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Facets of Non-Master Additive System Polyhedra

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Abstract

The lifting of the facet is a technique used to generate polyhedral facet for the integer optimization problem. For groups, semigroups, and abelian additive systems master problems, homomorphism and sub-morphism can be used for lifting the facets. Another known methodology is the sequential lifting, which provides a new facet from a facet an element which not considered by facet, thus considering a new element. For abelian groups, there exist results for the sequential lifting of facets to consider the algebraic aspect and not the geometric aspect of the polyhedron. In this case of semigroups or additive systems master problems, the subadditive cone is important to the lifting facet. These results do not use the polyhedron polarity to lifting facet, in this paper we used the polarity polyhedra results to define sequential lifting facets of non-master associative, abelian, and b-complementary. The results presented here extend the known theorems of the sequential lifting for groups and semigroups. The sequential lifting of facets theorems for non-master problems doesn't consider the polarity of the polyhedron to characterize facets, as far as we know, this is the first result that establishes sequential lifting for associative, abelian and b-complementary additive system non-master problems.

Keywords: additive system, polyhedra-polarity, sequential lifting.
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1 Introduction

In general, for master additive system polyhedra (see [5]) there are results of lifting facets, but these cannot be used for non-master since when projecting a facet of a master polyhedra onto non-master polyhedra we do not necessarily obtain a facet of the non-master polyhedra. But, in the case where the algebraic structure associated with the polyhedra is an abelian group, Gastou (see [7]) showed how to lift facets in a sequential way. Gastou's results (see [7]) for sequential liftings of facets could be considered from an algebraic point of view but he does not consider the polarity of the polyhedron to characterize the facets. The central motivation of this work is that the polarity of polyhedra considers the polyhedron from a geometric point of view.

The main objective of this article is to define the sequential lift for the polyhedra of the associative, abelian, and b-complementary additive system. We extend the result of the sequential lift the semigroups (see [10]) for multivalued additive systems. The work is structured as follows: In section 2, we present the additive system and the optimization problem associated with this algebraic structure. The polarity results are presented in section 3. In the main part of the article, presented in Section 4, we show the sequential survey theorem for non-master polyhedra of the associative, abelian, and b-complementary additive system.

2 Associative, Abelian and b-complementary Additive System

Additive System. An *additive system* is defined to be a non-empty finite set A together with a function $\widehat{+}$ such that :

(i) $\{g\}\widehat{+}\{h\} \subseteq A$, for all g and h in A;

(ii) $S\widehat{+}T = \cup_{s\in S, t\in T}(\{s\}\widehat{+}\{t\})$, for all $S, T \subseteq A$.

In this paper we denoted an additive system by the pair $(A, \widehat{+})$.

Usually, we consider closed functions $\widehat{+}$ and we use

$s\widehat{+}t$ by $\{s\}\widehat{+}\{t\}$. On the other hand, an additive system $(A,\widehat{+})$ is *associative* if it satisfies

$$(S\widehat{+}T)\widehat{+}U = S\widehat{+}(T\widehat{+}U)$$

for all $S,T,U \subseteq A$, and an *abelian* additive system if

$$S\widehat{+}T = T\widehat{+}S$$

for all $S,T \subseteq A$.

We assume, without loss of generality, that there exists a *zero* $\widehat{0} \in A$ such that $\widehat{0}\widehat{+}g = g\widehat{+}\widehat{0} = g$, for all $g \in A$. If there were no such element in A we could adjoin one to A without changing $(A,\widehat{+})$. The zero is clearly unique.

Subsystem. Define a *sub-system* $(T,\widehat{+})$ of $(A,\widehat{+})$ to be a subset T of A, with the same function $\widehat{+}$ from $(A,\widehat{+})$, such that $(T,\widehat{+})$ is itself an additive system.

Expressions. An *expression* E of an additive system $(A,\widehat{+})$ is defined recursively by

(i) The empty string ξ is an expression called the *empty expression*.

(ii) (g) is an expression, for all $g \in A$, called a *primitive expression*.

(iii) $E = (E_1\widehat{+}E_2)$ is an expression, whenever E_1 and E_2 are non-empty expressions, and E_1 and E_2 are subexpressions of E.

A *primitive subexpression* of E is any subexpression of E which is a primitive expression.

Evaluation. The *evaluation* of an expression E is a function γ from expressions to subsets of A defined by

(i) $\gamma(\xi) = \widehat{0}$;

(ii) $\gamma((g)) = \{g\}$;

(iii) $\gamma((E_1)\widehat{+}(E_2)) = \gamma((E_1))\widehat{+}\gamma((E_2))$

To *evaluate* E means to find $\gamma(E)$, which can be done recursively using the definition.

