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Linear-Quadratic Stochastic Differential Games on Directed Chain Networks *

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Abstract

We study linear-quadratic stochastic differential games on directed chains inspired by the directed chain stochastic differential equations introduced by Detering, Fouque and Ichiba. We solve explicitly for Nash equilibria with a finite number of players and we study more general finite-player games with a mixture of both directed chain interaction and mean field interaction. We investigate and compare the corresponding games in the limit when the number of players tends to infinity. The limit is characterized by Catalan functions and the dynamics under equilibrium is an infinite-dimensional Gaussian process described by a Catalan Markov chain, with or without the presence of mean field interaction.

Key Words and Phrases: Linear-quadratic stochastic games, directed chain network, Nash equilibrium, Catalan functions, Catalan Markov chain, mean field games.
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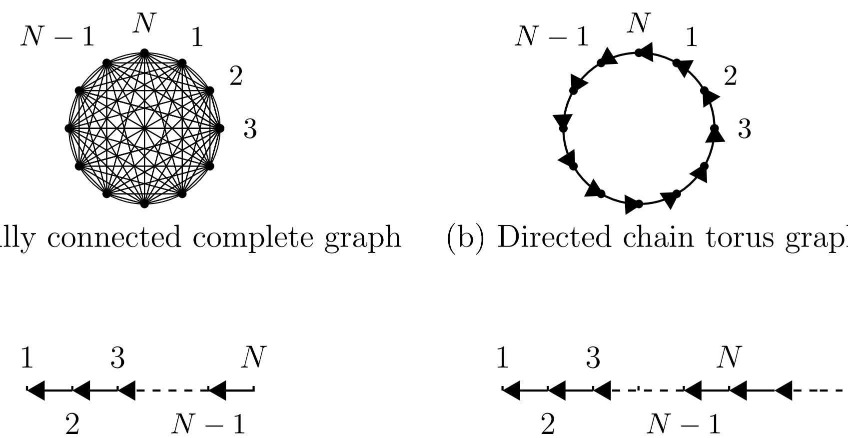
1 Introduction

The study of stochastic differential games on networks is a broad area. In a stochastic differential game on a network, the state process of each player is associated with a vertex of the network graph and each player minimizes the individual cost function by controlling its state, where the state processes are described by a stochastic differential system. The interactions among the players through the network is encoded in the individual cost functions. Roughly speaking, if player i (vertex i) is connected to player j (vertex j) in the network, then the cost function of player i depends on the state process of player j, and the cost function of player j depends on the state process of player i. If the graph is directed, and if there is an arrow from j to i, then the cost function of player i depends on the state process of player j. The goal of study of stochastic differential game problem on networks is to determine and analyze the Nash equilibrium of the game for different types of networks. There are the following two extreme situations of the networks.

On one hand, we can consider a fully connected network (complete graph), described in fig. 1 (a), with interaction of mean-field type. When the number N of players goes to infinity, i.e., $N \to \infty$, with appropriate scalings, this kind of game can be approximated by a mean field game. The approximation problem of mean field games has been discussed widely, for instance in Lacker [7]. Stochastic games on infinite random networks have been proposed and studied. Delarue [5] investigated an example of a game with a large number of players in mean-field interaction when the graph connection between them is of Erdős-Rényi type. More recently, Caines and Huang [1] [2] explored stochastic differential games under dense graphs.

On the other hand, we can consider a very sparse, structured network such as a directed, torus chain of N vertices in fig. 1 (b), where there are arrows from $i+1$ to i for $i = 1, \ldots, N-1$ and an arrow from 1 to N. There are only N directed edges in the network in contrast to the fully connected graph, where there are $\binom{N}{2}$ undirected edges. It is a complete opposite to the mean field games, since, on a directed chain network, each player interacts only with its neighbor in a given direction. The finite directed chain of N vertices in fig. 1 (c) is obtained as a graph with even a fewer number of directed edges, by removing the directed edge from 1 to N in the directed, torus chain of fig. 1 (b). The difference between them is how to deal with the boundary vertices (vertices 1 and N).

In this paper, we introduce a stochastic differential game aspect of the directed chain structures and identify Nash equilibria. We consider the limit, when the number of players goes to infinity as in fig. 1 (d), and generalize the results to the stochastic differential games on a directed tree structure.



(a) Fully connected complete graph (b) Directed chain torus graph

(c) Finite directed chain graph (d) Infinite directed chain graph

Figure 1: (a) Fully connected graph, (b) Directed torus chain graph, (c) Finite directed chain, (d) Infinite directed chain.

Recently, the stochastic processes on one-dimensional, infinite directed chain have been studied in Detering, Fouque and Ichiba [6] without the game aspect. Similarly, Lacker, Ramanan and Wu [8] studied the limit of an interacting diffusive particle system on a large sparse interaction graph with finite average degree. Interestingly, the equilibrium dynamics on the network discussed in this paper turns out to be different from the dynamics suggested in [6]. Particularly, the long time variance behavior is different. The equilibrium dynamics for the infinite-player game is described by a Catalan Markov chain introduced in this paper.

Our goal is to consider tractable stochastic differential games on directed chain networks and to find their Nash equilibria explicitly in a similar spirit of the work by Carmona, Fouque and Sun [4]. We focus on open-loop Nash equilibria, discuss briefly closed loop Nash equilibria and examine how the structure of the network affects this Nash equilibrium. We propose three directed chain networks shown in fig. 1 (b)-(d) first and then consider the stochastic differential games on the directed tree structure as an extension of directed chain graphs. In these considerations, all these graphs are not considered as geometric graphs. In other words, the graph represents interactions among players through the cost functions but not necessarily reflects physical (spatial) distance among players.

The paper is organized as follows. In section 2, we propose a finite-player game model on a directed chain of fig. 1 (b), and construct an open-loop Nash equilibrium. We discuss general boundary conditions on the boundary vertex of the network graph as well as two special cases to illustrate that the boundary condition actually affects weakly the Nash equilibrium. We also observe that for this type of games open-loop and closed-loop Nash equilibria coincide.

Section 3 is devoted to the analysis of an infinite-player stochastic differential game on a directed chain of fig. 1 (c). We find an open-loop Nash equilibrium from a similar Riccati system to that of the finite-player game. The solutions of the infinite-dimensional Riccati system are called Catalan functions. We use them to build a Catalan Markov chain and introduce an infinite-dimensional Ornstein-Uhlenbeck process in section 4. We find that its long-time asymptotic variance and covariance are finite.

In section 5, we shall incorporate the mean-field interactions to the stochastic differential games on the directed chain. We call it a mixed system of directed chain and mean-field interactions. We discuss both finite-player and infinite-player games for the mixed system. By choosing a tuning parameter $u \in [0, 1]$, we may adjust the model to be a purely mean field game (studied in [4]), or a purely directed chain game, or a mixture of the two interactions. For it, we repeat the same steps as in section 2, section 3, and section 4 to find the Nash equilibria and we construct a generalized Catalan Markov chain describing the two effects. We find that the long-time asymptotic variance of the process with the purely directed chain interaction is finite, which is different from the case with mean-field interaction as it was shown in Table 1 in [6].

In section 6, we propose an N player stochastic differential game under the directed chain torus graph fig. 1 (a). It corresponds to the periodic boundary condition. We construct an open-loop Nash equilibrium. We conjecture that as $N \to \infty$, its infinite-player limit is the same as the one found for other boundary condition. This conjecture is supported by numerical results.

In section 7, we extend our results to tree structures (fig. 3) with fixed finite number of descendants. Section 8 gives a conclusion and open problems. Appendix A includes some technical proofs and discussions.

2 N-Player Directed Chain Game

2.1 Setup and Assumptions

In fig. 1 (c), we consider a stochastic differential game in continuous time, involving N players indexed from 1 to N. Each player i is controlling its own, real-valued private state X_t^i by taking a real-valued action α_t^i at time $t \in [0, T]$. The dynamics of the states of the N individual players are given by N stochastic differential equations of the form:

$$dX_t^i = \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t^i, \quad i = 1, \cdots, N, \tag{1}$$

where $0 \leq t \leq T$ and $(W_t^i)_{0 \leq t \leq T}$, $i = 1, \cdots, N$ are independent standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ generated by the noises and augmented with an initial σ-algebra $\mathcal{F}_0$, independent of the Brownian motions.

Here and throughout the paper, the argument in the superscript represents index or label but not the power. For simplicity, we assume that the diffusion is one-dimensional and the diffusion coefficients are constant and identical denoted by $\sigma > 0$. The drift coefficients α^i's are progressively measurable with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ and satisfy the square integrability $\mathbb{E}[\int_0^T |\alpha_t^i|^2 dt] < \infty$ for $i = 1, \ldots, N$. The system starts at time $t = 0$ from *i.i.d.* square-integrable, $\mathcal{F}_0$-measurable random variables $X_0^i = \xi_i$ for $i = 1, \ldots, N$, independent of the Brownian motions. For simplicity, we assume $\mathbb{E}(\xi_i) = 0$ for $i = 1, \ldots, N$.

In this model, among the first $N - 1$ players, each player i chooses its own strategy α^i, in order to minimize its objective function given by: for $1 \leq i \leq N - 1$

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\Big\{ \int_0^T \Big(\frac{1}{2}(\alpha_t^i)^2 + \frac{\epsilon}{2}(X_t^{i+1} - X_t^i)^2 \Big) \mathrm{d}t + \frac{c}{2}(X_T^{i+1} - X_T^i)^2 \Big\}, \tag{2}$$

for some constants $\epsilon > 0$ and $c \geq 0$. The running cost and the terminal cost functions are defined by

$$f^i(x, \alpha^i) := \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2, \quad \text{and} \quad g^i(x) := \frac{c}{2}(x^{i+1} - x^i)^2, \tag{3}$$

respectively for $x := (x^1, \ldots, x^N) \in \mathbb{R}^N$ and $\alpha^i \in \mathbb{R}$, $i = 1, \ldots, N$. This is a *Linear-Quadratic* game on a directed chain network in fig. 1 (c), since the state X^i of each player i interacts only with X^{i+1} through the quadratic cost functions for $i = 1, \ldots, N-1$. The system is completed by describing the behavior of player N, which will be done in the following section, when we discuss it as the *boundary condition* of the system.

2.2 Open-Loop Nash Equilibrium

In this section, we search for an open-loop Nash equilibrium of the system of N players among the admissible strategies $\{\alpha_t^i, i = 1, \cdots, N, t \in [0, T]\}$ by the Pontryagin stochastic maximum principle (see the monograph [10] for stochastic controls, and also see [3] for stochastic maximum principle in the mean-field games) and study the effect of boundary conditions induced by the behavior of player N.

Definition (Open-loop Nash equilibrium). *We call $\{\alpha^i, 1 \le i \le N\}$ an open-loop Nash equilibrium if for every player i and for any other $(\mathcal{F}_t)_{0\le t\le T}$ adapted and square-integrable control β. we have*

$$J^i(\alpha^1, \ldots, \alpha^{i-1}, \beta, \alpha^{i+1}, \ldots, \alpha^N) \ \ge\ J^i(\alpha^1, \ldots, \alpha^{i-1}, \alpha^i, \alpha^{i+1}, \ldots, \alpha^N). \quad (4)$$

We discuss a general boundary condition first in section 2.2.1 and then show two particular choices in sections 2.2.2 -2.2.3.

2.2.1 General Boundary Condition

We consider a setup with a general boundary condition for the directed chain where the last player N does not depend on the other players. The expected cost functional for player N is defined by:

$$J^N(\alpha^N) := \mathbb{E}\left\{\int_0^T \left(\frac{1}{2}(\alpha_t^N)^2 + q_2(X_t^N)\right) \mathrm{d}t + Q_2(X_T^N)\right\}, \quad (5)$$

where $\quad q_2(x) := \dfrac{a_1}{2}(x-m)^2 + a_2, \quad$ and $\quad Q_2(x) := \dfrac{c_1}{2}(x-m)^2 + c_2, \quad x \in \mathbb{R} \quad (6)$

are non-degenerate convex quadratic functions in x, where a_1, a_2, m, c_1, c_2 are some constants with $a_1 > 0$ and $c_1 > 0$. The running cost and terminal cost functions are $f^N(x, \alpha^N) := \frac{1}{2}(\alpha^N)^2 + q_2(x)$ and $g^N(x) := Q_2(x)$, respectively. This can be seen as a control problem for the player N and we assume its state is attracted to some constant level $m \in \mathbb{R}$. We define the Hamiltonian for each player. The Hamiltonian for player $i \leq N-1$ is given by:

$$H^i(x^1, \cdots, x^N, y^{i,1}, \cdots, y^{i,N}, \alpha^1, \cdots, \alpha^N) := \sum_{k=1}^{N} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2,$$

while the Hamiltonian for player N is:

$$\begin{aligned} &H^N(x^1, \cdots, x^N, y^{i,1}, \cdots, y^{i,N}, \alpha^1, \cdots, \alpha^N) \\ &:= \sum_{k=1}^{N} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^N)^2 + \frac{a_1}{2}(x^N - m)^2 + a_2 \end{aligned}$$

for $x^k, y^{i,k}, \alpha^k \in \mathbb{R}$, $i, k = 1, \ldots, N$. For $i = 1, \ldots, N$ the value of α^i minimizing the Hamiltonian $H^i(\cdot)$ with respect to α^i, when all the other variables including α^j for $j \neq i$ are fixed, is given by the first order condition

$$\partial_{\alpha^i} H^i = y^{i,i} + \alpha^i = 0 \quad \text{leading to the choice:} \quad \hat{\alpha}^i = -y^{i,i}.$$

