"Science Stays True Here"
Journal of Mathematics and Statistical Science (ISSN 2411-2518, USA), Vol.7, Issue 9, 214-228 | Science Signpost Publishing

COMPARISON OF NUMERICAL METHODS FOR SOLVING STOCHASTIC DIFFERENTIAL EQUATIONS WITH TIME DELAY

HAMIT MIRTAGIOGLU[1], AKBAR BARATI CHIYANEH[2]

Abstract. In this paper, we have presented computational methods for solving stochastic differential equations (SDDEs) with distuributed delay term in the drift and diffusion cofficient. Our purpose is comparatively investigates some numerical methods to solve stochastic delay differential equations. We indicate the nature of the methods of interest and examine convergence of them. By presenting some numerical experiments we illustrate the theorical results and finally the results numerical methods are supported with graphs and error tables and discussed about which method is useful and superior and in which cases these methods can be used.

1. Introduction

In this paper we will study the problem of the numerical solution of SDDEs of Itô form

$$\begin{aligned} dX(t) &= f(t, X(t), X(t-\tau))dt + g(t, X(t), X(t-\tau))dW(t), \quad t \in [0, T] \\ (1.1) X(t) &= \psi(t), \quad t \in [-\tau, 0] \end{aligned}$$

with given f, g drift and diffusion cofficient, Wiener noise W and given delay parameter $\tau > 0$, with a prescribed initial function $\psi(t)$.

In recent years there has been interest in stochastic differential equatins and applied fields such as population dynamics, population growth with incubation/gestation period, logistic growth, advertising models, financial mathematics, computational biology, epidemiology, physiology, optics and mechanics. In many areas of science we need to construct mathematical models to understand the structure and behavior of systems. Because of the noise, we do not have enough information about the parameters of the system. So, we construct SDEs with the addition of that noise term into the deterministic models. A careful study of SDEs is a clear understanding of the behavoir and effects of randomness on change and stability. A basic stochastic concepts are considered in [1]. For the theoretical analysis on SDEs refer to ([2], [3]) and for applications of numerical methods for SDEs, see ([6]-[13]).

In many applications of SDEs, it is assumed that behavior of the system does not depend on the past. However, in many applications of science phenomena do not show their effect at the moment of their occurrence ([16]- [17]). So, we use new terms that, namely, time delay that is obtained from the past states of the system could be added in the model to create a more realistic one. SDDEs give a mathematical formulation for such a system and in many areas of science, there is an increasing interest in the investigation of SDDEs. The analysis of numerical

Key words and phrases. Stochastic delay differential equations; Strong Solutions; Numerical Methods; Strong and weak convergence.

methods for SDDEs is based on the numerical analysis of delay differential equatins and the numerical analysis of SDEs. For the theory of SDDEs, see ([4], [5]) and for other aspects of the qualitative theory of it refer to ([5], [14]) and the references therein. For the application of SDDEs, one could refer to ([15]-[25]). For the SDDEs, explicit solutions cannot easlly be obtained. So, we need numerical techniques to approximate solution process and understand behavior of numerical solution. The numerical analysis methods for SDDEs are similar to the numerical methods for DDEs but it is not a straight forward generalization of numerical analysis of DDEs and SDEs. For the numerical treatment of DDEs, one can see ([26]-[32]).

Although SDDEs can be a special class of SDEs, but delay parameter in SDDEs may intrigue instabilities in the underlying SDDEs while the corresponding SDEs are stable ([29], [33]). The formulation of numerical methods requires a slightly different calculus because of the delay nature of SDDEs or anticipative calculus ([18], [34]-[35]). The research on numerical methods for SDDEs is still new. The convergence of Euler method for SDDEs in mean-square sense was studied by Baker and Buckwar ([52]). Improving the rate of convergence for numerical methods in SDDEs by introducing a way of expanding the Stratonovich Taylor expansion of SDDEs and then proposed a Milstein scheme with the rate of convergence of 1.0 was done by Kloeden & Platen ([12]). Introduced Itô formula with tame function in order to derive the same order of convergence but with different scheme was introduced by Hu et al. ([34]). Modification of Milstein scheme was contributed by Hofmann & Muller ([53]). Strong discrete time approximations of SDDEs are derived by Kühler and Platen ([38]). Convergence analysis for explicit one-step methods and a number of numerical stability results have been derived by Baker and Buckwar ([37]-[38]). Some definitions and Euler Maruyama scheme for SDDEs with detailed information and proofs, one can see ([33], [39], [40]-[43]). Most numerical methods for SDDEs have been investigated the convergence and stability of time-discretization schemes ([20], [22]-[44]). The Euler-type schemes are stated in ([36], [40], [45]-[48]), the split-step schemes ([47]-[49]), the Milstein schemes in ([4], [10], [12], [54]-[60],), the Runge-Kutta Method ([61]-[76], [22], [42]), the Heun Method ([86]-[93]), and also some multistep schemes ([29], [37], [39]-[40]). Numerical stability plays an important role in numerical analysis. Stability theory for numerical simulations of SDDEs typically deals with mean-square behavior. The delay dependent stability of the stochastic theta method was studied by Huang et al. ([69]). The mean-square stability of the stochastic theta method for a linear scalar SDDE was studied by Liu et al. ([60]). Developed A pth mean stability analysis of the Euler-Maruyama type methods for a linear SDDE was developed by Baker and Buckwar ([65]). The mean-square exponential stability of a split-step Euler method was investigated by Wang and Gan ([47]). Some stability conditions for the Milstein method was obtained by Wang and Zhang ([67]). Nonlinear stability of numerical methods, including Euler-type and the theta methods, has also received attention ([83]-[87]).

In this paper we shall be interested in obtaining numerical solutions of an SDDE by several numerical methods and comprasin results of methods. We indicate the nature of numerical methods of the stochastic delay differential equation and give convergence for the methods. A illustrative numerical example using a strong Euler-Maruyama scheme, Milstein method, Heun Method and Runge-Kutta Method are

provided and comprasion with together. We give a comparison study between some numerical methods for SDDEs. Then, some numerical methods for finding approximate solution of SDDEs are introduced and the properties of these methods are studied.

2. Discrete time approximation

In this section, we note defination and property of a discrete time approximation, and increment of time and standard Brownian motion for SDDEs. Then, we will construct discrete time approximations to establish of the discrete stochastic differential equation with time delay for numerical methods.