Solution vector. Let $(A,\widehat{+})$ be an additive system. A vector $(t(g); g \in A)$ is the *incidence vector of an expression E* if $t(g)$ is equal to the number of times (g) appears as a primitive subexpression of E. Now, let $g \in A$. A vector t *represents* g if there is some expression E for which t is the incidence vector of E, and $g \in \gamma(E)$.

Let b be one fixed element in A, we call b the *right-hand side*. An expression E is a *solution expression* for b if $b \in \gamma(E)$. And, the incidence vector t of an expression E is a *solution vector* for b, if E is a solution expression. That is, t is a solution vector if it represents b.

We denote by $P(A,b)$ the *convex hull* of solutions vector, that is:

$$P(A,b) = convexhull\,\{(t(g), g \in A : t \text{ is a solution vector}\}$$

Araóz and Johnson [5] described the characterization for vertices and facets of the $P(A,b)$.

Let $(A, \widehat{+})$ be an abelian associativity additive system. For any positive integer k and any $g \in A$, we define kg by

$$k * g = \gamma(g \widehat{+} .. \widehat{+} g)$$

where $g \widehat{+} .. \widehat{+} g$ is taken k times.

The elements of the subsystem generated by g are the $h \in A$ such that $h \in kg$, for some $k \geq 0$. Clearly

$$0g = \gamma(\xi) = \left\{\widehat{0}\right\} \text{ and } 1g = \gamma(g) = \{g\}.$$

Now, since there are only a finite number of subsets of A in the sequence of sets $0g, 1g, 2g, \ldots, kg, \ldots$ there are sets which appear infinitely many times, such sets are called *loop sets of* g.

The *loop of* g is the union of all the loop sets of g. We define g *goes to* ϕ and write $g \to \phi$ when the loop of g is empty, otherwise we write g not $\to \phi$.

Let $s = mg$ be the first occurrence of any set appearing for the second time in the sequence $(kg \mid k \geq 0)$. Since s appears the second time in the sequence, $s = pg$ for some $p < m$, and the sequence of distinct sets $(kg \mid p < k \leq m-1)$ is the same as $(kg \mid m \leq k \leq 2m-p-1)$. In fact we have $(p+k+il)g = (p+kg)$ (where $l = m-p$) for $0 \leq k \leq l-1$ and $i \geq 0$, since $(m+k)g = mg \widehat{+} kg = pg \widehat{+} kg = (p+k)g$.

The *loop order* of g is defined to be this l. Clearly $h \in A$ is in loop of g if and only if there exists $k \geq 0$ such that $h \in (k+il)g$ for all $i \geq 0$, where l is the loop order of g.

Let $(A, \widehat{+})$ be an additive system and $b, g \in A$. We define $b \sim g$ by

$$b \sim g = \{x \in A : b \in x \hat{+} g\}.$$

A definition previous to this indicates a partial order in A, that is, we say $h \widetilde{>} g$

when $b \sim g \subseteq b \sim h$. When the set $b \sim g$ has a minimum element, this minimum element is called the $b-complement$ of g. We denote by $\widehat{g}$ the $b-complement$ of g, and, A is *b-complementary* when every element has a $b-complement$.

An element $g \in A$ is *infeasible* if $b \sim g = \emptyset$. We can assume, without loss of generality, that the additive system has at most one infeasible element denoted by $\widehat{\infty}$.

In this paper we denote by: $A_+ = A - \{\hat{0}, \widehat{\infty}\}$ the set of proper elements. Let $(A, \widehat{+})$ be an abelian, associate and b-complementary additive system. Given $b \in A$ and $M \subseteq A_+$, a vector $t \in \mathbb{N}^M$ is a *solution vector* if

$$b \in \widehat{\sum_{g \in M}} t(g) * g.$$

We denoted the set solution vectors by

$$T(A, M, b) = \left\{ t \in \mathbb{N}^M : b \in \widehat{\sum_{g \in M}} t(g) * g \right\}$$

and $P(A, M, b)$ is the convex hull of $T(A, M, b)$.

We are interested in the optimization problem

$$min \sum_{g \in M} c(g) t(g)$$

$$\text{subject to: } b \in T(A, M, b).$$

where $c \in \mathbb{R}^M$.

The optimization problem is called the *Master Problem* if $M = A_+$, and when $M \neq A_+$ it's called the *Non-Master Problem*. We are interested in lifting facets of the $P(A, M, b)$ where $M \neq A_+$.