The adjoint processes $Y_t^i = (Y_t^{i,j}; j \leq N)$ and $Z_t^i = (Z_t^{i,j,k}; j, k \leq N)$ for $i = 1, \cdots, N$ are defined as the solutions of the system of backward stochastic differential equations (BSDEs): for $j = 1, \ldots, N$

$$\begin{aligned} \mathrm{d}Y_t^{i,j} &= -\partial_{x^j} H^i(X_t, Y_t^i, \alpha_t)\mathrm{d}t + \sum_{k=1}^{N} Z_t^{i,j,k} \mathrm{d}W_t^k \\ &= -\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j})\mathrm{d}t + \sum_{k=1}^{N} Z_t^{i,j,k} \mathrm{d}W_t^k, \quad 0 \leq t \leq T, \\ Y_T^{i,j} &= \partial_{x^j} g_i(X_T) = c(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}), \quad i \leq N-1; \\ \mathrm{d}Y_t^{N,j} &= -a_1(X_t^N - m)\delta_{N,j}\mathrm{d}t + \sum_{k=1}^{N} Z_t^{N,j,k} \mathrm{d}W_t^k, \quad 0 \leq t \leq T, \\ Y_T^{N,j} &= c_1(X_T^N - m)\delta_{N,j} \end{aligned} \tag{7}$$

where $\delta_{i,j} := 1$, if $i = j$, and 0, otherwise. Particularly, for $j = i$, $j = i+1$, it becomes:

$$
\begin{aligned}
\mathrm{d}Y_t^{i,i} &= \epsilon(X_t^{i+1} - X_t^i)\mathrm{d}t + \sum_{k=1}^N Z_t^{i,i,k}\mathrm{d}W_t^k, \quad Y_T^{i,i} = -c(X_T^{i+1} - X_T^i),\\
\mathrm{d}Y_t^{i,i+1} &= -\epsilon(X_t^{i+1} - X_t^i)\mathrm{d}t + \sum_{k=1}^N Z_t^{i,i+1,k}\mathrm{d}W_t^k, \quad Y_T^{i,i+1} = c(X_T^{i+1} - X_T^i),\\
\mathrm{d}Y_t^{N,N} &= -a_1(X_t^N - m)\mathrm{d}t + \sum_{k=1}^N Z_t^{N,N,k}\mathrm{d}W_t^k, \quad Y_T^{N,N} = c_1(X_T^N - m)
\end{aligned}
\tag{8}
$$

for $i \leq N-1$, $0 \leq t \leq T$. Thus, because of $Y_T^{i,i} = -Y_T^{i,i+1}$ and of the form of dynamics, it is reduced to

$$
Y_t^{i,i} = -Y_t^{i,i+1}, \quad Z_t^{i,i,k} = -Z_t^{i,i+1,k} \tag{9}
$$

for $i \leq N-1$, $k \leq N$, $0 \leq t \leq T$. For $j \neq i, i+1$, $i \leq N-1$, it becomes: $\mathrm{d}Y_t^{i,j} = \sum_{k=1}^N Z_t^{i,j,k}\mathrm{d}W_t^k$, $Y_T^{i,j} = 0$, and hence, the solution is

$$
Y_t^{i,j} \equiv 0\,, \quad Z_t^{i,j,k} \equiv 0, \quad 0 \leq t \leq T\,. \tag{10}
$$

Considering the BSDE (8) and its terminal condition, we make the ansatz:

$$
Y_t^{i,i} = \sum_{j=i}^{N-1} \phi_t^{N,i,j} X_t^j + (\phi_t^{N,i,N} X_t^N + \psi_t^{N,i}) = \sum_{j=i}^{N} \phi_t^{N,i,j} X_t^j + \psi_t^{N,i}, \tag{11}
$$

for some deterministic scalar functions ϕ_t (depending on N) satisfying the terminal conditions: for $1 \leq i \leq N-1$, $\phi_T^{N,i,i} = c$, $\phi_T^{N,i,i+1} = -c$, $\phi_T^{N,i,j} = 0$ for $j \geq i+2$, $\psi_T^{N,i} = 0$; and $\phi_T^{N,N,N} = c_1$, $\psi_T^{N,N} = -c_1 m$. With this ansatz, the optimal strategy $\hat{\alpha}_\cdot$ and the controlled forward equation for $X_\cdot$ in (1) become, for $i \leq N$

$$
\hat{\alpha}_t^i = -Y_t^{i,i}, \quad \mathrm{d}X_t^i = -\Big(\sum_{k=i}^N \phi_t^{N,i,k} X_t^k + \psi_t^{N,i}\Big)\mathrm{d}t + \sigma \mathrm{d}W_t^i, \quad t \geq 0. \tag{12}
$$

Differentiating the ansatz (11) and substituting (12) leads to: $\mathrm{d}Y_t^{i,i}$ has drifts

$$
\Big\{ \sum_{k=i}^N \big(\dot{\phi}_t^{N,i,k} - \sum_{j=i}^k \phi_t^{N,i,j}\phi_t^{N,j,k}\big) X_t^k + \big[\dot{\psi}_t^{N,i} - \sum_{j=i}^N \psi_t^{N,j}\phi_t^{N,i,j}\big]\Big\}\mathrm{d}t \tag{13}
$$

and martingale terms $\sigma \sum_{k=i}^{N} \phi_t^{N,i,k} \mathrm{d}W_t^k$. Here $\dot{\phi}_t$ represents the time derivative of ϕ_t. Comparing the martingale terms of two Itô's decompositions (8) and (13) of $Y_t^{i,i}$, we obtain the deterministic (and therefore adapted) processes $Z_t^{i,i,k}$:

$$Z_t^{i,i,k} = 0 \quad \text{for} \quad k < i, \quad \text{and} \quad Z_t^{i,i,k} = \sigma \phi_t^{N,i,k} \quad \text{for} \quad k \geq i; \tag{14}$$

Moreover, the drift terms show that the functions $\phi_t^{N,\cdot,\cdot}$ and $\psi_t^{N,\cdot}$ must satisfy the system of Riccati equations : $\dot{\phi}_t^{N,N,N} = \phi_t^{N,N,N} \cdot \phi_t^{N,N,N} - a_1$, $\phi_T^{N,N,N} = c_1$ and

$$\dot{\phi}_t^{N,i,j} = \sum_{\ell=i}^{j} \phi_t^{N,i,\ell} \phi_t^{N,\ell,j} + \varepsilon(-\delta_{i,j} + \delta_{i+1,j}), \quad \phi_T^{N,i,j} = c(\delta_{i,j} - \delta_{i+1,j}) \tag{15}$$

for $i \leq N-1$, $j \leq N$, and $\psi_\cdot^{N,j}$, $j \leq N$ are determined by $\dot{\psi}_t^{N,N} = \psi_t^{N,N} \phi_t^{N,N,N} + a_1 m$, $\psi_T^{N,N} = -c_1 m$ and for $i \leq N-1$

$$\dot{\psi}_t^{N,i} = \sum_{j=i}^{N} \psi_t^{N,j} \phi_t^{N,i,j}, \quad \psi_T^{N,i} = 0, \tag{16}$$

From the equations above, the functions $\phi_t^{N,i,i}$ for all $i = 1, \cdots, N-1$ are identical; the functions $\phi_t^{N,i,i+1}$ for all $i = 1, \cdots, N-2$ are identical ;$\cdots$; and the functions $\phi_t^{N,i,N-2} = \phi_t^{N,i+1,N-1}$. The functions $\phi_t^{N,i,N}$ for all i depend on $\phi_t^{N,N,N}$ of the last player which is determined by the boundary condition. However, the functions $\phi_t^{N,i,i}, \cdots, \phi_t^{N,i,N-1}$ are independent of $\phi_t^{N,i,N}$ and the boundary condition. The functions $\psi^{N,\cdot}$ depend on the ϕ functions and have no effect on $\phi^{N,i,j}$ $(j < N)$ as well.

In conclusion, these $\phi^{N,i,j}$ $(j < N)$ are solvable, identical and independent of the boundary condition as long as the boundary condition defines the last player as a self-controlled problem. The preceding argument is summarized as the following.

Proposition 1. *An open-loop Nash equilibrium for the linear quadratic stochastic game with cost functionals* (2)-(3) *for the first* $N-1$ *players and* (5)-(6) *for the* N*th player is given by* (12), *where* $\phi_\cdot^{N,i,j}$ *and* $\psi^{N,j}$ *are uniquely determined by the system* (15)-(16) *of Riccati equations.*

As the number of players goes to infinity, we can get rid of the boundary condition and get a sequence of functions $\{\phi_t^j, j = 1, 2, \cdots\}$, defined by $\phi_t^0 =$

$\phi_t^{N,i,i}$, $\phi_t^1 = \phi_t^{N,i,i+1}$, $\cdots$, $\phi_t^j = \phi_t^{N,i,i+j}$ for large N and so on. It indicates that the Nash equilibrium converges to a limit independent of the boundary condition. Therefore, it is natural to study a similar game with infinite players. We conjecture that in general, as the number N of players goes to infinity, the limit of the Nash equilibrium of the finite-player, linear-quadratic stochastic differential game under the directed chain graphs gives us the Nash equilibrium of the infinite-player game, and moreover, $\{\phi_t^j, i \in \mathbb{N}\}$ is the solution to the Riccati equation system of the infinite-player game. This will be discussed in section 3. Next, two particular examples are discussed to better illustrate the effect of the special boundary.

2.2.2 Boundary Condition 1: X^N is attracted to 0

Here, we discuss the case when X^N is attracted to 0 which is also the common mean $\mathbb{E}[\xi_i] = 0$ of the initial condition. It is equivalent to the general boundary condition (5)-(6) with $m = 0$. Without loss of generality, we can take constants: $a_1 = \epsilon$, $c_1 = c$ and $a_2 = c_2 = 0$. Then the cost functional for player N is given by:

$$J^N(\alpha^N) := \mathbb{E}\left\{ \int_0^T \left(\frac{1}{2}(\alpha_t^N)^2 + \frac{\epsilon}{2}(X_t^N)^2 \right) \mathrm{d}t + \frac{c}{2}(X_T^N)^2 \right\}.$$

The running cost function is defined by $f^N(x, \alpha^N) = \frac{1}{2}(\alpha^N)^2 + \frac{\epsilon}{2}x^2$ and the terminal cost function is defined by $g^N(x) = \frac{c}{2}x^2$. Then, X^N is independent of the other players and is the solution of a self-controlled problem. We then make the same ansatz as (11) with $\psi_t^{N,i} = 0$ for all i, $0 \le t \le T$. As a result, $Z_\cdot^{i,i,k}$ and $\phi^{N,i,j_\cdot}$ are as (14) and (15), respectively. Consequently, we have the same conclusion: the functions $\phi_t^{N,i,i+k} = \phi_t^{N,j,j+k}$ for all $i, j \ge 1, k \ge 1$ and $i + k < N, j + k < N$; and functions $\phi_t^{N,i,j}(j < N)$ are independent of the boundary condition.

Remark 1 (Shift invariance)**.** *Notice that in this case $\phi_t^{N,N,N}$ has the same solution as $\phi_t^{N,i,i}$ $(i < N)$. Thus, in the ansatz* (11)*, we can actually assume the solution $\phi_\cdot^{N,i,j}$ depends only on the difference $j - i$ for $j \ge i$.*

2.2.3 Boundary Condition 2: $\alpha^N = 0$

We study the case when there is no control for the last player X^N, i.e. the dynamics of the state is given by:

$$\mathrm{d}X_t^N = \sigma \mathrm{d}W_t^N, \quad 0 \le t \le T; \qquad X_0^N = \xi_N, \quad \mathbb{E}(\xi_N) = 0.$$

Player i chooses the strategy α_t^i ($i < N$) to minimize J^i given in (2) and the last player does not control, i.e., $\alpha_\cdot^N \equiv 0$. We make the same ansatz as in (11) with $\psi_t^{N,i} = 0$ for all i. Then $Z_\cdot^{i,i,k}$ are the same as in (14) for $i \leq N$, $k \leq N$, $0 \leq t \leq T$, and $\phi_\cdot^{N,i,j}$, $i \leq N-1$, $j \leq N$ satisfy (15), however, for $i = N$, $\dot{\phi}_t^{N,N,N} = -\epsilon$ for $0 \leq t \leq T$ with $\phi_T^{N,N,N} = c$. Thus, it is demonstrated again that the boundary condition does not affect the solutions $\phi_\cdot^{N,i,j}$ ($j < N$), however, the functions $\phi_\cdot^{N,i,N}$ for all i are different from those in section 2.2.2, which depends on the boundary.

2.3 Closed-loop Nash Equilibrium

In search for closed-loop Nash equilibria, the controls are of the form $\alpha^k(t, x)$. When computing $\partial_{x^j} H^i$ in the derivation of the BSDE for $Y^{i,j}$, one needs to pay attention in taking derivatives with respect to x^j in $\hat{\alpha}^k$ for $k \neq i$, using $\hat{\alpha}^k = -y^{k,k}$ and the ansatz (22). This is a tedious but straightforward computation which leads to the fact that the obtained closed-loop equilibrium coincides with the open-loop equilibrium identified before. We omit the details here as well as repeating this remark in the following sections. The only place where closed-loop and open-loop equilibria will be different is in section 5 when we will look at a mixture of directed chain and mean field interactions for finite player games, as it is already the case for pure mean field interaction studied in [4]. However, they will coincide again for the infinite-player games in section 5.2.

3 Infinite-Player Game Model

Motivated by the limit of the finite-player game discussed in section 2, we define the game with infinite players on a directed chain structure as shown in fig. 1. In remark 2 in section 3.1, we will see that the Hamiltonian only depends on finite players, which will make it well-defined. We assume that the state dynamics of all players are given by the stochastic differential equations of the form: for $i \geq 1$,

$$\mathrm{d}X_t^i = \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t^i, \quad 0 \leq t \leq T, \tag{17}$$

where $(W_t^i)_{0 \leq t \leq T}$, $i \geq 1$ are one-dimensional, independent Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. Similar to the setup for the finite-player games in section 2, we assume that the drift coefficients α^i are adapted to the filtration of the Brownian motions and satisfy $\mathbb{E}[\int_0^T |\alpha_t^i|^2 \mathrm{d}t] < \infty$. We also assume that the diffusion coefficients are constant and identically denoted by $\sigma > 0$. The system starts at time $t = 0$ from *i.i.d.* square-integrable random variables $X_0^i = \xi_i$ with $\mathbb{E}(\xi_i) = 0$, independent of the Brownian motions. In this model, player i chooses its own strategy α^i in order to minimize its expected cost function of the form:

$$J^i(\boldsymbol{\alpha}) := \mathbb{E}\Big[\int_0^T f^i(X_s, \alpha_s^i)\mathrm{d}s + g^i(X_T)\Big], \tag{18}$$

where the running and terminal cost functions $f^i(x, \alpha^i)$, $g^i(x)$ are the same as in (3).

3.1 Open-Loop Nash Equilibrium

We search for an open-loop Nash equilibrium of the infinite system (17) among admissible strategies $\{\alpha_t^i, i = 1, 2, \cdots, 0 \le t \le T\}$.