Let us consider the SDDE that is given in (1.1), where we have a constant finite time delay $\tau > 0$. We formulate Lipschitz and growth conditions to ensure the existence of a unique solution of (1.1). We are going to consider our general SDDE in equation (2.5) in the autonomous form for simplicity, i.e., functions f and g do not depend explicitly on t in (1.1). Suppose the delay time $\tau > 0$ is fixed and the observation time T is greater than τ. Consider a partition of the interval $[0, T]$, $0 = t_0 < t_1 < ... < t_N = T$ with uniform step size h then $h = T/N$ and $t_n = nh$ where $n = 0, 1, ..., N$. Moreover, we define a positive integer number N_τ such that $N_\tau h = \tau$. We define the increment of time and standard Brownian motion with a uniform step size h :

$$\begin{aligned} \Delta t_{n+1} &= t_{n+1} - t_n = h, \\ \Delta W_{n+1} &= W(t_{n+1}) - W(t_n) = W(h) = \sqrt{h} Z_{n+1}, \end{aligned}$$

for some random variable $Z_{n+1} \in N(0, 1)$, where $0 \leq n \leq N - 1$. Suppose that $\tilde{X}_n$ is an approximation of the strong solution to equation

$$\begin{aligned} dX(t) &= f(X(t), X(t-\tau))dt + g(X(t), X(t-\tau))dW(t), \quad t \in [0, T] \\ X(t) &= \psi(t), \quad t \in [-\tau, 0] \end{aligned} \tag{2.1}$$

using a stochastic explicit one step method with an increment function Γ,

$$\begin{aligned} &\tilde{X}_{n+1} = \tilde{X}_n + \Gamma(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \triangle W_{n+1}), \quad 0 \leq n \leq N-1 \\ &\tilde{X}_{n-N_\tau} = \psi(t_n - \tau), \quad 0 \leq n \leq N_\tau, \end{aligned} \tag{2.2}$$

where

$$\Gamma(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \Delta W_{n+1}) = f(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta t_{n+1} + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta W_{n+1}.$$

We will assume that for any $x; x', y, y' \in \mathbb{R}$, the increment function Γ fulfills the following conditions:

$$\begin{aligned} &|E\left(\Gamma(h, x, y, \Delta W_{n+1}) - \Gamma(h, x', y', \Delta W_{n+1})\right)| \leq C_1 h(|x - x'| + |y - y'|), \\ &E\left(\left|\Gamma(h, x, y, \Delta W_{n+1}) - \Gamma(h, x', y', \Delta W_{n+1})\right|^2\right) \leq C_2 h(|x - x'|^2 + |y - y'|^2), \end{aligned} \tag{2.3}$$

where C_1 and C_2 are some positive constant numbers. $\tilde{X}(t_{n+1})$ denotes the locally approximate value obtained after just one step of equation (2.2) i.e.,

$$\tilde{X}\left(t_{n+1}\right) = X\left(t_n\right) + \Gamma(h, X\left(t_n\right), X\left(t_{n-N_\tau}\right), \Delta W_{n+1}),$$

where $X(t_{n+1})$ and $\tilde{X}_{n+1}$denotes the values of the exact solution of equation (2.1) at the point t_{n+1} and the value of approximate solution using equation (2.2), respectively.

Now, we are going to provide some definitions that are related to the way of measuring the accuracy of a numerical approximate solution to SDDE.

Definition 1. *The local error of* $\left\{\tilde{X}\left(t_n\right)\right\}$ *between two consecutive time is the sequence of random variables:*

$$\delta_n = X\left(t_n\right) - \tilde{X}\left(t_n\right), \quad n = 1, 2, ..., N.$$

The local error measures the difference between the approximation and the exact solution on a subinterval of the integration.

Definition 2. *The global error of* $\left\{\tilde{X}_n\right\}$ *from the beginning point to the end point is the sequence of random variables:*

$$\epsilon_n = X\left(t_n\right) - \tilde{X}_n, \quad n = 1, 2, ..., N.$$

The global error measures the difference between the approximation and the exact solution over the entire integration range.

Definition 3. *If the explicit one step method defined in equation (2.2) satisfies the following conditions:*

$$\begin{aligned} \max_{1\leq n\leq N} |E(\delta_n)| &\leq Ch^{p_1} \quad as \quad h \longrightarrow 0, \\ \max_{1\leq n\leq N} \left(E\left|\delta_n\right|^2\right)^{\frac{1}{2}} &\leq Ch^{p_2} \quad as \quad h \longrightarrow 0, \end{aligned}$$

for some positive constants $p_2 \geq \frac{1}{2}$, $p_1 \geq p_2 + \frac{1}{2}$*and* C *which does not depend on* h *but may depend on the initial condition* ψ*and* T *then it is called consistent with order* p_1*in the mean and with order* p_2 *in the mean square sense.*

Definition 4. *The method in equation (2.2) is convergent in the mean with order* p_1 *and in the mean square with order* p_2 *if the following conditions are satisfied*

$$\begin{aligned} \max_{1\leq n\leq N} |E(\epsilon_n)| &\leq Ch^{p_1} \quad as \quad h \longrightarrow 0, \\ \max_{1\leq n\leq N} \left(E\left|\epsilon_n\right|^2\right)^{\frac{1}{2}} &\leq Ch^{p_2} \quad as \quad h \longrightarrow 0, \end{aligned}$$

again the constant C *is independent of* h*, but may depends on the initial function and* T*.*

Theorem 1. *Assume that drift and diffusion terms namely functions* f *and* g *fulfill local Lipschitz condition and linear growth condition. Moreover, suppose the increment function in equation (2.2) satisfies conditions in equation (2.3) and the method in equation (2.2) is consistent with order* p_1 *in the mean and order* p_2 *in the mean square sense. Then approximation in equation (2.2) for the equation (2.1) is convergent in* L_2 *with order* $p = p_2 - \frac{1}{2}$*which means that convergence occurs in the mean square sense and we can write*

$$\max_{1\leq n\leq N} \left(E\left|\epsilon_n\right|^2\right)^{\frac{1}{2}} \leq Ch^p \quad as \quad h \longrightarrow 0,$$

Proof. The detailed proof can be found in ([59]). □

Suitably appropriate numerical methods for SDDE's should take into account a detailed analysis of the order of convergence as well as stability of the numerical scheme and the behavior of the errors. Now, we give some definitions about convergence and consistency of discrete time approximations of a numerical approximate solution to SDDE.