3 Lifting Polarity

The polarity of polyhedra given by general bilinear relations have been described in Araóz, Edmonds and Griffin [3]. There they showed that there are six different polarity relations, four of which have been extensively studied: The Cone Polarity, The Minkowski Polarity, The Reverse Minkowski Polarity, and the Polarity of Convex Sets.

Let N and H be finite sets. Given $W \in \mathbb{R}^{N\times H}$, $u \in \mathbb{R}^N$, $v \in \mathbb{R}^H$ and $\alpha \in \mathbb{R}$, the generalized bilinear inequality (see [9]) $\Omega \subset \mathbb{R}^N \times \mathbb{R}^H$ is defined by

$$x\Omega y \text{ if and only if } xWy + xu + vy \leq \alpha. \tag{1}$$

For any set $P \subset \mathbb{R}^N$ the Ω -polar of P is $P^\Omega = \left\{y \in \mathbb{R}^H : x\Omega y \text{ for all } x \in P\right\}$, and, P is Ω-closed when $P = P^{\Omega\Omega}$.

A polyhedron is defined to be the solution set of a finite system of linear inequalities, that is $P \subset \mathbb{R}^N$ is a polyhedron if and only if there exist matrix $A \in \mathbb{R}^{N\times H}$ and $b \in \mathbb{R}^H$ such that $P = \{x \in \mathbb{R}^N : Ax \leq b\}$. On this definition of polyhedron in this work, we will use the following notation. For any system of linear inequalities $LI = Ax \leq b$, $P(LI)$ denote the polyhedron $\left\{x \in \mathbb{R}^N : Ax \leq b\right\}$. For any finite sets S and T we denote the convex hull of S by $CONV(S)$, the conical hull of T by $CONE(T)$ and $CONV(S) + CONE(T)$ by $C(S,T)$. $dim(P)$ will denote the dimension of P.

Let P be the polyhedron $C(S,T)$. Then the following theorem characterizes P^Ω.

Theorem 3.1 *(3.12 [2]) P^Ω equals $P(LI)$, where LI is the system of linear inequalities defined by*

$$LI = \begin{cases} sWy + su + vy \leq \alpha \text{ for all } s \in S; \\ tWy + tu \leq 0 \text{ for all } t \in T. \end{cases}$$

Corollary 3.1 *(2.4 [3]) The Ω-polar of a polyhedron is a polyhedron.*

In general Ω defines two sets which provide the different polarity types (see [2]), they are

$$X_\Omega = \left\{x \in \mathbb{R}^N : xW + v = 0\right\};\; Y_\Omega = \left\{y \in \mathbb{R}^H : Wy + u = 0\right\}. \tag{2}$$

For these sets, we have the following results

Lemma 3.1 *(3.20 [2]) $X_\Omega \neq \emptyset$ and $Y_\Omega \neq \emptyset$ implies for all $x \in X_\Omega$, xu is a constant value denoted by $\widehat{\alpha} = xu$.*

Lemma 3.2 *(3.21 [2]) $X_\Omega \neq \emptyset$ and $Y_\Omega = \emptyset$ implies for all β there exists $x \in X_\Omega$ such that $xu = \beta$.*

The *Lifting Polarity* is an example for $X_\Omega \neq \emptyset$ and $Y_\Omega = \emptyset$, when $N = H \cup L$ and Ω is given by$x_H y + x_L u_L \leq \alpha + ry$. In [2] Aráoz, Edmond and Griffin give the relation between the vertices and extreme rays, that is, the generators, of a pointed Ω-closed polyhedron P and the facets of P^Ω for the two cases of polarity.

In this paper we will use the following result.

Theorem 3.2 *(2.14 [3]) Let $P = P(\{xA \leq b\})$ be full dimensional and Ω be a bilinear inequality relation expressed in the form 1 such that $dim(P \cap X_\Omega) = dim(X_\Omega) > 0$, $xu \leq \alpha$ is a facet of $P \cap X_\Omega$ and 0 is the only solution to $Wz = 0$. Then $P^\Omega = C(S,T)$ and is pointed with S being the set of vertices of P^Ω and T the set of the extreme rays of the recessional cone of P^Ω, where*

$$\begin{aligned} S &= \left\{ s \in \mathbb{R}^H : xWs + xu + vs \leq \alpha \text{ is a facet of } P \right\}; \\ T &= \left\{ t \in \mathbb{R}^H : xWt + vt \leq 0 \text{ is a facet of } P \right\}. \end{aligned} \tag{3}$$

Let P be a polyhedron in $\mathbb{R}^{\{j\} \cup L}$ and $r, \alpha \in \mathbb{R}$ such that $P_L = \left\{ x_L \in \mathbb{R}^L : (r, x_L) \in P \right\}$ is full dimensional and $x_L u_L \leq \alpha$ is a facet of the P_L.