Definition (Open-loop Nash equilibrium). *We call $\boldsymbol{\alpha} := \{\alpha^i, i \ge 1\}$ an open-loop Nash equilibrium if for every player i and for any other $(\mathcal{F}_t)_{0\le t\le T}$ adapted and square-integrable control β. we have*

$$J^i(\alpha^1, \dots, \alpha^{i-1}, \beta, \alpha^{i+1}, \dots) \ \ge\ J^i(\alpha^1, \dots, \alpha^{i-1}, \alpha^i, \alpha^{i+1}, \dots). \tag{19}$$

First, we define the Hamiltonian H^i of the form:

$$\begin{aligned} &H^i(x^1, x^2, \cdots, y^{i,1}, \cdots, y^{i,n_i}, \alpha^1, \alpha^2, \cdots) \\ &:= \sum_{k=1}^{n_i} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2, \end{aligned} \tag{20}$$

assuming it is defined on real numbers x^i, $y^{i,k}$, α^i, $i \ge 1$, $k \ge 1$, where only finitely many $y^{i,k}$ are non-zero for every given i. Here, n_i is a finite number depending on i with $n_i > i$. This assumption is checked in remark 2 below. Thus, the Hamiltonian H^i is well defined for $i \ge 1$.

The adjoint processes $Y_t^i = (Y_t^{i,j}; j \le n_i)$ and $Z_t^i = (Z_t^{i,j,k}; j \le n_i, k \ge 1)$ for $i \ge 1$ are the solutions of the following BSDEs for $0 \le t \le T$, $i \ge 1$, $1 \le j \le n_i$,

$$\begin{cases} dY_t^{i,j} &= -\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j})\mathrm{d}t + \sum_{k=1}^{\infty} Z_t^{i,j,k}\mathrm{d}W_t^k, \\ Y_T^{i,j} &= \partial_{x^j} g_i(X_T) = c(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}). \end{cases} \tag{21}$$

Remark 2. *For every $j \neq i$ or $i+1$, $dY_t^{i,j} = \sum_{k=1}^{\infty} Z_t^{i,j,k} dW_t^k$ and $Y_T^{i,j} = 0$ implies $Z_t^{i,j,k} = 0$ for all k. This observation is consistent with (10) in the finite player game case. Note also that $Y^{i,i+1} = Y^{i,i}$. There must be finitely many non-zero $Y^{i,j}$'s for every i. Hence, the Hamiltonian H^i in (20) can be rewritten as*

$$H^i(x^1, x^2, \cdots, y^{i,i}, y^{i,i+1}, \alpha^1, \alpha^2, \cdots) = \alpha^i y^{i,i} + \alpha^{i+1} y^{i,i+1} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1} - x^i)^2.$$

Since each H^i is minimized at $\hat{\alpha}^i = -y^{i,i}$, inspired by the conclusion from the finite-player game (see also remark 1), we then make the ansatz of the form:

$$Y_t^{i,i} = \sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j, \quad 0 \leq t \leq T \tag{22}$$

for some deterministic scalar functions ϕ_t^i satisfying the terminal conditions: $\phi_T^0 = c, \phi_T^1 = -c, \phi_T^i = 0$ for $i \geq 2$. Substituting the ansatz (22), the optimal strategy $\hat{\alpha}^i$ and the forward equation for $X_\cdot^i$ in (17) are

$$\hat{\alpha}_t^i = -Y_t^{i,i} = -\sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j, \quad \mathrm{d}X_t^i = -\sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j \mathrm{d}t + \sigma \mathrm{d}W_t^i \tag{23}$$

for $i \geq 1$, $0 \leq t \leq T$. Differentiating the ansatz (22), we obtain

$$\mathrm{d}Y_t^{i,i} = \sum_{\ell=0}^{\infty} \dot{\phi}_t^{\ell} X_t^{i+\ell} \mathrm{d}t - \sum_{\ell=0}^{\infty} \Big(\sum_{j=0}^{\ell} \phi_t^j \phi_t^{\ell-j} \Big) X_t^{i+\ell} \mathrm{d}t + \sigma \sum_{\ell=i}^{\infty} \phi_t^{\ell-i} \mathrm{d}W_t^{\ell}. \tag{24}$$

Now by comparing the two Itô's decompositions (24) and (21) of $Y_t^{i,i}$, we obtain

$$Z_t^{i,i,k} = 0 \quad \text{for} \quad k < i \quad \text{and} \quad Z_t^{i,i,k} = \sigma \phi_t^{k-i} \quad \text{for} \quad k \geq i$$

and the system of Riccati equations: for $i \geq 0$, $0 \leq t \leq T$

$$\dot{\phi}_t^i = \sum_{j=0}^{i} \phi_t^j \phi_t^{i-j} + \varepsilon(-\delta_{0,i} + \delta_{1,i}), \quad \phi_T^i = c(\delta_{0,i} - \delta_{1,i}). \tag{25}$$

The solutions to this Riccati system coincide with the limit of the solutions to the ODE system (15) of the N-player directed chain game in section 2, i.e., $\phi_\cdot^i = \lim_{N\to\infty} \phi_\cdot^{N,i,i+j}$ in the supremum norm. The Riccati system (25) is solvable.

Proposition 2 (Catalan functions). *With $c > 0$, $\varepsilon > 0$, the solution to* (25) *satisfies*

$$\sum_{j=0}^{\infty} \phi_t^j = 0, \qquad \phi_t^0 = \frac{(-\epsilon - c\sqrt{\epsilon})e^{2\sqrt{\epsilon}(T-t)} + \epsilon - c\sqrt{\epsilon}}{(-\sqrt{\epsilon} - c)e^{2\sqrt{\epsilon}(T-t)} - \sqrt{\epsilon} + c} > 0, \tag{26}$$

for $0 \le t \le T$. Moreover, the functions $\phi_\cdot^k$'s are obtained by a series expansion of the generating function $S_t(z) = \sum_{k=0}^{\infty} z^k \cdot \phi_t^k$, $z \le 1$ of $\{\phi^\ell\}$ given by $S_t(1) \equiv 0$, and

$$S_t(z) = \frac{\big(-\epsilon(1-z) - c\sqrt{\epsilon(1-z)}(1-z)\big)\, e^{2\sqrt{\epsilon(1-z)}(T-t)} + \epsilon(1-z) - c\sqrt{\epsilon(1-z)}(1-z)}{\big(-\sqrt{\epsilon(1-z)} - c(1-z)\big)\, e^{2\sqrt{\epsilon(1-z)}(T-t)} - \sqrt{\epsilon(1-z)} + c(1-z)} \tag{27}$$

for $0 \le t \le T$, $z < 1$. We call ϕ^k's Catalan functions.

Proof. Given in appendix A.1. □

Remark 3. *It follows from* (2) *that the forward dynamics* (23) *can be written as:*

$$\mathrm{d}X_t^i = -\sum_{j=0}^{\infty} \phi_t^j X_t^{i+j} \mathrm{d}t + \sigma \mathrm{d}W_t^i = \phi_t^0 \cdot \Big(\sum_{j=1}^{\infty} \frac{-\phi_t^j}{\phi_t^0} X_t^{i+j} - X_t^i\Big)\mathrm{d}t + \sigma \mathrm{d}W_t^i \tag{28}$$

for $i \ge 1$, $0 \le t \le T$. This is a mean-reverting type process with $\phi_t^0 > 0$. We also see that this system is invariant under the shift of indices of individuals, i.e., the law of X^i is the same as that of X^1 for every i and also X^i is independent of $(W^1, \cdots, W^{i-1})$.

We end with a summary of this section on the infinite player game.

Proposition 3. *An open-loop Nash equilibrium for the infinite-player stochastic game with cost functionals* (18) *with* (3) *is determined by* (28)*, where $\{\phi^j, j \ge 0\}$ are the unique solution to the infinite system* (25) *of Riccati equations.*

4 Catalan Markov Chain

In order to simplify our analysis, we look at the stationary solution $\{\phi^j, j \ge 0\}$ of (25) and the corresponding dynamics of (28), as $T \to \infty$. For simplicity, we assume $\epsilon = 1$. By taking $T \to \infty$, we obtain the stationary long-time behavior satisfying $\dot{\phi}^j_\cdot = 0$ for all j. Then, (25) gives the recurrence relation for the stationary solution $\{\phi^j, j \geq 0\}$:

$$\phi^0 = 1 \quad \text{and} \quad \sum_{j=0}^{n} \phi^j \phi^{n-j} = \delta_{0,n} - \delta_{1,n}; \quad n \geq 0. \tag{29}$$

This is closely related to the recurrence relation of *Catalan* numbers. By using a moment generating function method as in appendix A.1, we get the stationary solutions

$$\phi^0 := 1, \quad \phi^1 := -\frac{1}{2}, \quad \phi^j := -\frac{(2j-3)!}{(j-2)!\, j!\, 2^{2j-2}} \quad \text{for } \ j \geq 2. \tag{30}$$

We consider the continuous-time Markov chain $M(\cdot)$ with state space $\mathbb{N}$ and generator matrix $\mathbf{Q} = (q_{i,j})$, where (i,j) element $q_{i,j}$ of $\mathbf{Q}$ is given by $q_{i,j} := p_{j-i} \cdot \mathbf{1}_{\{j \geq i\}}$ with $p_k := -\phi^k$, $k \geq 0$, $i, j \geq 1$. Note that the transition probabilities of the continuous-time Markov chain $M(\cdot)$, called a Catalan Markov chain, are $p_{i,j}(t) = \mathbb{P}(M(t) = j | M(0) = i) = (e^{t\mathbf{Q}})_{i,j}$, $i, j \geq 1$, $t \geq 0$. For simplicity, we assume $\sigma = 1$. Then with replacement of ϕ^j_t, $t \geq 0$ by the stationary solution ϕ^j in (30), the infinite particle system $(X^i_\cdot, i \geq 1)$ in (28) can be represented formally as a linear stochastic evolution equation:

$$\mathrm{d}\mathbf{X}_t = \mathbf{Q}\, \mathbf{X}_t \mathrm{d}t + \mathrm{d}\mathbf{W}_t; \quad t \geq 0, \tag{31}$$

where $\mathbf{X}_\cdot = (X^i_\cdot, i \geq 1)$ with $\mathbf{X}_0 = \mathbf{x}_0$ and $\mathbf{W}_\cdot = (W^k_\cdot, k \geq 1)$. Its solution is

$$\mathbf{X}_t = e^{t\mathbf{Q}} \mathbf{x_0} + \int_0^t e^{(t-s)\mathbf{Q}} \mathrm{d}\mathbf{W}_s; \quad t \geq 0. \tag{32}$$

Without loss of generality, let us assume $\mathbf{X}_0 = \mathbf{0}$. Then,

$$\begin{aligned} X^i_t &= \int_0^t \sum_{j=i}^{\infty} p_{i,j}(t-s) \mathrm{d}W^j_s = \int_0^t \sum_{j=i}^{\infty} \mathbb{P}(M(t-s) = j | M(0) = i) \mathrm{d}W^j_s \\ &= \mathbb{E}^M \Big[\int_0^t \sum_{j=i}^{\infty} \mathbf{1}_{(M(t-s)=j)} \mathrm{d}W^j_s | M(0) = i \Big]; \quad t \geq 0, \end{aligned} \tag{33}$$

where the expectation is taken with respect to the probability induced by the Catalan Markov chain $M(\cdot)$, independent of the Brownian motions $(W_{\cdot}^{j},\, j \in \mathbb{N}_0)$. This is a Feynman–Kac representation formula for the infinite particle system $\mathbf{X}_{\cdot}$ in (33) associated with the continuous-time Markov chain $M(\cdot)$ with the generator $\mathbf{Q}$. Interestingly, we may compute quite explicitly the corresponding transition probability.

Proposition 4. *With $\mathbf{x}_0 = \mathbf{0}$, the Gaussian process $X_t^i\,,\ i \ge 1\,,\ t \ge 0$ in (33), corresponding to the Catalan Markov chain with the generator $\mathbf{Q}$, is*

$$X_t^i \,=\, \sum_{j=i}^{\infty} \int_0^t \frac{(t-s)^{2(j-i)}}{(j-i)!} \cdot \rho_{j-i}(-(t-s)^2)\, e^{-(t-s)} \cdot \mathrm{d}W_s^j\,, \tag{34}$$

where $W_{\cdot}^{j}\,,\ j \in \mathbb{N}$ are independent standard Brownian motions and $\rho_i(\cdot)$ is defined by

$$\rho_i(x) := \frac{1}{2^i} \sum_{j=i}^{2i-1} \frac{(i-1)!}{(2j-2i)!!(2i-j-1)!} \cdot (-x)^{-\frac{j}{2}}\,, \tag{35}$$

for $i \ge 1$, and $\rho_0(x) \,=\, 1$ for $x \le 0$.

Proof. Given in appendix A.2. □

Remark 4. *To evaluate the asymptotic properties, it can be shown that*

$$\rho_j(-\nu^2) = \frac{1}{2^j \nu^j} \cdot \sqrt{\frac{2\nu}{\pi}} \cdot e^{\nu} \cdot K_{j-(1/2)}(\nu)\,; \quad j \ge 1\,, \tag{36}$$

where $K_n(x)$ is the modified Bessel function of the second kind defined by

$$K_n(x) \,=\, \int_0^{\infty} e^{-x\cosh t} \cosh(nt)\mathrm{d}t\,(>0)\,; \quad n > -1, x > 0.$$

The asymptotic behaviors of $X_{\cdot}^{i}$ in (34) are derived rather straightforwardly from its explicit expression and are summarized in the following with proofs in appendix A.3 - appendix A.5.

4.1 Asymptotic Behavior of the Variances as $t \to \infty$

It follows from (34) that for $t \geq 0$, the variance of the Gaussian process $X^i_\cdot$, $i \geq 1$, in (33) is given by

$$\begin{aligned} \mathrm{Var}(X_t^i) = \mathrm{Var}(X_t^1) &= \sum_{j=0}^{\infty} \int_0^t \frac{(t-s)^{4j}}{(j!)^2} |\rho_j(-(t-s)^2)|^2 e^{-2(t-s)} \mathrm{d}s \\ &= \sum_{k=1}^{\infty} \int_0^t \frac{2}{\pi} \frac{\nu^{2k+1}}{(k!)^2\, 4^k} \big(K_{k-(1/2)}(\nu)\big)^2 \mathrm{d}\nu + \frac{1-e^{-2t}}{2} \end{aligned} \tag{37}$$

Proposition 5. *The asymptotic variance is* $\lim_{t\to\infty} \mathrm{Var}(X_t^1) = 1/\sqrt{2}$*.*

4.2 Asymptotic Independence

The auto-covariance and cross-covariance are given respectively by: for $s \leq t$

$$\begin{aligned} \mathbb{E}[X_s^1 X_t^1] &= \sum_{j=0}^{\infty} \int_0^s \frac{((t-s+u)u)^{j+1/2}}{\pi (j!)^2 2^{2j-1}} K_{j-1/2}(t-s+u) K_{j-1/2}(u) \mathrm{d}u, \\ \mathbb{E}[X_t^1 X_t^{j+1}] &= \sum_{\ell=0}^{\infty} \int_0^t \frac{s^{j+2\ell+1}}{\pi (j+\ell)!\ell! 2^{j+2\ell-1}} K_{j+\ell-1/2}(s) K_{\ell-1/2}(s) \mathrm{d}s, \quad t \geq 0. \end{aligned} \tag{38}$$

The following propositions give two results about these covariances and the details of the proofs are given in Appendix appendix A.5.