2.1. **Convergence and consistency of discrete time approximations.** The strong order of convergence gives the rate at which the mean of the errors decreases as the time step tends to zero and it is very demanding to implement, as it requires that the whole path is known. However, we do not always need that much information, in just knowing the probability distribution of the solution $X(t)$. So, it would suffice to know the rate the at which error of the means decreases, as the time step tends to zero. Now, we can consider the way of measuring the accuracy of a numerical solution of the SDDE. The most used ones are strong convergence and weak convergence.

Definition 5. *The time discretized approximation $\tilde{X}$ with step size h converges strongly to X at time T if*

$$\lim_{h \longrightarrow 0} E\left[\left|X(T) - \tilde{X}(T)\right|\right] = 0$$

$\tilde{X}$ is said to converge strongly to X with (global) order p if we have

$$E\left[\left|X(T) - \tilde{X}(T)\right|\right] \leq Ch^p,$$

for some $C > 0$ which does not depend on h. We denote the error at final time T in the strong sense as

$$e_h^{Strong} = E\left[\left|X(T) - \tilde{X}(T)\right|\right].$$

Definition 6. *The approximation $\tilde{X}$ with uniform step size h converges weakly to X at time T if the following condition is satisfied for any continuously differentiable function f*

$$\lim_{h \longrightarrow 0} E(f(X(T))) - E(f(\tilde{X}(T))) = 0.$$

$\tilde{X}$ converges weakly to X with order p means

$$\left|E(f(X(T))) - E(f(\tilde{X}(T))\right| \leq Ch^p,$$

for some positive constant number C which is independent of h. We define the error at the final time T as

$$e_h^{Weak} = \left|E\left[X(T)\right] - E\left[\tilde{X}(T)\right]\right|.$$

Strong convergence measures mean of the error while weak convergence measures error of the means of solution and approximation with given any continuously differentiable function g.

We wish to study the consistency of the schemes, that is, we want to know if the truncation errors vanish as the time step goes to zero. Let us introduce the concept of consistency for the SDDE (1.1). As with ODEs numerical schemes, the concept of consistency of a discrete time approximation is closely related to that

of convergence, but, is easier to verify. Our analysis will be based on a filtered probability space

$$\left(\Omega,\ \mathcal{F},\ \underline{\mathcal{F}} = (\mathcal{F}_t)_{t\in[0,T]}, P\right)$$

that fulfils the usual conditions, see ([10]). We shall say that a discrete time approximation $\tilde{X}_{n+1}$ corresponding to a time discretisation

$$r_h = \{r_j : j = -j, -j+1, ..., 0, 1, ..., N\}$$

constant step size h is strongly consistent if there exists a non-negative function $c = c(h)$ with

$$\lim_{h\longrightarrow 0} c(h) = 0,$$

such that

$$E\left[\left|E\quad \frac{\tilde{X}_{j+1} - \tilde{X}_j}{h}\right| \mathcal{F}r_j) - f(r_j, \tilde{X}_{j+1}(r_j), \tilde{X}_{j+1}(r_{j-T}))\right|^2\right] \leq c(h)$$

and

$$E\left(\frac{1}{h}\left|\tilde{X}_{j+1} - \tilde{X}_j - E\left(\tilde{X}_{j+1} - \tilde{X}_j|\mathcal{F}_{r_j}\right)\right.\right.$$
$$\left.\left.-g(r_n, \tilde{X}_j(r_j), \tilde{X}_{j+1}(r_{j-T}))\Delta W\right|^2\right) \leq c(h)$$

of the time interval $[-\tau; T]$ with

$$r_j = jh,$$

where $h = T/N$, $j = 0, 1, ...$ and where $\{\mathcal{F}_r, r \geq 0\}$ is a preassigned increasing family of ς-fields.

Let us suppose that the assumptions including Lipschitz and growth conditions of the existence and uniqueness theorem in the previous section are satisfied. In particular, we assume that the coefficients a and b satisfy a uniform Lipschitz condition and a growth bound with respect to $X(t)$ and $X(t-\tau)$. Then, the following theorem allows us to conclude strong convergence from strong consistency.

Theorem 2. *Under the assumptions of Theorem 2.1, every strongly consistent discrete time approximation $\tilde{X}_{n+1}$ of the solution of a one-dimensional, autonomous SDDE with time delay and initial condition $\tilde{X}_{n+1}(t)$ for $t \in [-\tau, 0]$ converges strongly to X for $T \longrightarrow \infty$ that is $h \longrightarrow 0$.*

Proof. The detailed proof can be found in ([5]). □

3. Numerical Methods for SDDEs

In practice we use implicit schemes to obtain numerically stable approximate solutions for SDDEs. In the following, we will give introduce several specific discrete schemes for the SDDE (1.1) with time delay.

3.1. **Euler Maruyama Method.** Consider approximation with uniform step size h on the interval $[0,T]$, i.e., $h = T/N$ and $t_n = nh$ where $n = 0, 1, ..., N$. We define a positive integer number N_τ such that $N_\tau h = \tau$. The increment function Γ in equation (2.2) for the Euler Maruyama method is defined as

$$\begin{aligned}\Gamma(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \Delta W_{n+1}) &= f(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta t_{n+1} + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta W_{n+1}, \\ n &= 0, 1, ..., N-1.\end{aligned}$$

Then, equation (2.2) becomes

$$\begin{aligned}&\tilde{X}_{n+1} = \tilde{X}_n + f(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta t_{n+1} + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta W_{n+1} \\ &= \tilde{X}_n + f(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta t_{n+1} + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\sqrt{h}Z_{n+1}, \qquad (3.1)\end{aligned}$$

for all $n - N_\tau \geq 0$, where Z_{n+1} corresponds to normally distributed random variable with mean 0 and variance 1, and for all indices $n - N_\tau \leq 0$ we define $\tilde{X}_{n-N_\tau} = \psi(t_n - \tau)$.

From the practical and theoretical point of view it is of interest to obtain an order of strong convergence for the Euler Maruyama Method. It can be shown that the Euler Maruyama approximation converges under rather general assumption.

