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Common Fixed Point Theorem under the pair of T-Zamfirescu Mappings in Cone Metric Space

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Abstract

In this paper we find sufficient conditions for the existence of unique common fixed point for pairs of T-Zamfirescu contraction mappings.
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1. Introduction

We know that the classic contraction mapping principle of Banach is a fundamental result in fixed point theory. Several authors have obtained various extensions and generalizations of Banach's theorem by considering contractive mappings on many different metric spaces. Recently, Guang and Xian[5] introduce the notion of cone metric spaces. He replaced real number system by ordered Banach space. He also gave the condition in the setting of cone metric spaces. They also described the convergence of sequence in the cone metric spaces and introduce the corresponding notion of completeness. The results in [5] were generalized by Sh. Rezapour and R. Hamlbarani omitting the assumption of normality on the cone. Subsequently, many authors have generalized the results of Guang and Xian and have studied fixed point theorems for normal and non-normal cone.

In 2009 [1], A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh introduced new classes of contractive functions T-Contractive and T-contractive mappings and then they established and extended the Banach contraction principle and the Edelstein's fixed point theorems. S. Moradi[8] introduce the T-Kannan contractive mapping which extended the well known Kanna's fixed point theorem given in[6]. In sequel, J.R. Morales and E. Rojas[3] and[4] obtained sufficient conditions for the existence of unique fixed point of T-Kannan contractive and T-contractive mappings respectively, on complete cone metric spaces.

In this paper we study the existence of unique common fixed point for pair of T-Zamfirescu contraction mappings in complete cone metric space. Or results is generalizing of [5],[2] and [10].

2. **Preliminary**

Definition 2.1: Let E be a real Banach space and P a subset of E. P is called cone if and only if:

(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real number a, b;
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in intP, intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y\ implies\ \|x\| \leq K\|y\|$.

The least positive number K satisfying the above is called the normal constant of P.

Definition 2.2: Let X be a non-empty set and $d: X \times X \to E$ be a mapping such that

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3: Let (X, d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$;

(i) (x_n) converges to x if for every $c \in E$ with $0 \ll c$ there is an n_0 such that for all $n > n_0, d(x_n, x) \ll c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x, (n \to \infty)$.
(ii) If for any $c \in E$ with $0 \ll c$ there is an n_0 such that for all $n, m \geq n_0, d(x_n, x_m) \ll c$, then (x_n) is called a Cauchy sequence in X. Let (X, d) be a cone metric space. If every Cauchy sequence in convergent in X, then X is called a complete cone metric space.

Lemma 2.4: Let (X, d) be a cone metric space, $P \subset E$ a normal cone with normal constant K. Let $(x_n), (y_n)$ be a sequence in X and $x, y \in X$.

(i) (x_n) converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$;
(ii) If (x_n) converges to x and (x_n) converges to y then $x = y$. That is the limit of (x_n) is unique;
(iii) If (x_n) converges to x, then (x_n) is Cauchy sequence;
(iv) If $x_n \to x$ and $y_n \to y, (n \to \infty)$ then $d(x_n, y_n) \to d(x, y)$.

Definition 2.5: Let (X, d) be a cone metric space, P a normal cone with normal constant K and $T: X \to X$. Then

(i) T is said to be continuous if $\lim_{n\to\infty} x_n = x$, implies that $\lim_{n\to\infty} T(x_n) = T(x)$, for all (x_n) in X;

(ii) T is said to be subsequentially convergent, if we have, for every sequence (y_n) that $T(y_n)$ is convergent, implies (y_n) has a convergent subsequence;

(iii) T is said to be sequentially convergent, if we have, for every sequence (y_n), if $T(y_n)$ is convergent, then (y_n) is also convergent.

Definition 2.6: Let (x, d) be a cone metric space $T, S: X \to X$ two mappings

(i) The mapping S is called a T-Banach contraction, if there is $a \in [0,1)$ such that $d(TSx, TSy) \leq ad(Tx, Ty)$ for all $x, y \in X$.

(ii) A mapping S is called a T-Kannan contraction, if there is $b \in \left[0, \frac{1}{2}\right)$ such that $d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)]$ for all $x, y \in X$.

(iii) A mapping S is said to be Chatterjea contraction, if there is $c \in [0, \frac{1}{2})$ such that $d(TSx, TSy) \leq c[d(Tx, Ty) + d(Ty, TSx)]$ for all $x, y \in X$.

Definition 2.7: Let (X, d) be a cone metric space and $T, S: X \to X$ two mappings, f and g are T-Zamfirescu mapping (TZFS-mapping), if and only if, there are real numbers, $0 \leq a < 1, 0 \leq b, c < \frac{1}{2}$ such that for all $x, y \in X$, at least one of the next conditions are true;

(i) $d(Tfx, Tgy) \leq ad(Tx, Ty)$;

(ii) $d(Tfx, Tgy) \leq b[d(Tx, Tfx) + d(Ty, Tgy)]$;

(iii) $d(Tfx, Tgy) \leq c[d(Tx, Tgy) + d(Ty, Tfx)]$

Lemma 2.8: Let (X, d) b a cone metric space and $T, S: X \to X$ two mappings. If f, g are TZFS-mapping, then there is $0 \leq \delta < 1$ such that

$$d(Tfx, Tgy) \leq \delta d(Tx, Ty) + 2\delta d(Tx, Tfx) \text{ for all } x, y \in X$$