Proposition 6 (Ergodicity)**.** *The auto-covariance* $\mathbb{E}[X_s^1 X_t^1]$ *is positive. For every* $s > 0$*, as* $t \to \infty$*, it converges to* 0*, i.e., the process is ergodic.*

Proposition 7 (Asymptotic behavior of the cross-covariance)**.** *Similarly, for every* $k \geq 0$ *and for any* $t > 0$ *the cross-covariance* $\mathbb{E}[X_t^1 X_t^{k+1}]$ *is positive, and* $0 < \lim_{t\to\infty} \mathbb{E}[X_t^1 X_t^{k+1}] \leq 1/\sqrt{2}$*. The asymptotic cross-covariance is positive and bounded above, which means the states are asymptotically dependent.*

5 Mixture of Directed Chain and Mean Field Interaction

In the spirit of the paper, we shall look at the game on a mixed system, including the directed chain interaction and the mean field interaction for finite players. This section repeats the same steps as before to analyze the mixed system game.The state dynamics of all the payers are of the form: $\mathrm{d}X_t^i = \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t^i$ for $i \geq 1$ as in the previous sections.

5.1 Finite-Player Game

In this N-player model, player i chooses its own strategy α^i in order to minimize its objective function of the mixed form: $i \leq N$

$$J^i(\alpha^1, \cdots, \alpha^N) := \mathbb{E}\Big\{ \int_0^T f^i(X_t, \alpha^i)\mathrm{d}t + g^i(X_T) \Big\}, \tag{39}$$

where the running cost and terminal cost functions are defined by

$$f^i(x, \alpha^i) := \frac{1}{2}(\alpha^i)^2 + u \cdot \frac{\epsilon}{2}(x^{i+1} - x^i)^2 + (1-u) \cdot \frac{\epsilon}{2}(\bar{x} - x^i)^2, \tag{40}$$

$$g^i(x) := u \cdot \frac{c}{2}(x^{i+1} - x^i)^2 + (1-u) \cdot \frac{c}{2}(\bar{x} - x^i)^2, \tag{41}$$

for some positive constants ϵ, c and a weight $u \in [0,1]$. Here, $\bar{x}$ is defined by $\bar{x} = (x_1 + \cdots + x_N)/N$ and we use the convention $x^{N+1} \equiv 0$ for notational simplicity.

Each player optimizes the cost determined by the mixture of two criteria: distance from the neighbor in the directed chain with weight u and distance from the empirical mean $\bar{X}.$ with weight $1-u$. The system is again completed by describing the behavior of player N. For simplicity, we consider the boundary condition of the system where X^N is attracted to 0 (cf. section 2.2.2). If $u = 1$, the system becomes the directed chain system discussed before. If $u = 0$, it becomes a mean-field system where each player is attracted towards the mean.

5.1.1 Open-Loop Nash Equilibrium

As before, we find an open-loop Nash equilibrium of the system among strategies $\{\alpha_t^i, i = 1, \cdots, N\}$ by reiterating the previous procedure and solving the corresponding BSDE system. It is desribed by

$$\mathrm{d}X_t^i = \Big[-u\sum_{k=i}^{N}\phi_t^{N,i,k}X_t^k + (1-u)(\bar{X}_t - X_t^i)\theta_t\Big]\mathrm{d}t + \sigma\mathrm{d}W_t^i, \tag{42}$$

where $\bar{X}_\cdot := (X_\cdot^1 + \cdots + X_\cdot^N)/N$, $\phi_\cdot^{N,i,j}$ and $\theta_\cdot$ are determined by the ODE system

$$\begin{aligned}
&u\dot{\phi}_t^{N,i,\ell} - u^2\sum_{j=i}^{\ell}\phi_t^{N,i,j}\phi_t^{N,j,\ell} - 2u(1-u)\theta_t\phi_t^{N,i,\ell}\\
&+ u\epsilon(\delta_{i,\ell} - \delta_{i+1,\ell}) + [(1-u)\dot{\theta}_t - (1-u)^2\theta_t^2]\delta_{i,\ell}\\
&+ \frac{1}{N}(1-u)\Big[-\dot{\theta}_t + (1-u)\theta_t^2 + u\theta_t(\sum_{j=1}^{\ell}\phi_t^{N,j,\ell} + \sum_{k=i}^{N}\phi_t^{N,i,k})\Big] = 0,\\
&u\theta_t\sum_{k=i}^{N}\phi_t^{N,i,k} - \dot{\theta}_t + (1-u)\theta_t^2 - \varepsilon\Big(1 - \frac{1}{N}\Big) = 0
\end{aligned} \tag{43}$$

with terminal condition $\phi_T^{N,i,\ell} = c(-\delta_{i,\ell} + \delta_{i+1,\ell})$, $\theta_T = c(1 - N^{-1})$ for $\ell \geq i$, $0 \leq t \leq T$ and for fixed $u \in (0,1)$.

The Hamiltonian is denoted by $H^i(x^1, \cdots, x^N, y^{i,1}, \cdots, y^{i,N}, \alpha^1, \cdots, \alpha^N) :=$

$$\sum_{k=1}^{N}\alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + u\frac{\epsilon}{2}(x^{i+1} - x^i)^2 + (1-u)\frac{\epsilon}{2}(\bar{x} - x^i)^2,$$

for player $i \leq N-1$, and $H^N(x^1, \cdots, x^N, y^{i,1}, \cdots, y^{i,N}, \alpha^1, \cdots, \alpha^N) :=$

$$\sum_{k=1}^{N}\alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + u\frac{\epsilon}{2}(x^N)^2 + (1-u)\frac{\epsilon}{2}(\bar{x} - x^i)^2$$

for player N. Minimizing the Hamiltonian with respect to α^i,

$$\partial_{\alpha^i}H^i = y^{i,i} + \alpha^i = 0 \quad \text{leading to the choice:} \quad \hat{\alpha}^i = -y^{i,i}.$$

The adjoint processes $Y_t^i = (Y_t^{i,j}; j = 1, \cdots, N)$ and $Z_t^i = (Z_t^{i,j,k}; j = 1, \cdots, N, k = 1, \cdots, N)$ for $i = 1, \cdots, N$ are defined as the solutions of the backward stochastic differential equations (BSDEs):

$$i<N: \quad \begin{cases} \mathrm{d}Y_t^{i,j} & = -\partial_{x^j} H^i(X_t, Y_t^i, \alpha_t)\mathrm{d}t + \sum_{k=0}^{N} Z_t^{i,j,k}\mathrm{d}W_t^k \\ & = -\Big[u\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j}) \\ & \qquad + (1-u)\epsilon(\bar{X}_t - X_t^i)\Big(\frac{1}{N} - \delta_{i,j}\Big)\Big]\mathrm{d}t + \sum_{k=0}^{N} Z_t^{i,j,k}\mathrm{d}W_t^k, \\ Y_T^{i,j} & = \partial_{x^j} g_i(X_T) = uc(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}) \\ & \quad + (1-u)c(\bar{X}_T - X_T^i)(\frac{1}{N} - \delta_{i,j}). \end{cases} \tag{44}$$

$$i=N: \quad \begin{cases} \mathrm{d}Y_t^{N,j} & = -\Big[u\epsilon X_t^N \delta_{N,j} \\ & \qquad + (1-u)\epsilon(\bar{X}_t - X_t^N)\Big(\frac{1}{N} - \delta_{N,j}\Big)\Big]\mathrm{d}t + \sum_{k=0}^{N} Z_t^{N,j,k}\mathrm{d}W_t^k, \\ Y_T^{N,j} & = ucX_T^N \delta_{N,j} + (1-u)c(\bar{X}_T - X_T^N)\Big(\frac{1}{N} - \delta_{N,j}\Big). \end{cases} \tag{45}$$

When $j=i$, it becomes:

$$\begin{cases} \mathrm{d}Y_t^{i,i} & = \Big[u\epsilon(X_t^{i+1} - X_t^i) + (1-u)\epsilon(\bar{X}_t - X_t^i)(1 - \frac{1}{N})\Big]\mathrm{d}t \\ & \quad + \sum_{k=0}^{N} Z_t^{i,i,k}\mathrm{d}W_t^k, \\ Y_T^{i,i} & = -uc(X_T^{i+1} - X_T^i) - (1-u)c(\bar{X}_T - X_T^i)\Big(1 - \frac{1}{N}\Big), \quad i<N \end{cases} \tag{46}$$

$$\begin{cases} \mathrm{d}Y_t^{N,N} & = \Big[-u\epsilon X_t^N + (1-u)\epsilon(\bar{X}_t - X_t^N)\Big(1 - \frac{1}{N}\Big)\Big]\mathrm{d}t \\ & \quad + \sum_{k=0}^{N} Z_t^{N,N,k}\mathrm{d}W_t^k, \\ Y_T^{N,N} & = ucX_T^N - (1-u)c(\bar{X}_T - X_T^N)\Big(1 - \frac{1}{N}\Big). \end{cases} \tag{47}$$

Considering the BSDE system and the initial condition, we then make the following ansatz with function parameters depending on N:

$$Y_t^{i,i} = u\sum_{j=i}^{N} \phi_t^{N,i,j} X_t^j - (1-u)(\bar{X}_t - X_t^i)\theta_t^N, \tag{48}$$

for some deterministic scalar functions ϕ_t, θ_t satisfying the terminal condition: when $i < N$, $\phi_T^{N,i,i} = c, \phi_T^{N,i,i+1} = -c, \phi_T^{N,i,j} = 0$ for $N \geq j \geq i+2$; $\phi_T^{N,N,N} = c$ and $\theta_T^N = c(1-\frac{1}{N})$. For simplicity of notation, we denote $\theta_t = \theta_t^N$. Using the ansatz (48), the optimal strategy and forward equation become:

$$\begin{cases} \hat{\alpha}^i = -Y_t^{i,i} = -u\sum_{j=i}^{N}\phi_t^{N,i,j}X_t^j + (1-u)(\bar{X}_t - X_t^i)\theta_t, \\ \mathrm{d}X_t^j = \Big[-u\sum_{k=j}^{N}\phi_t^{N,j,k}X_t^k + (1-u)(\bar{X}_t - X_t^j)\theta_t\Big]\mathrm{d}t + \sigma\mathrm{d}W_t^j. \end{cases} \tag{49}$$

By taking the averages, we obtain

$$\begin{aligned} \mathrm{d}\bar{X}_t &= -u\cdot\frac{1}{N}\sum_{j=1}^{N}\sum_{k=j}^{N}\phi_t^{N,j,k}X_t^k\mathrm{d}t + \sigma\cdot\frac{1}{N}\sum_{j=1}^{N}\mathrm{d}W_t^j \\ &= -u\cdot\frac{1}{N}\sum_{k=1}^{N}(\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t + \sigma\cdot\frac{1}{N}\sum_{k=1}^{N}\mathrm{d}W_t^k \end{aligned}$$

and then

$$\begin{aligned} \mathrm{d}(\bar{X}_t - X_t^i) = {} & -u\cdot\frac{1}{N}\sum_{k=1}^{i-1}(\sum_{j=1}^{k}\phi_t^{N,j,k})X_t^k\mathrm{d}t + u\sum_{k=i+1}^{N}\Big(\phi_t^{N,i,k} - \frac{1}{N}\sum_{j=1}^{k}\phi_t^{N,j,k}\Big)X_t^k\mathrm{d}t \\ & + \Big(u\phi_t^{N,i,i} - u\frac{1}{N}\sum_{j=1}^{i}\phi_t^{N,j,i} + (1-u)\theta_t\Big)X_t^i\mathrm{d}t - (1-u)\bar{X}_t\theta_t\mathrm{d}t \\ & + \sigma\Big(\frac{1}{N}\sum_{k=1}^{N}\mathrm{d}W_t^k - ud W_t^i\Big). \end{aligned} \tag{50}$$

Differentiating the ansatz eq. (48) and using eq. (50), we obtain

$$\begin{aligned} \mathrm{d}Y_t^{i,i} &= u\cdot\sum_{j=i}^{N}[X_t^j\dot{\phi}_t^{N,i,j}\mathrm{d}t + \phi_t^{N,i,j}dX_t^j] \\ &\quad - (1-u)\cdot\Big(\dot{\theta}_t(\bar{X}_t - X_t^i)\mathrm{d}t + \theta_t\mathrm{d}(\bar{X}_t - X_t^i)\Big) \\ &\stackrel{\text{def}}{=} u\cdot\mathrm{I} - (1-u)\cdot\mathrm{II} \end{aligned} \tag{51}$$

For the first term, we have

$$\begin{aligned}
\mathrm{I} &= \sum_{j=i}^{N} [X_t^j \dot{\phi}_t^{N,i,j} \mathrm{d}t + \phi_t^{N,i,j} \mathrm{d}X_t^j] \\
&= \sum_{k=i}^{N} \big(\dot{\phi}_t^{N,i,k} - u \sum_{j=i}^{k} \phi_t^{N,i,j} \phi_t^{N,j,k} - (1-u)\theta_t \phi_t^{N,i,k}\big) X_t^k \mathrm{d}t \\
&\quad + (1-u)\theta_t \sum_{k=i}^{N} \phi_t^{N,i,k} \cdot \bar{X}_t dt + \sigma \sum_{k=i}^{N} \phi_t^{N,i,k} dW_t^k.
\end{aligned}$$

Then, for the second term, we have

$$\begin{aligned}
\mathrm{II} &= \dot{\theta}_t(\bar{X}_t - X_t^i)\mathrm{d}t + \theta_t \mathrm{d}(\bar{X}_t - X_t^i) \qquad (52) \\
&= -u\theta_t \frac{1}{N} \sum_{k=1}^{i-1} (\sum_{j=1}^{k} \phi_t^{N,j,k}) X_t^k dt + u\theta_t \sum_{k=i+1}^{N} (\phi_t^{N,i,k} - \frac{1}{N} \sum_{j=1}^{k} \phi_t^{N,j,k}) X_t^k dt \\
&\quad - [\dot{\theta}_t - u\theta_t(\phi_t^{N,i,i} - \frac{1}{N} \sum_{j=1}^{i} \phi_t^{N,j,i}) - (1-u)\theta_t^2] X_t^i dt \\
&\quad + (\dot{\theta}_t - (1-u)\theta_t^2)\bar{X}_t dt + \sigma(\frac{1}{N} \sum_{k=1}^{N} dW_t^k - dW_t^i).
\end{aligned}$$