Theorem 3. *Assume that the coefficient functions f and g in equation (2.1) satisfy the conditions of existence and uniqueness theorem, namely local Lipschitz and linear growth conditions. Then the Euler Maruyama scheme is consistent with order $p_1 = 2$ in the mean and order $p_2 = 1$ in the mean square sense.*

Proof. The complete proof can be found in ([52]). □

Lemma 1. *If equation (2.1) has a unique strong solution, then the increment function in equation (3.1) satisfies the conditions in equation (2.3).*

Proof. Assume that we have a unique strong solution which means the coefficient functions f and g satisfy the local Lipschitz and linear growth conditions. We show that, there exists constant numbers C_1 and C_2 so that the conditions in equation (2.3) hold. We know that $E(W(t) = 0)$,so:

$$\begin{aligned}&|E(\Gamma(h, x, y, \Delta W_{n+1}) - \Gamma(h, x', y', \Delta W_{n+1}))| \\ = \;& |E(f(x,y)h + g(x,y)\Delta W_{n+1} - f(x',y')h - g(x',y')\Delta W_{n+1})| \\ \leq \;& h|f(x,y) - f(x',y')| \\ \leq \;& C_1 h(|x - x'| + |y - y'|),\end{aligned}$$

and

$$\begin{aligned}&E\left(|\Gamma(h, x, y, \Delta W_{n+1}) - \Gamma(h, x', y', \Delta W_{n+1})|^2\right) \\ = \;& E\left(|f(x,y)h - g(x,y)\Delta W_{n+1} - f(x',y')h - g(x',y')\Delta W_{n+1}|^2\right) \\ \leq \;& E\left((|f((x,y) - f(x',y'))h| + |(g(x,y) - g(x',y'))\Delta W_{n+1}|)^2\right) \\ \leq \;& 2h^2|f(x,y) - f(x',y')|^2 + 2|g(x,y) - g(x',y')|^2 E(\Delta W_{n+1}^2) \\ \leq \;& 2h^2(2|x - x'| + 2|y - y'|)^2 + 2h(2|x - x'| + 2|y - y'|)^2 \\ \leq \;& C_2 h(|x - x'|^2 + |y - y'|^2).\end{aligned}$$

□

for any $x, x', y, y' \in \mathbb{R}$. According to Theorem 1 and Lemma 1, the Euler Maruyama method fulfills Theorem 1 with order of convergence $p = 1/2$ in the mean square sense, so:

$$\max_{1\leq n\leq N}\left(E\left|\epsilon_n\right|^2\right)^{\frac{1}{2}} \leq Ch^{\frac{1}{2}} \quad \text{as} \quad h \longrightarrow 0.$$

If in equation 2.1 function g does not depend on X, the Euler Maruyama method is consistent with order $p_1 = 2$ in the mean and order $p_2 = 3/2$ in the mean square sense. In this case, method is converge with order $p = 1$ in the mean square sense and we get:

$$\max_{1\leq n\leq N}\left(E\left|\epsilon_n\right|\right) \leq Ch \quad \text{as} \quad h \longrightarrow 0.$$

It was easy to shown that, autonomous case that the Euler approximation strongly converges. Thus, it represents a stochastic numerical method that can be used to simulate solutions of SDDEs.

3.2. **Milstein method.** The Milstein method was first derived by Milstein, who used the Itô formula to expand an integrand involving the solution in one of the error terms of the Euler-Maruyama scheme. It is the numerical scheme that achieves a strong order of convergence higher than the Euler-Maruyama scheme. The method is the scheme for SDEs with a strong order of convergence one. The systematic derivation of stochastic Taylor expansions and numerical schemes of high strong and weak orders expounded in Kloeden and Platen ([12]), see also Milstein ([13]).

In this paper, we develop a strong Milstein approximation scheme for solving SDDEs. The scheme has convergence order one. Because of the nonanticipating nature of the SDDE, the use of predicting calculus methods in the strong approximation schemes appears to be newfound. The Milstein schemes for SDEs and SDDEs have the same complexity. Although the solution of the SDDE is adapted to the filtration of the driving noise, methods from anticipating stochastic analysis and the Malliavin calculus are necessary in order to derive an Itô formula for the segment of the solution process.

Consider the same SDDE and time interval partition. The increment function Γ in equation (2.2) for the Milstein method is defined as

$$\begin{aligned}\Gamma(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \Delta W_{n+1}) &= f(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta t_{n+1} + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta W_{n+1} \\ &\quad + \frac{1}{2}g(\tilde{X}_n, \tilde{X}_{n-N_\tau})g'(\tilde{X}_n, \tilde{X}_{n-N_\tau})\left(\Delta W_{n+1}^2 - \Delta t_{n+1}\right),\end{aligned}$$

for $n = 0, 1, ..., N-1$. Then for mesh with uniform step h on the interval $[0, T]$, the equation (3.1) can be rewritten while using this increment function as:

$$\begin{aligned}&\tilde{X}_{n+1} = \tilde{X}_n + f(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta t_{n+1} + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta W_{n+1} \\ &+ \frac{1}{2}g(\tilde{X}_n, \tilde{X}_{n-N_\tau})g'(\tilde{X}_n, \tilde{X}_{n-N_\tau})\left(\Delta W_{n+1}^2 - \Delta t_{n+1}\right) \\ = \;& \tilde{X}_n + f(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta t_{n+1} + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\sqrt{h}Z_n \\ &+ \frac{1}{2}g(\tilde{X}_n, \tilde{X}_{n-N_\tau})g'(\tilde{X}_n, \tilde{X}_{n-N_\tau})h\left(Z_n - 1\right).\end{aligned}$$

Note that Milstein Method and Euler Maruyama Method give same result whenever the derivative of g with respect to X namely $g'(\tilde{X}_n, \tilde{X}_{n-N_\tau})$ is 0 .

3.3. **Runge-Kutta Method.** This paper proposes 2-stage stochastic Runge-Kutta (SRK2) to SDDEs with a constant time delay, $\tau > 0$. This method has 1.0 order of convergence and does not require the computation of high level partial derivatives. In general, formulation of stochastic Runge-Kutta for SDDEs is introduced and high order Stratonovich Taylor series expansion of SDDEs for numerical solution of SRK2 is presented by Rosli et al. ([35]). By adding stochastic integral terms from Stratonovich Taylor series expansion, more accurate method can be obtained. With order increases, the complexity of implementing those numerical methods can become more complicated. This is at the cost of requiring more partial derivatives of drift function, f and diffusion function, g. So, To overcome this problem, it is natural to consider a derivative-free method for solving SDDEs. There is no derivative-free method to facilitate the approximation of SDDEs. Here, we implementing a derivative-free method, that is 2-stage method for solving SDDEs.

In general, 2-stage SRK method developed in this paper has better performance than the Euler method and Milstein scheme. Moreover, it is a derivative-free method which does not require the computation of partial derivative for drift and diffusion functions of SDDEs. The Runge-Kutta method for SDDEs is rarely mentioned in the literature. In deterministic case, we can improve the convergence rate by increasing the stage number of a Runge-Kutta type method, while, discretization of the stochastic integrals ([73]-[74], [90]-[93]). An approach for simulating the multiple stochastic integrals with time-delay was proposed, but it is unfeasible to achieve first order because one needs a great many terms in the Karhunen-Loéve expansion for a small time step-size ([34]).