Above inequality can be replaced by

$$d(Sfx, Sgy) \leq \delta d(Sx, Sy) + 2\delta d(Sx, Sgy) \text{ for all } x, y \in X.$$

3. Main result

Let (X,d) be a complete cone metric space , P be a normal cone with normal constant K. Moreover, let $T, S: X \to X$ be a continuous and one-to-one self mappings and $f, g: X \to X$ be a pair of T-Zamfirescu (TZFS) continuous mappings. Then

(1) For every $x \in X$,

$$\lim_{n\to\infty} d(Tf^{2n+1}x, Tg^{2n+2}x) = 0$$

and

$$\lim_{n\to\infty} d(Sf^{2n+2}x, Sg^{2n+3}x) = 0$$

(2) There is $u, v \in X$ such that

$$\lim_{m\to\infty} Tf^{2m+1}x = u = \lim_{m\to\infty} Tg^{2m+2}x$$

and

$$\lim_{n\to\infty} Sg^{2n+2}x = v = \lim_{n\to\infty} Sg^{2n+3}x$$

(3) There is unique common fixed point $u \in X$ such that $fu = gu = u$.

Proof: (1) Since f and g are pair of TZFS mappings, then by Lemma (2.8), there exist $0 < \delta < 1$ such that

$$d(Tfx, Tgy) \leq \delta d(Tx, Ty) + 2\delta d(Tx, Tfx) \text{ for all } x, y \in X$$

and

$$d(Sfx, Sgy) \leq \delta d(Sx, Sy) + 2\delta d(Sx, Sfx) \text{ for all } \quad x, y \in X$$

Suppose $x \in X$ be an arbitrary point and the Picard iteration associated to f and g. (x_{2n+1}) and $(x_{2n+2)}$ are defined by

$$x_{2n+2} = fx_{2n+1} = f^{2n+1}x \text{ , n=0,1,2,…}$$

$$x_{2n+3} = gx_{2n+2} = g^{2n+2}x \text{ , n=0,1,2,…}$$

$$d(Tf^{2n+1}x, Tg^{2n+2}x) \leq hd(Tf^{2n}x, Tg^{2n+1}x)$$

where $h = \frac{\delta}{1-2\delta} < 1$.

Therefore, for all n we have

$$d(Tf^{2n+1}x, Tg^{2n+2}x) \leq h^{2n+1}d(Tfx, Tgx) \qquad (1)$$

Similarly, we have

$$d(Sf^{2n+2}x, Sg^{2n+3}x) \leq k^{2n+2}d(Sfx, Sgx) \qquad (2)$$

From (1) and the fact the cone P is a normal cone we obtain that

$$\|d(Tf^{2n+1}x, Tg^{2n+2}x\| \leq kh^{2n+1}\|d(Tfx, Tgx)\|$$

taking limit $n \to \infty$ in the above inequality, we can conclude that

$$\lim_{n\to\infty} d(Tf^{2n+1}x, Tg^{2n+2}x) = 0$$

Similarly, from (2) we have

$$\lim_{n\to\infty} d(Sf^{2n+2}x, Sg^{2n+3}x) = 0$$

(2) Now for $m, n \in N$ with $m > n$, we get

$$d(Tf^{2m+1}x, Tg^{2n+1}x) \leq (h^{2n+1} + \cdots + h^{2m)}$$

$$\leq \frac{h^{2n+1}}{1-h} d(Tfx, Tgx)$$

Again, as above since P is a normal cone we obtain

$$\lim_{n,m\to\infty} d(Tf^{2m+1}x, Tg^{2n+1}) = 0$$

Hence, the fact that (X, d) is complete cone metric space, imply that $(Tf^{2m+1}x)$ is a Cauchy sequence in X, therefore there is $u \in X$ such that

$$\lim_{m\to\infty} Tf^{2m+1}x = u = \lim_{m\to\infty} Tg^{2m+2}x$$

Similarly we can conclude that

$$\lim_{n\to\infty} Sf^{2n+2}x = v = \lim_{n\to\infty} Sg^{2n+3}x$$

(3) Since T and f are continuous mappings, we obtain;

$$\lim_{i\to\infty} Tf^{(2n+1)i}x = Tu, \qquad \lim_{i\to\infty} Tf^{(2n+1)i}x = Tfu$$

therefore, $Tu = y = Tfu$ and since T is one-to-one, then $fu = u$. So f has a fixed point.

Now, suppose that $fu = u$ and $fu_1 = u_1$,

$$d(Tfu, Tfu_1) \leq \delta d(Tu, Tu_1) + 2\delta d(Tu, Tfu)$$

$$d(Tu, Tu_1) \leq \delta d(Tu, Tu_1)$$

from the fact that $0 \leq \delta < 1$ and that T is one-to-one obtain that $u = u_1$.

Again since S and g are also continuous mappings, we obtain;

$$\lim_{i\to\infty} Sg^{(2n+1)i}x = Su, \lim_{i\to\infty} Tg^{(2n+1)i}x = Tgu$$

Therefore $Su = z = Sgu$ and since S is one-to-one, then $gu = u$, so g has a fixed point.

Now suppose that $Su = u$ and $Su_1 = u_1$,

$$d(Sfu, Sfu_1) \leq \delta d(Su, Su_1) + 2\delta d(Su, Sgu)$$

$$d(Su, Su_1) \leq \delta d(Su, Su_1)$$

from the fact that $0 \leq \delta < 1$ and that δ is one-to-one we obtain that $u = u_1$.

Thus $fu = gu = u$.

This completes the proof.

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