Thus $\mathrm{d}Y_t^{i,i} = u \cdot \mathrm{I} - (1-u) \cdot \mathrm{II}$ in (51) can be written as:

$$\begin{aligned}
&\sum_{k=1}^{i-1} \Big(u(1-u)\theta_t \frac{1}{N} \sum_{j=1}^{k} \phi_t^{N,j,k}\Big) X_t^k \mathrm{d}t \\
&+ \sum_{k=i+1}^{N} \Big[u\dot{\phi}_t^{N,i,k} - u^2 \sum_{j=i}^{k} \phi_t^{N,i,j} \phi_t^{N,j,k} \qquad (53) \\
&- u(1-u)\theta_t \phi_t^{N,i,k} - u(1-u)\theta_t(\phi_t^{N,i,k} - \frac{1}{N} \sum_{j=1}^{k} \phi_t^{N,j,k})\Big] X_t^k \mathrm{d}t \\
&+ \Big[u\dot{\phi}_t^{N,i,i} - u^2(\phi_t^{N,i,i})^2 - 2u(1-u)\theta_t \phi_t^{N,i,i} \\
&\quad + (1-u)\dot{\theta}_t + u(1-u)\theta_t \frac{1}{N} \sum_{j=1}^{i} \phi_t^{N,j,i} - (1-u)^2\theta_t^2\Big] X_t^i \mathrm{d}t
\end{aligned}$$

$$+\Big[u(1-u)\theta_t\sum_{k=i}^{N}\phi_t^{N,i,k}-(1-u)\dot\theta_t+(1-u)^2\theta_t^2\Big]\bar X_t\mathrm{d}t$$

$$+u\sigma\sum_{k=i}^{N}\phi_t^{N,i,k}\mathrm{d}W_t^k-(1-u)\sigma\theta_t\Big(\frac{1}{N}\sum_{k=1}^{N}\mathrm{d}W_t^k-dW_t^i\Big).$$

Now we compare the two Itô's decompositions (46) and (53). The martingale terms give the processes $Z_t^{i,j,k}$:

$Z_t^{i,i,k}=-(1-u)\sigma\theta_t\frac{1}{N}$ for $k<i$,
$Z_t^{i,i,i}=u\sigma\phi_t^{N,i,i}+(1-u)\sigma\theta_t(1-\frac{1}{N})$ and $Z_t^{i,i,k}=u\sigma\phi_t^{N,i,k}$ for $k>i$.

And from the drift terms, we get the following system of ordinary differential equations for $\phi^{N,i,k}$:
when $i<N$, $k=i$

$$\begin{aligned}&u\dot\phi_t^{N,i,i}-u^2(\phi_t^{N,i,i})^2-2u(1-u)\theta_t\phi_t^{N,i,i}\\&+(1-u)\dot\theta_t\Big(1-\frac{1}{N}\Big)-(1-u)^2\theta_t^2\Big(1-\frac{1}{N}\Big)\\&\quad+u(1-u)\theta_t\frac{1}{N}\Big(\sum_{j=1}^{i}\phi_t^{N,j,i}+\sum_{\ell=i}^{N}\phi_t^{N,i,\ell}\Big)\\&=-u\epsilon-(1-u)\epsilon\Big(1-\frac{1}{N}\Big)^2,\quad \phi_T^{N,i,i}=c,\end{aligned}\tag{54}$$

for $k=i+1$

$$\begin{aligned}&u\dot\phi_t^{N,i,i+1}-u^2(\phi_t^{N,i,i}\phi_t^{N,i,i+1}+\phi_t^{N,i,i+1}\phi_t^{N,i+1,i+1})\\&\quad-2u(1-u)\theta_t\phi_t^{N,i,i+1}-(1-u)\dot\theta_t\frac{1}{N}+(1-u)^2\theta_t^2\frac{1}{N}\\&\quad+u(1-u)\theta_t\frac{1}{N}(\sum_{j=1}^{i+1}\phi_t^{N,j,i+1}+\sum_{\ell=i}^{N}\phi_t^{N,i,\ell})\\&=u\epsilon+(1-u)\epsilon\Big(1-\frac{1}{N}\Big)\frac{1}{N},\quad \phi_T^{N,i,i+1}=-c,\end{aligned}\tag{55}$$

for $k \geq i+2$

$$
\begin{aligned}
& u\dot{\phi}_t^{N,i,k} - u^2 \sum_{j=i}^{l} \phi_t^{N,i,j} \phi_t^{N,j,k} - 2u(1-u)\theta_t \phi_t^{N,i,k} \\
& - (1-u)\dot{\theta}_t \frac{1}{N} + (1-u)^2 \theta_t^2 \frac{1}{N} + u(1-u)\theta_t \frac{1}{N}\Big(\sum_{j=1}^{k} \phi_t^{N,j,k} + \sum_{\ell=i}^{N} \phi_t^{N,i,\ell} \Big) \\
& = (1-u)\epsilon\Big(1 - \frac{1}{N}\Big)\frac{1}{N}, \quad \phi_T^{N,i,k} = 0,
\end{aligned} \tag{56}
$$

and

$$
\begin{aligned}
& u(1-u)\theta_t \sum_{k=i}^{N} \phi_t^{N,i,k} - (1-u)\dot{\theta}_t + (1-u)^2\theta_t^2 = (1-u)\epsilon\Big(1 - \frac{1}{N}\Big), \\
& \theta_T = c\Big(1 - \frac{1}{N}\Big);
\end{aligned} \tag{57}
$$

When $i = k = N$,

$$
\begin{aligned}
& u\dot{\phi}_t^{N,N,N} - u^2(\phi_t^{N,N,N})^2 - 2u(1-u)\theta_t\phi_t^{N,N,N} \\
& + (1-u)\dot{\theta}_t\Big(1 - \frac{1}{N}\Big) - (1-u)^2\theta_t^2\Big(1 - \frac{1}{N}\Big) \\
& \quad + u(1-u)\theta_t \frac{1}{N}\Big(\sum_{j=1}^{N} \phi_t^{N,j,N} + \phi_t^{N,N,N}\Big) \\
& = -u\epsilon - (1-u)\epsilon\Big(1 - \frac{1}{N}\Big)^2, \quad \phi_T^{N,N,N} = c,
\end{aligned} \tag{58}
$$

and

$$
\begin{aligned}
& u(1-u)\theta_t\phi_t^{N,N,N} - (1-u)\dot{\theta}_t + (1-u)^2\theta_t^2 \\
& = (1-u)\epsilon\Big(1 - \frac{1}{N}\Big), \quad \theta_T = c\Big(1 - \frac{1}{N}\Big).
\end{aligned} \tag{59}
$$

Proposition 8. *An open-loop Nash equilibrium for the finite-player stochastic game with cost functionals* (39) *with* (40)-(41) *is determined by* (49)*, where* $\{\phi^{N,i,j}, \theta\}$ *are the unique solution to the finite system* (54)-(57) *of Riccati equations with* (58)-(59)*.*

When $u = 1$, the systems are exactly what we obtained for finite-player directed chain game in section 2. We have the similar conclusion that the boundary condition does not affect the functions $\phi_t^{N,i,j}$ $(j < N)$ for all $i < N$. We can also compare the system with the system (63) we introduce later. Under suitable assumptions, the system may converge, as the number N of players goes to infinity.

5.2 Infinite-Player Game Model with Mean-Field Interaction

Motivated by section 5.1 and following section 3, we can define a game with infinite players on a mixed system, including the directed chain interaction and the mean field interaction. This section searches for an open-loop Nash equilibrium and repeats the same steps as before to analyse the infinite mixed system game. We have a more general Catalan Markov chain and table 1 below shows the asymptotic behaviors of the variances and covariances as $t \to \infty$ for the process with different types of interactions. Comparing it with Table 1 in [6], we have similar conclusions except that our asymptotic variance of purely directed chain does not explode.

We assume the same drift and diffusion coefficients and the initial conditions for $X_\cdot^i$ as the finite-player game. By choosing α_t^i, player i tries to minimize:

$$\begin{aligned} J^i(\alpha^1, \alpha^2, \cdots) :=& \mathbb{E}\Big\{ \int_0^T \Big(\frac{1}{2}(\alpha_t^i)^2 + u \cdot \frac{\epsilon}{2}(X_t^{i+1} - X_t^i)^2 \\ &\qquad\qquad + (1-u) \cdot \frac{\epsilon}{2}(m_t - X_t^i)^2\Big)\mathrm{d}t \\ &\qquad + u \cdot \frac{c}{2}(X_T^{i+1} - X_T^i)^2 + (1-u) \cdot \frac{c}{2}(m_T - X_T^i)^2 \Big\}, \end{aligned} \tag{60}$$

for some positive constants ϵ, c and $u \in [0, 1]$. Here, there is an issue in the choice of m_t. Intuitively, it should come from the finite-player mixed game described in section 5.1 as the limit of $\bar{X}_\cdot$ as $N \to \infty$. Combined with the fact that we had $\mathbb{E}\{X_t^i\}$ independent of i, it is natural to set $m_t = \mathbb{E}\{X_t^i\}$ and check afterwards that this mean value does not depend on i de facto after solving the fixed point step. Note that the case $u = 0$ is very particular, and consists in solving the same mean field game problem for every i. The case $u = 1$ has already been studied in section 3, and therefore, in what follows, we concentrate on the case $u \in (0, 1)$.

5.2.1 Open-Loop Nash Equilibrium

We search for a Nash equilibria of the system among strategies $\{\alpha_t^i, i \geq 1\}$. For $i \geq 1$, minimizing the Hamiltonian

$$\sum_{k=1}^{\infty} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + u\frac{\epsilon}{2}(x^{i+1} - x^i)^2 + (1-u)\frac{\epsilon}{2}(m_t - x^i)^2, \tag{61}$$

with respect to α^i, and following closely to Carmona, Fouque, and Sun [4], we obtain

$$\mathrm{d}X_t^i = \big(- u \sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j + (1-u)(m_t - X_t^i)\psi_t \big)\mathrm{d}t + \sigma \mathrm{d}W_t^i, \tag{62}$$

where $\phi_\cdot^k$ and $\psi_\cdot$ are determined by the following system of Riccati equation: $k \geq 0$

$$\begin{aligned} \dot{\phi}_t^k &= u \sum_{j=0}^{k} \phi_t^j \phi_t^{k-j} + 2(1-u)\psi_t \phi_t^k + \varepsilon(-\delta_{0,k} + \delta_{1,k}), \\ \phi_T^k &= c(\delta_{0,k} - \delta_{1,k}), \\ \dot{\psi}_t &= u\psi_t \sum_{j=0}^{\infty} \phi_t^j + (1-u)(\psi_t)^2 - \varepsilon, \quad \psi_T = c. \end{aligned} \tag{63}$$

In appendix A.6 we show the following result which simplifies it considerably.

Proposition 9. *$\phi_\cdot^j$ satisfies $\sum_{j=0}^{\infty} \phi_t^j = 0$ for $0 \leq t \leq T$ and thus, $\psi_\cdot$ is the unique solution to $\dot{\psi}_t = (1-u)(\psi_t)^2 - \varepsilon$, $0 \leq t \leq T$, $\psi_T = c$ in eq. (63).*

Proposition 10. *An open-loop Nash equilibrium for the infinite-player stochastic game with cost functionals (60) is determined by (62), where $\{\phi_\cdot^{N,i,j}, \psi_\cdot\}$ are the unique solution to the infinite system (63) of Riccati equations.*

Looking at the stationary solution in the limit ($T \to \infty$), and without loss of generality assuming $\epsilon = 1$ again, the recurrence relation can be solved by the method of moment generating function to obtain:

$$\begin{cases} \psi = \sqrt{\frac{1}{1-u}}, \quad \phi^0 = \frac{1-\sqrt{1-u}}{u}, \\ \phi^1 = -\frac{1}{2}, \quad \phi^k = -\dfrac{(2k-3)!}{(k-2)!k!2^{2k-2}} u^{k-1}, \quad for\ k \geq 2. \end{cases} \tag{64}$$

5.2.2 Catalan Markov Chain for the Mixed Model

As in section 4, letting $T \to \infty$, with replacement of ϕ_t^j by the stationary solution ϕ^j in (64), $X_\cdot^i$, $i \geq 1$ defined by eq. (62) can be represented by the form of eq. (31) but now with a new matrix $\mathbf{Q}^{(u)}$, where its (i,j) element $q_{i,j}$ is given by $q_{i,i} = -1$, $q_{i,j} = -u\phi^{j-i} \cdot \mathbf{1}_{\{j>i\}}$ with ϕ^j in (64) for $i, j \geq 1$. Since $u^2 \sum_{i=1}^{k-1} \phi^i \phi^{k-i} = -2u\phi^k$, we have $(\mathbf{Q}^{(\mathbf{u})})^2 = I - uB$ with B having 1 's on the upper second diagonal and 0 's elsewhere.

With a smooth function $F(x) := \exp(-\sqrt{-x})$, $x \in \mathbb{C}$, the matrix exponential of $\mathbf{Q}^{(\mathbf{u})}t$ can be written formally $\exp(\mathbf{Q}^{(\mathbf{u})}t) = F((-I + uB)t^2)$. With a slight modification of proof of proposition 4 in appendix A.2, we may compute it explicitly. We can summarize our finding on the limiting process as $T \to \infty$:

Proposition 11. *With $\mathbf{x}_0 = \mathbf{0}$, the Gaussian process X_t^i, $i \in \mathbb{N}$, $t \geq 0$, corresponding to the (Catalan) general Markov chain with generator $\mathbf{Q}^{(u)}$, is*

$$X_t^i = \sum_{j=i}^{\infty} \int_0^t \frac{u^{j-i}(t-s)^{2(j-i)}}{(j-i)!} \cdot \rho_{j-i}(-(t-s)^2)\, e^{-(t-s)} \cdot \mathrm{d}W_s^j, \quad t \geq 0. \tag{65}$$

where $\rho_j(\cdot)$ is defined in eq. (35).