This paper is devoted to present a generalization of SRK2 to approximate the solution of SDDE (1.1).

$$X_{i+1} = X(t_i) + F(X(t_i), X(t_{i-N_\tau}))\Delta t + G(X(t_i), X(t_{i-N_\tau}))\Delta W_i$$
$$+\frac{1}{2\sqrt{\Delta t}}\left(G(\tilde{X}(t_i), X(t_{i-N_\tau})) - G(X(t_i), X(t_{i-N_\tau}))\right)\left\{(\Delta W_i)^2 - \Delta t\right\},$$

where $t_{i+1} = t_i + h$ and $0 \leq i \leq N-1$. Then, we obtain $\tilde{X}(t_i)$ as

$$\tilde{X}(t_i) = X(t_i) + F(X(t_i), X(t_{i-N_\tau}))\Delta t + G(X(t_i), X(t_{i-N_\tau}))\sqrt{\Delta t},$$

where, $X(t_{i-N_\tau}) = \psi(t_i - \tau)$.

Local truncation error of SRK2 is measured by comparing the Stratonovich Taylor expansion of the exact solution with the computed solution. Numerical experiment is performed to assure the validity of the method in simulating the strong solution of SDDEs.

3.4. **Heun Method.** Heun's method that is called modified Euler's method provides an early account for constructing a numerical method for solving SDEs ([90], [85]). One of the simplest discretization schemes for solving SDDEs is the Heun method. Numerical methods for SDDE's constructed by representing a deterministic numerical method like the Heun's method applying it to a SDDE. However, translating a deterministic numerical method and applying it to an SDDE will not provide accurate methods. The Heun's method for SDDE's is the simplest method which is a direct express of the deterministic Heun's method, but this method is not very accurate. However, this method is useful in that it provides a starting point for more advanced numerical methods for SDDE's ([2], [12], [54], [77]-[78], [80]).

In this paper our work is solving SDDE with time delay, by using Heun's method. This is a predictor-corrector method that given the value of X at a time t_n of the discretization, we first obtain the predictors, or supporting values, with the Euler integration scheme,

$$X_{i+1} = \hat{X}(t_i) + F(\hat{X}(t_i), \hat{X}(t_{i-N_\tau}))\Delta t + G(\hat{X}(t_i), \hat{X}(t_{i-N_\tau}))\Delta W_i,\ 0 \le i \le N-1$$

where $t_{i+1} = t_i + h$. Then, we obtain $\hat{X}(t_{i+1})$ as

$$\begin{aligned}\hat{X}(t_{i+1}) &= \hat{X}(t_i) + \frac{1}{2}\left\{F(\hat{X}(t_i), \hat{X}(t_{i-N_\tau})) + F(\hat{X}(t_{i+1}), \hat{X}(t_{i+1-N_\tau}))\right\}\Delta t \\ &\quad + \frac{1}{2}\left\{G(\hat{X}(t_i), \hat{X}(t_{i-N_\tau})) + G(\hat{X}(t_{i+1}), \hat{X}(t_{i+1-N_\tau}))\right\}\Delta W_i,\end{aligned}$$

where, $\hat{X}(t_{i-N_\tau}) = \psi(t_i - \tau)$.

4. Numerical Results

Let us consider the SDDE,

$$\begin{aligned}dX(t) &= 0.5X(t-\tau)dt + 0.1X(t-\tau)dW(t),\ 0 \le t \le T, \\ X(t) &= (t),\ t \in [-\tau, 0].\end{aligned}$$

Since the calculation of exact solution is not easy, we simulate the solutions process using Euler Maruyama, Milstein, Runge-Kutta and Heun methods on SDDE. In order to see the effect of the initial value on SDDE, we provide our simulations while setting $\psi(t) = 1 + t$ and $\psi(t) = e^{-t}$.

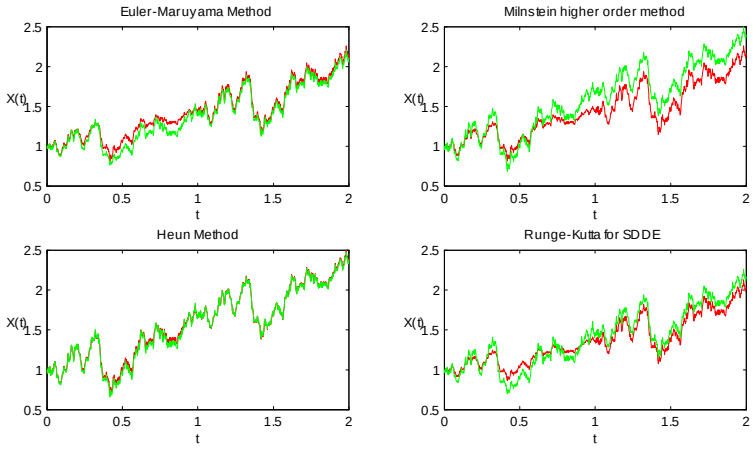

Figure 4.1. Sample path with different initial functions

The Figure 4.1 shows two sample path for the different choice of initial values with $N = 2^9$, $\tau = 1, T = 2$, $\mu = 0.1$ and $\beta = 0.5$. Up to time 0, the path with initial data $\psi(t) = e^{-t}$ decreases while path with initial data $\psi(t) = 1 + t$ increases. At time equal to 0, it is seen that both graphs take the same value, 1. After time 0, we observe that both graphs have the same structure and their values is always near to each other.

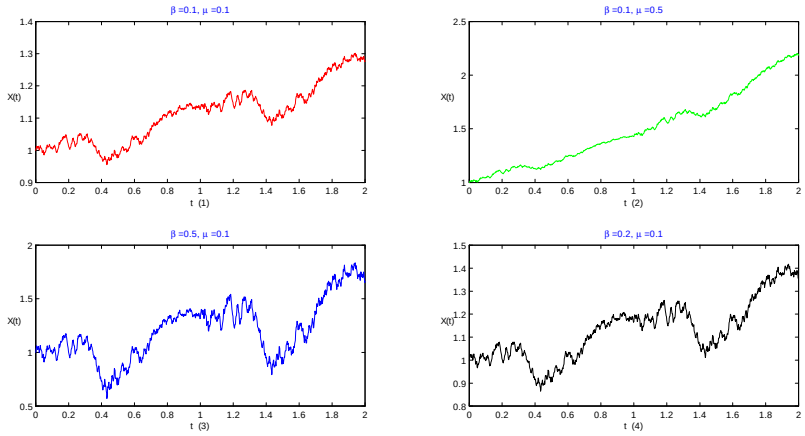

Figure 4.2. Sample path with different coefficients β and μ.