Any facet of P of the form

$$x_j y + x_L u_L \leq \alpha + ry \tag{4}$$

is called a *lifting* of $x_L u_L \leq \alpha$.

For $(x_j, x_L) \in \mathbb{R}^{\{j\} \cup L}$ and $y \in \mathbb{R}$, let $x * y$ if and only if $x_j y + x_L u_L \leq \alpha + ry$, where $u_L \neq 0$, and $\alpha, r \in \mathbb{R}$. The corresponds general bilinear polarity from $*$ is defined by $x * y$ if and only if

$$(x_j, x_L) \begin{bmatrix} 1 \\ 0_{L \times \{j\}} \end{bmatrix} y + (x_j, x_L) \begin{bmatrix} 0 \\ u_L \end{bmatrix} + (-r)y \leq \alpha. \tag{5}$$

From 3.2 we have, $X_* = \{(x_j, x_L) \in \mathbb{R}^{\{j\} \cup L} : x_j = r\} \neq \emptyset$ and $dim(X_*) = \mid L \mid > 0$, since $(r, x_L) \in X_*$ for all $x_L \in \mathbb{R}^L$ such that $x_L u_L \leq \alpha$. $Y_* = \emptyset$, since $u_L \neq 0$, and $\{y \in \mathbb{R}^{\{j\} \cup L} : \begin{bmatrix} 1 \\ O_{L \times \{j\}} \end{bmatrix} y = 0\} = \{0\}$. Then, $*$ is of type 6 defined in [3], therefore for any polyhedra P satisfying $dim(P \cap X_*) = dim(X_*) = \mid L \mid > 0$ and $x_L u_L \leq \alpha$ is a facet to $P \cap X_*$ for the relation $*$, the theorem 3.5 specializes to

$$\begin{aligned} S &= \{ s \in \mathbb{R} : x_j s + x_M u_M \leq \alpha + rs, \text{is a facet of } P \}, \text{and} \\ T &= \{ t \in \mathbb{R} : x_j t \leq rt, \text{is a facet of } P \} \end{aligned} \tag{6}$$

So using theorems 2.14 and 3.4 of [3] we have the following results.

Theorem 3.3 *Let P be a full dimensional polyhedron such that $dim(P \cap X_*) = dim(X_*)$, and $x_M u_M \leq \alpha$ is a facet of $P \cap X_*$. Then we have that the set S and T of* (3.6) *are: S is the set of vertices of P^*, T is the set of extreme rays of the recession cone of P^*.*

The theorem 3.6 gives a bijection between the facet extensions of the facet $x_L u_L \leq \alpha$ of $P \cap X_*$ and the vertices of the $*$-polar of P. However, the theorem also relates other facets of P to the extreme rays of the recessional cone of P^* and suggests the following generalization of the definition of the extension of facet.

Given P, r, u_L and α such that verity the hypotheses of theorem 3.6, an *facet extension* of $x_L u_L \leq \alpha$ is a facet of P of the form $x_j s + \rho x_M u_M \leq \rho\alpha + rs$ where $\rho \geq 0$. So, we can restate theorem 3.6 in terms of this definition as

Theorem 3.4 *Given the hypotheses of theorem* 3.6*, $x_j s + \rho x_M u_M \leq \rho\alpha + rs$ is a facet extension of $x_M u_M \leq \alpha$ if and only if $\rho > 0$ and s/ρ is a vertex of P^*; or $\rho = 0$ and s is an extreme ray of the recession cone of P^**

4 Sequential Lifting

In this section we consider $(A, \widehat{+})$ to be an abelian, associate and b-complementary additive system, $b \in A$, $M \subseteq A_+$, $P(A,M,b)$ is full dimensional and π is a facet of $P(A,M,b)$ such that:

i. $\pi(g) \geq 0$, for all $g \in A$;

ii. $\pi(g) = 0$, for all ;$g \in A \setminus M$.