5.2.3 Asymptotic Behavior

Table 1 exhibits the asymptotic behaviors of their variances and covariances as $t \to \infty$. The calculation is given in appendix A.7. We find that only when $u = 0$ (i.e., pure mean field game), the asymptotic cross-covariance is zero, which means the states are asymptotically independent. Otherwise, they are dependent and their covariance is finite. Note in the purely nearest neighbor interaction studied in Detering, Fouque, and Ichiba [6], i.e., in the case $u = 0$, the variance is not stabilized as in our "Catalan" interaction equilibrium dynamics.

6 Periodic Directed Chain Game

We consider a stochastic game with finite players on a periodic ring structure in fig. 1 (b). Assume the dynamics of the states of the individual players are

u	Interaction Type	Asymptotic Variance	Asymptotic Independence between two players
$u=0$	Purely mean-field	Stabilized	Independent
$u\in(0,1)$	Mixed interaction	Stabilized	Dependent
$u=1$	Purely directed chain	Stabilized	Dependent

Table 1: Asymptotic behaviors as $t\to\infty$

given by N stochastic differential equations of the form:

$$\mathrm{d}X_t^i = \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t^i, \quad i=1,\cdots,N, \quad 0\le t\le T, \tag{66}$$

where $(W_t^i)_{0\le t\le T}$, $i=1,\cdots,N$ are one-dimensional independent standard Brownian motions. The drift coefficient function, the diffusion coefficient and the initial conditions are assumed to be the same as those in section 2. In this model, player i chooses its own strategy α^i in order to minimize its objective function of the form:

$$J^i(\alpha^1,\cdots,\alpha^N) := \mathbb{E}\Big\{\int_0^T \Big[\frac{1}{2}(\alpha_t^i)^2 + \frac{\epsilon}{2}(X_t^{i+1}-X_t^i)^2\Big]\mathrm{d}t + \frac{c}{2}(X_T^{i+1}-X_T^i)^2\Big\}, \tag{67}$$

with some constants $\epsilon>0$, $c\ge 0$, where we define $X_\cdot^{N+1}=X_\cdot^1$ or more generally, $X_t^{i+j}=X_t^{(i+j) \mod N}$, because of the periodic ring structure, for $i,j=1,\dots,N$.

6.1 Construction of an Open-Loop Nash Equilibrium

We construct an open-loop Nash equilibria of the system among strategies $\{\alpha_t^i, 1\le i\le N\}$ by the Pontryagin stochastic maximum principle. The Hamiltonian H^i for player i is

$$\sum_{k=1}^N \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x^{i+1}-x^i)^2. \tag{68}$$

The adjoint processes $Y_t^i = (Y_t^{i,j}; j=1,\cdots,N)$ and $Z_t^i=(Z_t^{i,j,k}; j,k=1,\cdots,N)$ for $i=1,\cdots,N$ are defined as the solutions of the system of

BSDEs:

$$\begin{cases} dY_t^{i,j} & = -\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j})\mathrm{d}t + \sum_{k=1}^{N} Z_t^{i,j,k}\mathrm{d}W_t^k, \\ Y_T^{i,j} & = c(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}). \end{cases} \tag{69}$$

Based on the sufficiency part of the Pontryagin stochastic maximum principle, we can get an open-loop Nash equilibrium by minimizing the Hamiltonian H^i with respect to α^i: $\partial_{\alpha^i} H^i = y^{i,i} + \alpha^i = 0$ leadings to the choice $\hat{\alpha}^i = -y^{i,i}$ for each i. With this choice for the controls α^i's, the forward equation (66) becomes coupled with the backward equation (69). We make the ansatz: for $t \geq 0$, $i \geq 1$,

$$Y_t^{i,i} = \sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j}, \tag{70}$$

for some deterministic scalar functions $\phi_t^{N,j}$ satisfying the terminal conditions: $\phi_T^{N,0} = c, \phi_T^{N,1} = -c, \phi_T^{N,k} = 0$ for $k \geq 2$ and $X_t^{i+j} := X_t^{(i+j) \mod N}$. Using the ansatz, the optimal strategy $\hat{\alpha}^i$ and the forward equation (66) become:

$$\hat{\alpha}^i = -Y_t^{i,i} = -\sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j}, \quad \mathrm{d}X_t^i = -\sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j}\mathrm{d}t + \sigma \mathrm{d}W_t^i. \tag{71}$$

Using the equations (71), we can differentiate the ansatz (70): for $1 \leq i \leq N$, $t \leq 0$,

$$\mathrm{d}Y_t^{i,i} = \sum_{j=0}^{N-1} X_t^{i+j} \dot{\phi}_t^{N,j}\mathrm{d}t - \sum_{j=0}^{N-1} \phi_t^{N,j} \sum_{k=0}^{N-1} \phi_t^{N,k} X_t^{i+j+k}\mathrm{d}t + \sum_{j=0}^{N-1} \sigma \phi_t^{N,j}\mathrm{d}W_t^{i+j} \tag{72}$$

Now comparing the two Itô's decompositions (72) and (69) of $Y_t^{i,i}$, we obtain

$$\begin{aligned} &\dot{\phi}_t^{N,i} = \sum_{j=0}^{N-1} \phi_t^{N,j} \phi_t^{N,N+i-j} - \epsilon(\delta_{0,i} - \delta_{1,i}), \quad \phi^{N,i}(T) = c(\delta_{0,i} - \delta_{1,i}), \\ &Z_t^{i,i,k} = \sigma \phi_t^{N,N+k-i}, \quad 0 \leq t \leq T, \ 1 \leq i \leq N \end{aligned} \tag{73}$$

(cf. (25)), where we use a convention $\phi_\cdot^{N,N+i-j} = \phi_\cdot^{N,i-j}$, if $i - j \geq 0$.

It can be written as a matrix Ricatti equation:

$$\dot{\Phi}^N(t) = \Phi^N(t)\Phi^N(t) - \mathbf{E}\,, \quad \Phi^N(T) := \mathbf{C}\,, \tag{74}$$

where we denote by $\Phi^N(\cdot)$ the $N \times N$ matrix-valued function with (i,j) element being $\phi_t^{N,N+i-j}$ for $1 \leq i,j \leq N$ by using the convention $\phi_t^{N,N+i-j} = \phi_t^{N,i-j}$, if $i \geq j$, and by $\mathbf{C} = (c_{i,j})$ and $\mathbf{E} = (\epsilon_{i,j})$, respectively the $N \times N$ matrices with (i,j) element being $c_{i,j} := c(\delta_{i,j} - \delta_{i,j+1})$ and $\epsilon_{i,j} := \epsilon(\delta_{i,j} - \delta_{i,j+1})$ for $1 \leq i,j \leq N$.

Proposition 12. *The solution $\phi_\cdot^{N,k}$, $k = 1, \ldots, N$ to the system of Riccati equations* (73) *satisfies the relation $\sum_{k=0}^{N-1} \phi_t^{N,k} = 0$ for $0 \leq t \leq T$.*

Proof. Given in appendix A.8. □

Proposition 13. *An open-loop Nash equilibrium for the linear quadratic stochastic game with cost functionals* (67) *for the N players with a periodic boundary condition $X_\cdot^{N+1} = X_\cdot^1$ is given by* (71)*, where $\{\phi_\cdot^{N,k}\}$ are uniquely determined by the system* (73) *of Riccati equations.*

With finite N, these equations (73) are not easy to solve explicitely. If we let $N \to \infty$, we expect that the system converges to the Riccati system of the infinite-player game studied in section 3.

Conjecture 1. *The limit of each element in $\Phi^N(\cdot)$ in* (74) *exists as $N \to \infty$, i.e., $\Phi^N(t) \to \Phi^\infty(t)$ and the limit $\Phi^\infty(t)$ is an infinite dimensional, lower triangular, matrix-valued function of $t \geq 0$ given by $\Phi^\infty(t) = (\Phi^{\infty,i,j}(t))_{i,j\in\mathbb{N}}$ with $\Phi^{\infty,i,j}(\cdot) \equiv 0$ if $i < j$; $\Phi^{\infty,i,j}(\cdot) \equiv \phi^{i-j}$ if $i \geq j$, where ϕ^k's are given in proposition 2.*

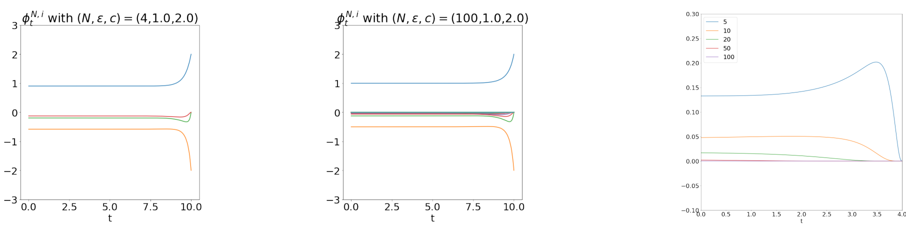
Remark 5. *Proving this conjecture is equivalent to verify that for every j, $\sum_{k=j+1}^{N-1} \phi_t^{N,k}\phi_t^{N,N+j-k} \to 0$ as $N \to \infty$. As of now, this remains an open problem.*

Our conjecture is substantiated by numerical evidences presented below.

6.2 Numerical Results

By the methods given in [9], we get the numerical solution of the matrix Riccati equation (74). Taking $\epsilon = 2$, $c = 1$, $T = 10$ (large terminal time), fig. 2

(a)-(b) shows the behaviors of the ϕ functions defined by the system (25) for $N = 4$ and $N = 100$. They converge to the constant solutions of the infinite game given in section 4, except in the tail close to maturity as T is large but not infinite. This result confirms our conjecture stated in the previous section. fig. 2 (c) shows the behavior of the function $\sum_{k=1}^{N-1} \phi_t^{N,k} \phi_t^{N,N-k}$ for different values of $N = 5, 10, 20, 50, 100$. As we can see, the sum converges to 0 when N becomes larger, which supports the statement with $j = 0$ in remark 5. Although these numerical results give us strong evidence and confidence that the conjecture is true, a mathematical proof is still needed and it is part of our ongoing research.



(a) $N = 4$ (b) $N = 100$ (c) $\sum_{k=1}^{N-1} \phi_t^{N,k} \phi_t^{N,N-k}$

Figure 2: The blue line (top) is $\phi_t^{N,0}$ and the orange line (bottom) is $\phi_t^{N,1}$ in (a)-(b). $\sum_{k=1}^{N-1} \phi_t^{N,k} \phi_t^{N,N-k}$ for different values of $N = 5, 10, 20, 50, 100$ from top to bottom in (c).

7 Directed Infinite Tree Game

We describe a stochastic game on a directed tree structure with $N \geq 2$ generations first. Starting with one player in the root node denoted by $(1, 1)$ in the first generation, recursively each parent has a fixed, common number of descendants, denoted by $d \geq 1$, and there are d^{n-1} players in the n-th generation for $n \geq 1$. For $1 \leq n \leq N, 1 \leq k \leq d^{n-1}$, $X^{n,k}$ represents the state of the k-th individual of the n-th generation, and its direct descendants in the $(n+1)$st generation are labelled as $\{X^{n+1,(k-1)d+1}, X^{n+1,(k-1)d+2}, \cdots, X^{n+1,kd}\}$. We consider the stochastic differential game of players in the N generations and then we generalize to a stochastic differential game in a directed infinite tree by considering its limit as $N \to \infty$. The network is shown in fig. 3.

We assume the dynamics of the states of the players are given by the

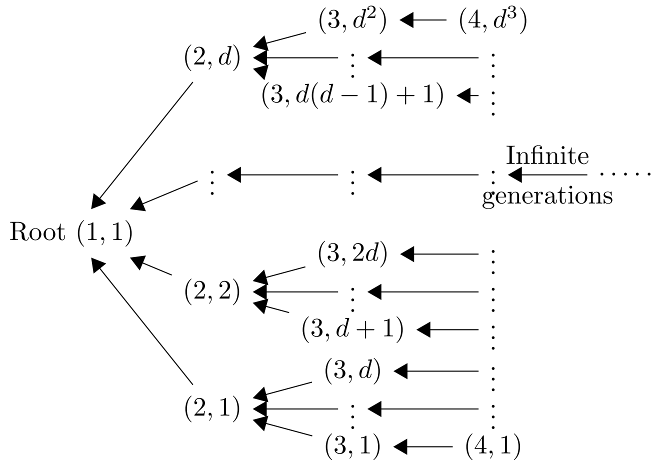


Figure 3: Directed Tree Network

stochastic differential equations of the form:

$$\mathrm{d}X_t^{n,k} = \alpha_t^{n,k}\mathrm{d}t + \sigma \mathrm{d}W_t^{n,k}, \quad 0 \le t \le T, \tag{75}$$

where $(W_t^{n,k})_{0\le t\le T}, 1 \le n \le N, 1 \le k \le d^{n-1}$ are one-dimensional independent standard Brownian motions. Similarly, we assume that the diffusion is one-dimensional and the diffusion coefficients are constant and identical denoted by $\sigma > 0$. The drift coefficients $\alpha^{n,k}$'s are adapted to the filtration of the Brownian motions and satisfy $\mathbb{E}[\int_0^T |\alpha_t^{n,k}|^2 dt] < \infty$. The system starts at time $t = 0$ from $i.i.d.$ square-integrable random variables $X_0^{n,k} = \xi_{n,k}$ independent of the Brownian motions and, without loss of generality, we assume $\mathbb{E}(\xi_{n,k}) = 0$ for every pair of (n, k).

In this model, among the first $N-1$ generations, each player (n, k) chooses its own strategy $\alpha^{n,k}$ in order to minimize its objective function of the form: for $1 \le n < N$

$$\begin{aligned} &J^{n,k}(\alpha^{m,\ell}; 1 \le m \le N, 1 \le \ell \le d^m) \\ &:= \mathbb{E}\bigg\{ \int_0^T \Big(\frac{1}{2}(\alpha_t^{n,k})^2 + \frac{\epsilon}{2}\big(\overline{X}_t^{n+1,k} - X_t^{n,k}\big)^2\Big)\mathrm{d}t + \frac{c}{2}\Big(\overline{X}_T^{n+1,k} - X_T^{n,k}\Big)^2\bigg\}, \end{aligned} \tag{76}$$

where $\overline{X}_\cdot^{n,k} := \sum_{i=(k-1)d+1}^{kd} X_\cdot^{n,i}/d$ for some constants $\epsilon > 0$ and $c \ge 0$ and for $n, k \ge 1$. The running cost and the terminal cost functions are defined by

$f^{n,k}(x,\alpha^{n,k}) := \frac{1}{2}(\alpha^{n,k})^2 + \frac{\epsilon}{2}(\overline{x}^{n+1,k} - x^{n,k})^2$ and $g^{n,k}(x) := \frac{c}{2}(\overline{x}^{n+1,k} - x^{n,k})^2$, respectively with $\overline{x}^{n,k} := \sum_{i=(k-1)d+1}^{kd} x^{n,i}/d$. For simplicity, the behaviours of the N-th generation are described by the boundary condition where all the players $\{X^{N,k}, 1 \le k \le d^{N-1}\}$ are attracted to 0. The cost functional for player (N,k) is given by:

$$J^{N,k}(\alpha^{N,k}) := \mathbb{E}\left\{ \int_0^T \left(\frac{1}{2}(\alpha_t^{N,k})^2 + \frac{\epsilon}{2}(X_t^{N,k})^2\right)\mathrm{d}t + \frac{c}{2}(X_T^{N,k})^2 \right\} \tag{77}$$

for $k = 1, \dots, d^{N-1}$. Since players of the last generation do not depend on the other players, the boundary condition defines a self-controlled problem.