Figure 4.2 provides the information about the effects of coefficients for the choice of initial function $\psi(t) = e^{-t}$. From the first and second graphs, it is seen that increasing the diffusion term increases the volatility. From the third and fourth graphs, we realize that change in the drift term only affects the value of the solution process and structure is preserved.

In this section, we consider Euler Maruyama, Milstein, Runge-Kutta and Heun methods for SDDEs and its convergence analysis. With the help of methods, simulations of the example is done. In that simulations, the effect of initial function is considered. It is observed that they have an important effect on the evolution of the solution process and corresponding expected value in the future states.

5. Conclusion

Stochastic delay differential equations (SDDEs) are become really important in many areas of science to understand the real world phenomena as well as to understand future behaviors of systems. They include both historical information and randomness. SDDEs are actually a generalization of deterministic delay differential equations and stochastic differential equations. SDDEs provide a more realistic model for many systems than deterministic delay differential equations (DDEs) and stochastic differential equations (SDEs). In this article, stochastic delay differential equations (SDDEs) are handled together with definitions and their numerical approaches. The properties of SDEs are provided to make easy to follow concept for SDDEs because of the complicated characteristic of them. The existence and properties of the solution process for SDDEs are discussed. In order to solve them, iteration is used and the time interval is divided into pieces with a length of the delay term. Numerical methods for the solution of stochastic differential equations are essential for the analysis of random phenomena. Several approaches exist for strong solvers. Independent of the choice of stochastic differential equation solver, methods of variance reduction exist that may increase computational efficiency. The replacement of pseudorandom numbers with quasi random analogues from low-discrepancy sequences is applicable as long as statistical independence along

trajectories is maintained. In addition, control variates offer an alternate means of variance reduction and increases inefficiency simulation of stochastic differential equations trajectories Heun. Since, in general, finding closed form solution is not easy for a model with delay, numerical treatments are handled in article. We consider Several Methoods for SDDEs and its convergence analysis. With the help of method, simulations of the example is done. In these simulations, the effect of initial function and length of delay term are considered. It is observed that they have an important effect on the evolution of the solution process and corresponding expected value in the future states. In general, the dynamics and the coefficients of the SDDEs are adapted, in fact, driven by Itô integrals and the formulation and implementation of the Methods do not require anticipating calculus ideas. Heun method is much more accurate than other methods.

References

[1] D.Williams, Probability with martingales (Cambridge University Press, Cambridge, 1991). 315, 317, 322.

[2] L. Arnold, Stochastic differential equations: theory and applications (Wiley- Interscience, New York, 1974). 315, 326.

[3] I. Karatzas and S. E. Shreve, Brownian motion and stochastic calculus (Springer, New York, 1991). 315.

[4] X. Mao, Stochastic differential equations and their applications (Horwood Publishing, Chichester, 1997). 315, 317, 319, 319, 329, 331, 331.

[5] S. E. A. Mohammed, Stochastic functional differential equations (Pitman Advanced Publishing Program, Boston, MA, 1984). 315, 319.

[6] C. T. H. Baker, 'Retarded differential equations', J. Comput. Appl. Math., in press. 315.

[7] M. Zennaro, 'Delay differential equations: theory and numerics', Theory and numerics of ordinary and partial differential equations, Leicester, 1994 (ed. M. Ainsworth, J. Levesley, W. A. Light and M. Marletta, Oxford University Press, New York, 1995) 291–333. 315.

[8] K. Burrage and E. Platen, 'Runge–Kutta methods for stochastic differential equations', Ann. Numer. Math. 1 (1994) 63-78. 315.

[9] J. M. C. Clark, 'The discretization of stochastic differential equations: a primer', Road Vehicle Systems and Related Mathematics, Proc. 2^{nd} DMV-GAMMWorkshop, Torino, 1987 (ed. H. Neunzert, Teubner, Stuttgart, and Kluwer Academic Publishers, Amsterdam, 1988) 163-179. 315.

[10] E. Platen, 'An introduction to numerical methods for stochastic differential equations', Acta Numerica 8 (1999) 195–244. 315.

[11] D. Talay, 'Simulation of stochastic differential systems', Probabilistic methods in applied physics, Lecture Notes in Phys. 451 (ed. P. Krée and W. Wedig, Springer, Berlin, 1995) 54–96. 315.

[12] P. E. Kloeden and E. Platen, Numerical solution of stochastic differential equations (Springer, Berlin, 1992). 315, 320.

[13] G. N. Milstein, Numerical integration of stochastic differential equations (translated and revised from the 1988 Russian original) (Kluwer, Dordrecht, 1995). 315, 318, 320, 322, 324, 329, 331, 331, 331, 333.

[14] S.-E. A. Mohammed, (1998). Stochastic differential systems with memory: Theory, examples and applications. In Stochastic Analysis and Related Topics VI (L. Decreusefond, J. Gjerde, B. Øksendal and A. S. Ustunel, eds.) 1–77. Birkhäuser, Boston.

[15] H. Gilsing and T. Shardlow, SDELab: A package for solving stochastic differential equations in MATLAB, J. Comput. Appl. Math., 205 (2007), pp. 1002–1018.

[16] K. Twardowska, Wong-Zakai approximations for stochastic differential equations, Acta Appl. Math., 43 (1996), pp. 317–359.

[17] M. Giles, Improved multilevel Monte Carlo convergence using the Milstein scheme, in Monte Carlo and Quasi-Monte Carlo Methods 2006, A. Keller, S. Heinrich, and H. Niederreiter, eds., Springer, Berlin, 2008, pp. 343–358.

