

$L'_3 = L_3 - \delta y\alpha$

$Q(\boldsymbol{a})$, then T and Λ remain invariant.

We calculate l'_1, l'_2 et l'_3, with $L(\boldsymbol{a}) = a(x\text{-}\ \delta\sigma x + \delta\rho y) + b(\delta\sigma x+ y(1\text{-}\ \delta))+zc(1\text{-}\delta\beta)$:

Where $L_1 = x(1\text{-}\ \delta\sigma) + \delta\rho y$; $L_2= \delta\sigma x+ y(1\text{-}\ \delta)$; $L_3= z(1\text{-}\delta\beta)$

And: $\boldsymbol{l'}u = (l'_1,\ l'_2 \text{ et } l'_3)\boldsymbol{u} = LT'\boldsymbol{u}= (L_1 + \delta(z\alpha\text{-}y\alpha^2/\beta), L_2 + \delta z\alpha,\ L_3 - \delta y\alpha)\frac{1}{\mu\sqrt{2}}\begin{bmatrix} 0 & y\sqrt{2} & z\sqrt{2} \\ \mu & -z & y \\ \mu & z & -y \end{bmatrix}$

The results are modified:

$l'_1 = l'_1 + \delta\alpha(z - y)/\sqrt{2}$

$l'_2 = l'_2 + \delta\alpha((z\text{-}y\alpha/\beta)y\sqrt{2}\text{-z}(z + y))/\ \mu\sqrt{2}$

$l'_3 = l_3 + \delta\alpha((z\text{-}y\alpha/\beta)z\sqrt{2}\text{+y}(z + y))/\ \mu\sqrt{2}$

The following calculations remains the same.

When $\boldsymbol{a}$ becomes $\boldsymbol{a}+\alpha_-$ $a"_1 = a + \delta\sigma(b\text{-}a) = a_1$

$b"_1 = b + \delta(\rho a\text{–}\ b\ \text{-}ac) +\delta(\alpha c\text{-}a\alpha^2/\beta) = b_1 + \delta(\alpha c\text{-}a\alpha^2/\beta)$

$c"_1 = c + \delta(\text{-}\beta c+ab)\ \text{-}\ \delta\alpha\ (a+b) = c_1\text{-}\ \delta\alpha\ (a+b)$

It remains the problems of domination and frontiers between the various distributions attached at each fixed point.

Remark. 9

We have to go back to the original coordinates. And the solution gives only probabilities of presence...

Conclusion

It is difficult to review all the perspectives of this first study, whether it's for mathematics or physics. But we can develop the theory without difficulties for all the ordinary differential equations quadratic, as it has been done for the Lorenz's equations. This will allow to accumulate experience useful to continue. We can also apply these methods to the partial derivatives equations.

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