Now, inspired by the conclusion in section 2, as the number N of generations goes to infinity, i.e., $N \to \infty$, the effect of the boundary condition should vanish. Thus it is natural and reasonable that we decide to pass the N-generation finite tree to an infinite tree with infinite number of generations, and study the Nash equilibrium of the infinite-tree game. We still assume each parent has d direct descendants. The dynamics of the states and the costs are the same as (75) and (76) with $n \ge 1$.

7.1 Open-Loop Nash Equilibria

We search for an open-loop Nash equilibrium of the directed infinite-tree system among strategies $\{\alpha^{n,k}; n \ge 1, 1 \le k \le d^{n-1}\}$. The Hamiltonian $H^{n,k}(x^{m,l}, y^{n,k;m,l}, \alpha^{m,l}; m \in \mathbb{N}, 1 \le l \le d^{m-1})$ for player (n,k) is

$$\sum_{m=1}^{M_n} \sum_{l=1}^{d^{m-1}} \alpha^{m,l} y^{n,k;m,l} + \frac{1}{2}(\alpha^{n,k})^2 + \frac{\epsilon}{2}(\overline{x}^{n+1,k} - x^{n,k})^2,$$

assuming it is defined on $Y_t^{n,k}$'s where *only finitely many* $Y_t^{n,k;m,l}$'s are nonzero for every given (n,k). Here, $M_n \in \mathbb{N}$ represents a depth of this finite dependence depending on n with $M_n > n$ for $n \ge 1$. This assumption is checked in remark 6 below. Thus, the Hamiltonian $H^{n,k}$ for player (n,k) is well defined for $n, k \ge 1$.

The adjoint processes $Y_t^{n,k} = (Y_t^{n,k;m,l}; m \in \mathbb{N}, 1 \le l \le d^{m-1})$ and $Z_t^{n,k} = (Z_t^{n,k;m,l;p,q}; m, p \in \mathbb{N}, 1 \le l \le d^{m-1}, 1 \le q \le d^{p-1})$ for $n \in \mathbb{N}, 1 \le k \le d^{n-1}$

are defined as the solutions of BSDEs

$$\begin{aligned} \mathrm{d}Y_t^{n,k;m,l} &= -\epsilon(\overline{X}_t^{n+1,k} - X_t^{n,k})(\overline{\delta}_{m,\ell}^{n+1,k} - \delta_{m,\ell}^{n,k})\mathrm{d}t + \sum_{p=1}^{\infty}\sum_{q=1}^{d^{p-1}} Z_t^{n,k;m,l;p,q}\mathrm{d}W_t^{p,q}, \\ Y_T^{n,k;m,l} &= \partial_{x^{m,l}} g_{n,k}(X_T) = c(\overline{X}_T^{n+1,k} - X_T^{n,k})(\overline{\delta}_{m,\ell}^{n+1,k} - \delta_{m,\ell}^{n,k}), \end{aligned} \tag{78}$$

where $\delta_{m,\ell}^{n,k} := 1$, if $(n,k) = (m,\ell)$; 0, otherwise, and $\overline{\delta}_{m,\ell}^{n,k} := \sum_{i=(k-1)d+1}^{kd} \delta_{m,\ell}^{n,i}/d$.

Remark 6. *For every $(m,l) \neq (n,k)$ or $(n+1,i)$ where $(k-1)d+1 \leq i \leq kd$, $\mathrm{d}Y_t^{n,k;m,l} = \sum_{p=1}^{\infty}\sum_{q=1}^{d^{p-1}} Z_t^{n,k;m,l;p,q}\mathrm{d}W_t^{p,q}$ and $Y_T^{n,k;m,l} = 0$ implies $Z_t^{n,k;m,l;p,q} = 0$ for all (p,q). Thus, there must be finitely many non-zero $Y^{n,k;m,l}$'s for every (n,k). Hence, the Hamiltonian can be rewritten as*

$$\begin{aligned} &H^{n,k}(x^{m,l}, y^{n,k;n,k}, y^{n,k;n+1,i}, \alpha^{m,l}; m \in N, 1 \leq l \leq d^{m-1}, (k-1)d+1 \leq i \leq kd) \\ &= \alpha^{n,k}y^{n,k;n,k} + \sum_{i=(k-1)d+1}^{kd} \alpha^{n+1,i}y^{n,k;n+1,i} + \frac{1}{2}(\alpha^{n,k})^2 + \frac{\epsilon}{2}(\overline{x}^{n+1,k} - x^{n,k})^2. \end{aligned}$$

By minimizing the Hamiltonian $H^{n,k}$ at $\hat{\alpha}^{n,k} = -y^{n,k;n,k}$ with respect to $\alpha^{n,k}$ for all (n,k), we can get an open-loop Nash equilibrium. We make the ansatz:

$$Y_t^{n,k;n,k} = \sum_{i=0}^{\infty} \phi_t^i \sum_{j=0}^{d^i-1} X_t^{n+i,d^ik-j} = \sum_{m=n}^{\infty} \phi_t^{m-n} \sum_{j=0}^{d^{m-n}-1} X_t^{m,d^{m-n}k-j}, \tag{79}$$

for some deterministic scalar function $\phi_\cdot^k$ satisfying the terminal conditions: $\phi_T^0 = c, \phi_T^1 = -\frac{c}{d}, \phi_T^k = 0$ for $k \geq 2$. Using the ansatz, the optimal strategy $\hat{\alpha}^{n,k}$ and the forward equation for $X_\cdot^{n,k}$ in (75) for an open-loop Nash equilibrium become:

$$\begin{aligned} \hat{\alpha}_t^{n,k} &= -Y_t^{n,k;n,k} = -\sum_{m=n}^{\infty} \phi_t^{m-n} \sum_{j=0}^{d^{m-n}-1} X_t^{m,d^{m-n}k-j}, \\ \mathrm{d}X_t^{n,k} &= -\sum_{m=n}^{\infty} \phi_t^{m-n} \sum_{j=0}^{d^{m-n}-1} X_t^{m,d^{m-n}k-j}\mathrm{d}t + \sigma \mathrm{d}W_t^{n,k}. \end{aligned} \tag{80}$$

Now comparing the two Itô's decompositions of $\mathrm{d}Y^{n,k;n,k}$ from (78) and (79)-(80), we obtain from the martingale terms:

$$Z_t^{n,k;n,k;p,q} = \sigma\phi_t^{p-n} \text{ for } p \geq n \text{ and } 1 \leq q \leq d^{p-n}\,; \quad Z_t^{n,k;n,k;p,q} = 0, \text{ otherwise,}$$

for $0 \leq t \leq T$, and we obtain from the drift terms: $k \geq 0$, $0 \leq t \leq T$

$$\dot{\phi}_t^k = \sum_{j=0}^{k} \phi_t^j \phi_t^{k-j} - \epsilon\Big(\delta_{0,k} - \frac{1}{d}\cdot\delta_{1,k}\Big), \quad \phi_T^k = c\Big(\delta_{0,k} - \frac{1}{d}\cdot\delta_{1,k}\Big). \tag{81}$$

This Riccati system is closely related to the one in (25) for the infinite-player directed chain game and we can have a similar result.

Proposition 14. *Let $\phi_\cdot^{(k)} := \phi_\cdot^k$ in (81) to avoid confusion from the power. We have $\sum_{k=0}^{\infty} d^k \phi_\cdot^{(k)} = 0$, and the functions ϕ^k's can be obtained by a series expansion.*

Proof. Given in appendix A.9. $\square$

Proposition 15. *An open-loop Nash equilibrium for the linear quadratic stochastic game with cost functionals (76) for the infinite players on the directed tree in fig. 3 is given by (80), where $\{\phi_\cdot^i\}$ are uniquely determined by the system (81) of Riccati equations.*

Without loss of generality, we assume $\epsilon = 1$ and $\sigma = 1$. Following section 4, by taking $T \to \infty$, we look at the stationary long-time behavior of the Riccati system (81) satisfying $\dot{\phi}^k = 0$ for all k. Then the system gives the recurrence relation: $\phi^0 = 1$, $\phi^1 = -1/(2d)$ and $\sum_{j=0}^{k} \phi^j \phi^{k-j} = 0$ for $k \geq 0$. By using a moment generating function method as in appendix A.10, we obtain the stationary solution (cf. (30)):

$$\phi^0 = 1, \quad \phi^1 = -\frac{1}{2d}, \quad \text{and} \quad \phi^k = -\frac{(2k-3)!}{(k-2)!k!2^{2k-2}}\cdot\frac{1}{d^k} \quad \text{for} \quad k \geq 2. \tag{82}$$

7.2 Catalan Markov Chain for the Directed Tree Model

As $T \to \infty$, the limit of average of the infinite particle system (80) can be rewritten as

$$\mathrm{d}\overline{\mathbf{X}}_t = \mathbf{Q}_{\mathbf{d}\text{-tree}}\,\overline{\mathbf{X}}_t\mathrm{d}t + \mathrm{d}\overline{\mathbf{W}}_t, \quad t \geq 0 \tag{83}$$

a linear stochastic evolution equation of Ornstein-Uhlenbeck type, where $\overline{\mathbf{X}}_\cdot := (\overline{X}^k_\cdot \equiv \sum_{i=1}^{d^{k-1}} X^{k,i}_\cdot / d^{k-1}, k \in \mathbb{N})$ with $\overline{\mathbf{X}}_0 = \overline{\mathbf{x}}_0$, and $\overline{\mathbf{W}}_\cdot := (\overline{W}^k_t = \sum_{i=1}^{d^{k-1}} W^{k,i}_\cdot / d^{k-1}, k \in \mathbb{N})$ is a vector of averaged Brownian motions with mean 0 and variance t/d^k in each generation $k \in \mathbb{N}$, and $\mathbf{Q}_{\mathbf{d}\text{-tree}}$ is exactly the same as $\mathbf{Q}$ in section 4. Its solution is

$$\overline{\mathbf{X}}_t = e^{t\mathbf{Q}_{d\text{-tree}}}\overline{\mathbf{x}}_0 + \int_0^t e^{(t-s)\mathbf{Q}_{d\text{-tree}}} \mathrm{d}\overline{\mathbf{W}}_s; \quad t \geq 0. \tag{84}$$

Similarly to proposition 4 we can find the following formulas for $X^{1,1}_\cdot \equiv \overline{X}^1_\cdot$ and its asymptotic variance. Proof is in appendix A.11.

Proposition 16. *With $\mathbf{x}_0 = \mathbf{0}$, the formula for the root node $X^{1,1}_t$ in* (83) *is:*

$$X^{1,1}_t = \sum_{j=1}^{\infty} \int_0^t \frac{(t-s)^{2(j-1)}}{(j-1)!} \cdot \rho_{j-1}(-(t-s)^2)\, e^{-(t-s)} \cdot \mathrm{d}\overline{W}^j_s, \quad t \geq 0, \tag{85}$$

where $\rho_i(\cdot)$ is in (35). *Moreover, the asymptotic variance of $X^{1,1}_t$ is finite, i.e.,*

$$\lim_{t\to\infty} Var(X^{1,1}_t) = \frac{\sqrt{2}}{2} \cdot \Big(1 + \Big(\frac{d-1}{d}\Big)^{1/2}\Big)^{-1/2} \in \Big(\frac{1}{2}, \frac{\sqrt{2}}{2}\Big]. \tag{86}$$

Remark 7 (Connection to the mean-field game)**.** *When d goes to infinity, we are in the regime of the mean field game. The asymptotic variance is $\frac{1}{2}$ which is consistent with the variance of an Ornstein–Uhlenbeck process where the particle is attracted to* 0 *and the volatility and the mean reversion constant are both* 1*.*

8 Conclusion

We studied a linear-quadratic stochastic differential game on a directed chain network. We were able to identify Nash equilibria in the case of finite chain with various boundary conditions and in the case of an infinite chain. This last case allows for more explicit computation in terms of Catalan functions and Catalan Markov chain. The Catalan open-loop Nash equilibrium that we obtained is characterized by interactions with all the neighbors in one direction of the chain weighted by Catalan functions, even though the interaction in the objective functions is only with the nearest neighbor. Under equilibrium the variance of a state converges in the infinite time limit as opposed to the diverging behavior observed in the nearest neighbor dynamics studied in Detering, Fouque & Ichiba [6]. Our analysis is extended to mixed games with directed chain and mean field interaction so that our game model includes the two extreme network interactions, fully connected and only one neighbor connection. It is also extended to game on a deterministic tree structure. Our ongoing and future research concerns games with interactions on directed tree-like stochastic networks modeled as branching processes.

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A Appendix

A.1 Proof of Proposition 2

Define $S_t(z) := \sum_{k=0}^{\infty} z^k \, \phi_t^{(k)}$ where $0 \le z < 1$ with $\phi_t^{(k)} = \phi_t^k$ in (25) to avoid confusion for $t \ge 0$, $k \ge 0$. Then substituting (25) into $S_t(z)$, we obtain the one-dimensional Riccati equation

$$\dot{S}_t(z) = \sum_{k=0}^{\infty} z^k \dot{\phi}_t^{(k)} = (S_t(z))^2 - \epsilon(1-z), \quad S_T(z) = c(1-z) \tag{87}$$

and its solution is given by (27). One needs to be careful when taking $z = 1$ because the series defining $S_t(1)$ may not converge a priori. Instead, we take a sequence $\{z_n\}$ converging to 1, the limit of $S_t(z_n)$ converges to the ODE $\dot{S}_t(1) = (S_t(1))^2$, $S_T(1) = 0$, and hence, solving this limiting ODE, we obtain (26).