[18] P. E. Kloeden and T. Shardlow, The Milstein scheme for stochastic delay differential equations without using anticipative calculus, Stochastic Anal. Appl., 30 (2012), pp. 181–202.
[19] C. Tudor and M. Tudor, On approximation of solutions for stochastic delay equations, Stud. Cerc. Mat., 39 (1987), pp. 265–274.
[20] E. Wong and M. Zakai, On the convergence of ordinary integrals to stochastic integrals, Ann. Math. Statist., 36 (1965), pp. 1560–1564.
[21] D. G. Hobson and L. C. G. Rogers, Complete models with stochastic volatility, Math. Finance, 8 (1998), pp. 27–48.
[22] M. Liu, W. Cao, and Z. Fan, Convergence and stability of the semi-implicit Euler method for a linear stochastic differential delay equation, J. Comput. Appl. Math., 170 (2004), pp. 255–268.
[23] K. Twardowska, T. Marnik, and M. Pas_lawaska-Po_luniak, Approximation of the Zakai equation in a nonlinear filtering problem with delay, Int. J. Appl. Math. Comput. Sci., 13 (2003), pp. 151–160.
[24] I. Karatzas and S. E. Shreve, Brownian Motion and Stochastic Calculus, 2nd ed., Springer-Verlag, New York, 1991.
[25] G. N. Milstein, Yu. M. Repin, and M. V. Tretyakov, Numerical methods for stochastic systems preserving symplectic structure, SIAM J. Numer. Anal., 40 (2002), pp. 1583–1604.
[26] E. Buckwar and T. Sickenberger, A comparative linear mean-square stability analysis of Maruyama- and Milstein-type methods, Math. Comput. Simulation, 81 (2011), pp. 1110–1127.
[27] W. Cao and Z. Zhang, On exponential mean-square stability of two-step Maruyama methods for stochastic delay differential equations, J. Comput. Appl. Math., 245 (2013), pp. 182–193.
[28] D. F. Kuznetsov, Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach, St. Petersburg State Polytechnic University, St. Petersburg, 2011, (in Russian).
[29] X. Mao and L. Shaikhet, Delay-dependent stability criteria for stochastic differential delay equations with Markovian switchin0g, Stab. Control Theory Appl., 3 (2000), pp. 88–102.
[30] K. Twardowska, On the approximation theorem of the Wong-Zakai type for the functional stochastic differential equations, Probab. Math. Statist., 12 (1991), pp. 319–334.
[31] R. Paley and N. Wiener, Fourier Transforms in the Complex Domain, AMS, Providence, RI, 1934.
[32] E.Wong and M. Zakai, On the relation between ordinary and stochastic differential equations, Internat. J. Engrg. Sci., 3 (1965), pp. 213–229.
[33] C. Huang, S. Gan, and D. Wang, Delay-dependent stability analysis of numerical methods for stochastic delay differential equations, J. Comput. Appl. Math., 236 (2012), pp. 3514–3527.
[34] Y. Hu, S.-E. A. Mohammed, and F. Yan, Discrete-time approximations of stochastic delay equations: The Milstein scheme, Ann. Probab., 32 (2004), pp. 265–314.
[35] N. Rosli, A. Bahar, S. H. Yeak, and X. Mao, A systematic derivation of stochastic Taylor methods for stochastic delay differential equations, Bull. Malays. Math. Soc. (2), 36 (2013), pp. 555–576.
[36] M. D. Paola and A. Pirrotta, Time delay induced effects on control of linear systems under random excitation, Probab. Engrg. Mech., 16 (2001), pp. 43–51.
[37] W. Cao and Z. Zhang, Simulations of two-step Maruyama methods for nonlinear stochastic delay differential equations, Adv. Appl. Math. Mech., 4 (2012), pp. 821–832.
[38] L. S. Tsimring and A. Pikovsky, Noise-induced dynamics in bistable systems with delay, Phys. Rev. Lett., 87 (2001), 250602.
[39] E. Buckwar and R. Winkler, Multi-step Maruyama methods for stochastic delay differential equations, Stochastic Anal. Appl., 25 (2007), pp. 933–959.
[40] E. Buckwar and R. Winkler, Multistep methods for SDEs and their application to problems with small noise, SIAM J. Numer. Anal., 44 (2006), pp. 779–803.
[41] M. Grigoriu, Control of time delay linear systems with Gaussian white noise, Probab. Engrg. Mech., 12 (1997), pp. 89–96.
[42] I. Gyöngy and A. Shmatkov, Rate of convergence of Wong-Zakai approximations for stochastic partial differential equations, Appl. Math. Optim., 54 (2006), pp. 315–341.
[43] V. Mackevičius, On approximation of stochastic differential equations with coefficients depending on the past, Liet. Mat. Rink., 32 (1992), pp. 285–298.