Given $j \in A \setminus M$ and $m \in \{1, ..., l_j\}$ where $l_j \in \mathbb{N}^*$ is the loop order of j, we define $V(m)$ by

$$V(m) = \left\{ t \in \mathbb{R}^M : (m,t) \in P(A, M \cup \{j\}, b) \right\}$$

If $V(m) \neq \emptyset$ then the problem $max\{-\pi t : t \in V(m)\}$ has a finite solution, in this case, we denoted by $Z(m)$ the $max\{-\pi t : t \in V(m)\}$.

In order to lift a facet for the $P(A, M \cup \{j\}, b)$, we define the function $\sigma_M : A \to \mathbb{R}$ by

$$\sigma_M(h) = \begin{cases} \min\left\{\sum_{g \in M} \pi(g) t(g) : t \in P(A,M,h)\right\} & \text{if } P(A,M,h) \neq \emptyset \\ +\infty & \text{if } P(A,M,h) = \emptyset \end{cases} \tag{7}$$

for all $h \in A$, and $\sigma_M(\hat{0}) = 0$, where $\hat{0}$ is the zero of A.

Lemma 4.1 *Given $j \in A \setminus M$ and $m \in \left\{1,...,l_j\right\}$. Let $\widehat{mj}$ be the b complement of mj. Then*

$$Z(m) \geq \max\left\{-\pi t : t \in P(A,M,\widehat{mj})\right\}.$$

Proof. For $t \in P(A,M,\widehat{mj})$, $\widehat{mj} \in \widehat{\sum}_{g\in M} t(g)g$, then

$$mj \hat{+} \widehat{mj} \in mj \hat{+} \widehat{\sum}_{g\in M} t(g)g.$$

Since
$b \in mj \hat{+} \widehat{mj}$,

$$b \in mj \hat{+} \widehat{\sum}_{g\in M} t(g)g. \tag{8}$$

Thus $P(A,M,\widehat{mj}) \subseteq V(m)$, and we have

$$\min\left\{\pi t : t \in V(m)\right\} \leq \min\left\{\pi t : t \in P(A,M,\widehat{mj})\right\}.$$

Therefore

$$Z(m) \geq \max\left\{-\pi t : t \in P(A,M,\widehat{mj})\right\}.$$

Corollary 4.1 *If $P(A,M,\widehat{mj}) \neq \emptyset$, then $Z(m) \geq -\sigma_M(\widehat{mj})$.*

Theorem 4.1 *Let $\pi' : M \cup \{j\} \longrightarrow R$ define to*

$$\pi'(g) = \begin{cases} \max\left\{0, \max_{m\in\{1,..,l_g\}}\left\{\frac{1-\sigma_M(\widehat{mg})}{m}\right\}\right\} & \text{if } g = j \\ \pi(g) & \text{if } g \neq j \end{cases} \tag{9}$$

for all $g \in M \cup \{j\}$. Then π' is a facet of $P(A,M \cup \{j\}, b)$.

Proof.

Notice that $P(A,M \cup \{j\}, b)$ is full dimensional, since $P(A,M \cup \{j\}$ is full dimensional. Now by the definitions 3.2 and 3.5, $X_* = \{(0,t) \in \mathbb{R}^{\{j\}\cup M} : t \in \mathbb{R}^M\}$, so we have to

$$dim(P(A,M \cup \{j\}, b) \cap X_*) = dim(X_*) > 0$$

therefore the hypotheses of the theorem 3.7 is hold. Them $(-s)m + (-\pi)t \leq -1$ is a facet of $P(A,M \cup \{j\}, b)$ if and only if $-s$ is a vertice of $P(A,M \cup \{j\}, b)^*$.

Let $t' \in V(m)$ such that $(-\pi)t' = Z(m)$, for $m \in \{1,..,l_j\}$ where l_j is the loop order of j. Them $(-s)m + (-\pi)t' \leq -1$ since $(m,t') \in P(A, M \cup \{j\}, b)$, and $(-s)m - \sigma_M(\hat{mj}) \leq -1$ by lemma (4.1), therefore

$$s \geq \frac{1 - \sigma_M(\widehat{mj})}{m}$$

, since $m > 0$.

Now, $\pi'(j)$ is a vertex of $P(A, M \cup \{j\}, b)^*$, therefore , π' is a facet of $P(A, M \cup \{j\}, b)$ by theorem 3.7, and the proof is complete.

If $(A, \widehat{+})$ is a finite abelian group, then $Z(m) = -\sigma_M(b - mj)$ and the function to be defined in (4.3) is a facet.Therefore we have the theorem II.1.4 of [7]. Moreover, if $(A, \widehat{+})$ is an associative, abelian and b-complementary semigroup, the function (4.3) is facet, too, and we have the theorem 4.3.2 of [10].

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