A.2 Catalan Markov Chain and Proposition 4

We have the Catalan probabilities $\{p_k > 0, k \ge 1\}$: $\sum_{k=1}^{\infty} p_k = 1$ and $p_k = \frac{1}{2}\sum_{i=1}^{k-1} p_i p_{k-i}$. It is easily seen then that $-\mathbf{Q}^2 = -I + B$ is an infinite Jordan block matrix with diagonal components -1, where B having 1 's on the upper second diagonal and 0 's elsewhere. Then as a smooth function $F(x) := \exp(-\sqrt{-x})$, $x \in \mathbb{C}$ of Jordan block matrix, we have

$$\exp(\mathbf{Q}t) = F((-I+B)t^2) = \sum_{k=0}^{\infty} \frac{t^{2k} F^{(k)}(-t^2)}{k!} B^k,$$

where the k-th derivative $F^{(k)}$ is given by $F^{(k)}(x) = \rho_k(x)F(x)$, $x \in \mathbb{C}$ with ρ_k in (35) from direct calculations and mathematical induction. Therefore, substituting them into (34), we obtain proposition 4.

A.3 Proof of Remark 4 in section 4

By ρ_k's formula in (36), we have for $\nu \geq 0$, $k \geq 1$,

$$\rho_k(-\nu^2) = \frac{1}{2^k} \sum_{j=k}^{2k-1} \frac{(j-1)!}{(2j-2k)!!(2k-j-1)!} = \frac{1}{2^k \nu^k} \cdot \sqrt{\frac{2\nu}{\pi}} \cdot e^{\nu} \cdot K_{k-(1/2)}(\nu),$$

where $K_n(x)$ is the modified Bessel function of the second kind, i.e.,

$$K_n(x) = \int_0^\infty e^{-x\cosh t} \cosh(nt) \mathrm{d}t\,; \quad n > -1, \;\; x > 0\,.$$

Then, by the change of variables, we obtain

$$\mathrm{Var}(X_t^1) = \sum_{k=0}^\infty \int_0^t \frac{(t-s)^{4k}}{(k!)^2} |\rho_k(-(t-s)^2)|^2 e^{-2(t-s)} \mathrm{d}s$$

$$= \sum_{k=1}^\infty \int_0^t \frac{2}{\pi} \frac{\nu^{2k+1}}{(k!)^2\, 4^k} \big(K_{k-(1/2)}(\nu)\big)^2 \mathrm{d}\nu + \frac{1-e^{-2t}}{2}\,; \quad t \geq 0\,.$$

A.4 Proof of Proposition 5 in section 4

Using the following identities from the special functions

$$\int_0^\infty t^{\alpha-1}(K_\nu(t))^2 dt = \frac{\sqrt{\pi}}{4\Gamma((\alpha+1)/2)} \Gamma\Big(\frac{\alpha}{2}\Big)\Gamma\Big(\frac{\alpha}{2}-\nu\Big)\Gamma\Big(\frac{\alpha}{2}+\nu\Big),$$

$$\frac{\sqrt{2}}{4} x\sqrt{x^2 - \sqrt{x^4-16}} = \sum_{k=0}^\infty \binom{4k}{2k} \frac{1}{2k+1} \frac{1}{x^{4k}}, \quad \text{for } x \geq 2,$$

based on remark 4, we obtain the limit of variance of X_t^1, as $t \to \infty$, i.e.,

$$
\begin{aligned}
\lim_{t\to\infty} \mathrm{Var}(X_t^1) &= \frac{1}{2} + \sum_{k=1}^{\infty} \int_0^{\infty} \frac{2\, s^{2k+1}}{\pi (k!)^2 4^k} \cdot [K_{k-(1/2)}(s)]^2 \mathrm{d}s \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi (k!)^2 4^k} \cdot \frac{\pi\, \Gamma(k+1)\, \Gamma(2k+(1/2))}{8\, \Gamma(k+(3/2))} \\
&= \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \binom{4k}{2k} \frac{1}{2k+1} \frac{1}{2^{4k}} = \frac{1}{2} \sum_{k=0}^{\infty} \binom{4k}{2k} \frac{1}{2k+1} \frac{1}{2^{4k}} \\
&= \frac{1}{2} \cdot \frac{\sqrt{2}}{4} 2\sqrt{2^2 - 0} = \frac{1}{\sqrt{2}}.
\end{aligned}
$$

A.5 Proofs of Propositions 6-7 in section 4

From the expression (34) for X_t^1, the auto-covariance $\mathbb{E}[X_s^1 X_t^1]$ and the cross covariance $\mathbb{E}[X_t^1 X_t^{j+1}]$ are

$$
\begin{aligned}
\mathbb{E}[X_s^1 X_t^1] &= \sum_{i=0}^{\infty} \frac{1}{\pi (i!)^2 2^{2i-1}} \int_0^s (t-v)^{i+1/2} (s-v)^{i+1/2} K_{i-1/2}(t-v) K_{i-1/2}(s-v) \mathrm{d}v \\
&= \sum_{i=0}^{\infty} \frac{1}{\pi (i!)^2 2^{2i-1}} \int_0^s ((t-s+v)v)^{i+1/2} K_{i-1/2}(t-s+v) K_{i-1/2}(v) \mathrm{d}v > 0;
\end{aligned} \tag{88}
$$

$$
\begin{aligned}
\mathbb{E}[X_t^1 X_t^{j+1}] &= \sum_{i=j}^{\infty} \int_0^t \frac{1}{\pi i! (i-j)!} \frac{(t-\nu)^{2i-j+1}}{2^{2i-j-1}} K_{i-1/2}(t-\nu) K_{i-j-1/2}(t-\nu) \mathrm{d}\nu \\
&= \sum_{i=0}^{\infty} \frac{1}{\pi (j+i)! j!} \frac{1}{2^{j+2i-1}} \int_0^t s^{j+2i+1} K_{j+i-1/2}(s) K_{i-1/2}(s) \mathrm{d}s \\
&\xrightarrow[t\to\infty]{} \sum_{i=0}^{\infty} \frac{1}{\pi (j+i)! j!} \frac{1}{2^{j+2i-1}} \int_0^{\infty} s^{j+2i+1} K_{j+i-1/2}(s) K_{i-1/2}(s) \mathrm{d}s > 0\,.
\end{aligned} \tag{89}
$$

By the Cauchy-Schwarz inequality, as $t \to \infty$, the asymptotic cross covariance between X_t^1 and X_t^{j+1} is bounded by

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}[X_t^1 X_t^{j+1}] &\le \lim_{t\to\infty} (\mathbb{E}[(X_t^1)^2])^{1/2} \cdot (\mathbb{E}[(X_t^{j+1})^2])^{1/2} \\ &= \lim_{t\to\infty} \mathrm{Var}(X_t^1) \;=\; \frac{1}{\sqrt{2}}\end{aligned} \tag{90}$$

for $j \ge 0$, because $X_\cdot^1$ and $X_\cdot^{j+1}$ have the same distribution.

To compute the asymptotic auto-covariance, fix $s > 0$ and let $t \to \infty$. By the asymptotic expansion of the modified Bessel function $K_\alpha(z)$, $z > 0$, there exists a positive constant $c > 0$ such that for every sufficiently large $t(> s)$

$$\sup_{i\ge 0} \frac{1}{i!\, t^{i+1}} \int_0^s ((t-s+v)v)^{i+1/2} K_{i-1/2}(t-s+v) K_{i-1/2}(v) \mathrm{d}v \le c \cdot e^{-(t-s)} \,.$$

Then combining this estimate with (88), we obtain

$$\mathbb{E}[X_s^1 X_t^1] \le \sum_{i=0}^{\infty} \frac{4ct^{i+1} e^{-(t-s)}}{\pi\, i!\, 4^i} \le \frac{4ct}{\pi} e^{-(t-s)+(t/4)} \xrightarrow[t\to\infty]{} 0.$$

A.6 Proof of Proposition 9

Define $S_t(z) := \sum_{k=0}^{\infty} z^k \phi_t^{(k)}$ for $0 \le z < 1$ and $\phi_t^{(k)} := \phi_t^k$ again to avoid confusion from the power. Then

$$\begin{aligned} u\dot{S}_t(z) &= \sum_{k=0}^{\infty} z^k u \dot{\phi}_t^{(k)} = u^2 (S_t(z))^2 + u(1-u)\psi_t S_t(z) - u(1-z)\epsilon, \\ uS_T(z) &= u(1-z)c \end{aligned} \tag{91}$$

as in appendix A.1. For $z \to 1$, we obtain the ODE: $u\dot{S}_t(1) = u^2(S_t(1))^2 + u(1-u)\psi_t S_t(1)$, $uS_T(1) = 0$, and hence, $S_\cdot(1) \equiv 0$ and conclude the proof.

A.7 Some details on Table 1

It follows from Proposition 11 that

$$\mathrm{Var}(X_t^1) = \mathrm{Var}\Big(\sum_{k=0}^{\infty} \int_0^t \frac{u^k (t-s)^{2k}}{k!} F^{(k)}(-(t-s)^2) \mathrm{d}W_k(s) \Big) \tag{92}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \int_0^t \frac{u^{2k}(t-s)^{4k}}{(k!)^2} |\rho_k(-(t-s)^2)|^2 e^{-2(t-s)} \mathrm{d}s \\
&= \sum_{k=1}^{\infty} \int_0^t \frac{2u^{2k}}{\pi(k!)^2 4^k} \nu^{2k+1} (K_{k-\frac{1}{2}}(\nu))^2 \mathrm{d}\nu + \frac{1-e^{-2t}}{2}
\end{aligned}
$$

for $t \geq 0$. As $t \to \infty$, we obtain

$$
\begin{aligned}
&\lim_{t\to\infty} \mathrm{Var}(X_t^1) \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \int_0^{\infty} \frac{2u^{2k}\, s^{2k+1}}{\pi(k!)^2 4^k} \cdot [K_{k-(1/2)}(s)]^2 \mathrm{d}s \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} u^{2k} \cdot \frac{\Gamma(2k+(1/2))}{4^{k+1}\, k!\, \Gamma(k+(3/2))} = \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \binom{2k}{k} \frac{u^{2k}}{8^k} \\
&= \frac{1}{2} + \frac{1}{2}((1-4\frac{u^2}{8})^{-\frac{1}{2}} - 1) = \frac{1}{2}\Big(1 - \frac{u^2}{2}\Big)^{-\frac{1}{2}} < \infty.
\end{aligned}
$$

A.8 Proof of Proposition 12

Define $S_t^N(z) = \sum_{k=0}^{N-1} z^k \phi_t^{N,k}$, then, by (73),

$$
\dot{S}_t^N(z) = (S_t^N(z))^2 + (1-z^N)\big[\sum_{j=0}^{N-2} z^j \cdot \sum_{k=j+1}^{N-1} \phi_t^{N,k} \phi_t^{N,N+j-k}\big] - (1-z)\epsilon, \quad (93)
$$

for $0 \leq t \leq T$ with $S_T^N(z) = (1-z)c$ for $0 \leq z < 1$. As $z \to 1$, $\dot{S}_t^N(1) = (S_t^N(1))^2$, $S_T^N(1) = 0$, and hence, $S_t^N(1) = \sum_{k=0}^{N-1} \phi_t^{N,k} = 0$.

A.9 Proof of Proposition 14 in section 7

Similar to the proof of lemma proposition 2 in appendix A.1, define $S_t(z) = \sum_{k=0}^{\infty} z^k \ \psi_t^{(k)}$where $0 \leq z < 1$ and $\psi_t^{(k)} := d^k \phi_t^{(k)}$ in (81). The Riccati system for $\psi_\cdot^{(k)}$ functions is now the same as (87). The conclusion follows directly from the Riccati equation.

A.10 Stationary Solution of (81) in section 7

Define $R_t(z) := \sum_{k=0}^{\infty} z^k \ \phi_t^{(k)}$ where $0 \leq z < 1$ and $\phi_t^{(k)} := \phi_t^k$ in (81) to avoid confusion. Without loss of generality, we assume $\epsilon = 1$. Then

$R_T(z) = c(1 - d^{-1}z)$ and for $0 \le t \le T$

$$\dot{R}_t(z) = \sum_{k=0}^{\infty} z^k \dot{\phi}_t^{(k)} = \sum_{k=0}^{\infty} z^k \sum_{j=0}^{k} \phi_t^{(j)} \phi_t^{(k-j)} - 1 + \frac{z}{d} = (R_t(z))^2 - \Big(1 - \frac{z}{d}\Big). \tag{94}$$

Thus, the stationary solution $\phi^{(k)}$ of (81), as $T \to \infty$, is obtained by the Taylor expansion of $\sqrt{1 - (z/d)}$.

A.11 Proof of Proposition 16

(85) follows directly from (84) and proposition 4. Taking the limit $t \to \infty$ in the variance formula

$$\mathrm{Var}(\overline{X}_t^1) \;=\; \sum_{k=1}^{\infty} \int_0^t \frac{2}{\pi} \frac{s^{2k+1}}{(k!)^2\, 4^k} \big(K_{k-(1/2)}(s)\big)^2 \cdot \frac{1}{d^k} \mathrm{d}s + \frac{1 - e^{-2t}}{2},$$

we obtain (86):

$$\begin{aligned}
\lim_{t\to\infty} \mathrm{Var}(\overline{X}_t^1) \;&=\; \frac{1}{2} + \sum_{k=1}^{\infty} \int_0^{\infty} \frac{2\, s^{2k+1}}{\pi (k!)^2 4^k} \cdot [K_{k-(1/2)}(s)]^2 \cdot \frac{1}{d^k} \mathrm{d}s \\
&=\; \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \binom{4k}{2k} \frac{1}{2k+1} \frac{1}{2^{4k} d^k} \;=\; \frac{1}{2} \sum_{k=0}^{\infty} \binom{4k}{2k} \frac{1}{2k+1} \frac{1}{(2d^{1/4})^{4k}} \\
&=\; \frac{1}{2} \cdot \frac{\sqrt{2}}{4}\, 2d^{1/4} \sqrt{4d^{1/2} - \sqrt{16d - 16}} \\
&=\; \frac{\sqrt{2}}{2}\, d^{1/4} \sqrt{\sqrt{d} - \sqrt{d-1}} = \frac{\sqrt{2}}{2} \frac{d^{1/4}}{\sqrt{\sqrt{d} + \sqrt{d-1}}} \\
&=\; \frac{\sqrt{2}}{2} \Big(1 + \sqrt{\frac{d-1}{d}}\Big)^{-1/2} \in \Big(\frac{1}{2}, \frac{\sqrt{2}}{2}\Big].
\end{aligned}$$

The limit is monotone in d with maximum of $1/\sqrt{2}$ at $d = 1$.