[44] M. Tudor, Approximation schemes for stochastic equations with hereditary argument, Stud. Cerc. Mat., 44 (1992), pp. 73–85.
[45] U. Küchler and E. Platen, Strong discrete time approximation of stochastic differential equations with time delay, Math. Comput. Simulation, 54 (2000), pp. 189–205.
[46] G. N. Milstein and M. V. Tretyakov, Stochastic Numerics for Mathematical Physics, Springer-Verlag, Berlin, 2004.
[47] X. Wang and S. Gan, B-convergence of split-step one-leg theta methods for stochastic differential equations, J. Appl. Math. Comput., 38 (2012), pp. 489–503.
[48] H. Zhang, S. Gan, and L. Hu, The split-step backward Euler method for linear stochastic delay differential equations, J. Comput. Appl. Math., 225 (2009), pp. 558–568.
[49] Q. Guo, W. Xie, and T. Mitsui, Convergence and stability of the split-step θ-Milstein method for stochastic delay Hopfield neural networks, Abstr. Appl. Anal., 2013 (2013), 169214.
[50] G. N. Milstein and M. V. Tret'yakov, Mean-square numerical methods for stochastic differential equations with small noises, SIAM J. Sci. Comput., 18 (1997), pp. 1067–1087.
[51] S. Ogawa, On a deterministic approach to the numerical solution of the SDE, Math. Comput. Simulation, 55 (2001), pp. 209–214.
[52] C.T.H. Baker and E. Buckwar, Numerical analysis of explicit one-step methods for stochastic delay differential equations, LMS J. Comput. Math. 3, 315-335, (2000).
[53] N. Hofmann and T. M¨uller-Gronbach, A modified Milstein scheme for approximation of stochastic delay differential equations with constant time lag, J. Comput. Appl. Math., 197 (2006), pp. 89–121.
[54] K. Burrage and P.M. Burrage, High strong order explicit Runge-Kutta methods for stochastic ordinary differential equations, Appl. Numer. Math. 22, 81-101, (1996).
[55] P.M. Burrage, Runge-Kutta methods for stochastic differential equations, Ph.D. Thesis, The University of Queenslaud, Australia, (1999).
[56] Y. Saito and T. Mitsui, T-stability of numerical schemes for stochastic differential equations, World Sci. Set. Appl. Anal. 2, 333-344, (1993).
[57] Y. Saito and T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, SIAM J. Numer. Anal. 33, 2254-2267, (1996).
[58] X. Zhu, 3. Li, H. Li and S. 3iang, The stability properties of Milstein scheme for stochastic differential equations, J. Huazhong Univ. of Sei. Tech. 31, 111-113, (2003).
[59] E. Buckwar, Introduction to the numerical analysis of stochastic delay differential equations, J. Comp. Appl. Math. 125, 297-307, (2000).
[60] W. Cao, M. Liu and Z. Fan, MS-stability of the Euler-Maruyama method for stochastic differential dalay equations, Appl. Math. Comp. 159, 127-135, (2004).
[61] D. Bratsun, D. Volfson, L.S. Tsimring, J. Hasty, Delay-induced stochastic oscillations in gene regulation, Proc. Natl. Acad. Sci. USA 102 (2005) 14593–14598.
[62] R. Schlicht, G. Winkler, A delay stochastic process with applications in molecular biology, J. Math. Biol. 57 (2008) 613–648.
[63] X. Mao, S. Sabanis, Numerical solutions of stochastic differential delay equations under local Lipschitz condition, J. Comput. Appl. Math. 151 (2003) 215–227.
[64] W.R. Cao, Z.Q. Zhang, G.E. Karniadakis, Numerical methods for stochastic delay differential equations via the Wong–Zakai approximation, SIAM J. Sci.Comput. 37 (2015) A295–A318.
[65] C.T.H. Baker, E. Buckwar, Exponential stability in pth mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations, J. Comput. Appl. Math. 184 (2005) 404–427.
[66] X. Wang, S. Gan, The improved split-step backward Euler method for stochastic differential delay equations, Int. J. Comput. Math. 88 (2011) 2359–2378.
[67] Z.Y. Wang, C.J. Zhang, An analysis of stability of Milstein method for stochastic differential equations with delay, Comput. Math. Appl. 51 (2006) 1445–1452.
[68] S.Q. Gan, A.G. Xiao, D.S. Wang, Stability of analytical and numerical solutions of nonlinear stochastic delay differential equations, J. Comput. Appl. Math. 268 (2014) 5–22.
[69] C.M. Huang, Mean square stability and dissipativity of two classes of theta methods for systems of stochastic delay differential equations, J. Comput. Appl. Math. 259 (2014) 77–86.
[70] X. Mao, Exponential stability of equidistant Euler–Maruyama approximations of stochastic differential delay equations, J. Comput. Appl. Math. 200 (2007) 297–316.

[71] L. Chen, F.K. Wu, Almost sure exponential stability of the backward Euler–Maruyama scheme for stochastic delay differential equations with monotone-type condition, J. Comput. Appl. Math. 282 (2015) 44–53.
[72] A. Abdulle, G. Vilmart, K.C. Zygalakis, Weak second order explicit stabilized methods for stiff stochastic differential equations, SIAM J. Sci. Comput. 35 (2013) A1792–A1814.
[73] A. Röβler, Second order Runge–Kutta methods for Itô stochastic differential equations, SIAM J. Numer. Anal. 47 (2009) 1713–1738.
[74] A. Röβler, Runge–Kutta methods for the strong approximation of solutions of stochastic differential equations, SIAM J. Numer. Anal. 48 (2010) 922–952.
[75] A. Rathinasamy, K. Balachandran, Mean-square stability of second-order Runge–Kutta methods for multi-dimensional linear stochastic differential systems, J. Comput. Appl. Math. 219 (2008) 170–197.
[76] V. Reshniak, A.Q.M. Khaliq, D.A. Voss, G. Zhang, Split-step Milstein methods for multi-channel stiff stochastic differential systems, Appl. Numer. Math. 89 (2015) 1–23.
[77] Evans,(2005): Lawrence C. Evance, "An Introduction to Stochastic Differential Equations", Version 1.2, Lecture Notes, Short Course at SIAM Meeting, July, 2005.
[78] Fridman,(1975) : AvnerFridman, "Stochastic Differential Equations and Applications", Volume 1, Academic Press, Inc., 1975.
[79] Gard, T. G. Gard, (1988). "Introduction to Stochastic Differential Equations", Marcel Dekker, New York, (1988).
[80] Glasserman P, Monte Carlo methods in numerical engineering. Springer, New York(2004).
[81] Higham DJ, An algorithmic introduction to numerical simulation of stochastic differential equation, s (2001).
[82] Higham DJ, Kloeden P, Numerical methods for nonlinear stochastic differential equations with jumps. Num Math.,(2005).
[83] Lamba H, Mattingly JC, Stuart A, An adaptive Euler-Maruyama scheme for SDEs : convergence and stability.(2008).
[84] F. Wu, X. Mao, L. Szpruch, Almost sure exponential stability of numerical solutions for stochastic delay differential equations, Numer. Math. 115 (2010) 681–697.
[85] Y. Saito, T. Mitsui, Mean-square stability of numerical schemes for stochastic differential systems, Vietnam J. Math. 30 (2002) 551–560.
[86] Abdu Khaled M, (2004)., "Mean Square Stability of Second-Order Weak Numerical Methods for Stochastic Differential Equations", Applied Numerical Mathematics, Vo.48, 127-134.
[87] Bernard P. and Fleury G. ,(2001), "Convergence of Schemes for Stochastic Differential Equations; Monte Carlo Methods", Applied, Vol.7(1), 35-53.
[88] Lamba H, Mattingly JC, Stuart A, An adaptive Euler-Maruyama scheme for SDEs : convergence and stability.(2008).
[89] M. Carletti, Numerical solution of stochastic differential problems in the biosciences, Journal of computational and applied mathematics, 185(2), pp. 422–440, 2006.
[90] A. Abdulle, S. Cirilli, S-ROCK: Chebyshev methods for stiff stochastic differential equations, SIAM J. Sci. Comput. 30 (2008) 997–1014.
[91] A. Abdulle, T. Li, S-ROCK methods for stiff Itô SDEs, Commun. Math. Sci. 6 (2008) 845–868
[92] Y. Komori, K. Burrage, Weak second order S-ROCK methods for Stratonovich stochastic differential equations, J. Comput. Appl. Math. 236 (2012) 2895–2908.
[93] Y. Komori, K. Burrage, Strong first order S-ROCK methods for stochastic differential equations, J. Comput. Appl. Math. 242 (2013) 261–274.

[1,2]Department of Mathematics, University of Bitlis Eren, Bitlis, Turkey
E-mail address: hamitsa86@gmail.com, baratiakbar@yahoo.com