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The Inverse Problem Relative to the Shapley Value and the Convexity

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Abstract

In an earlier work, for a given Shapley Value of a cooperative TU game we derived a procedure to find a new game in which the Shapley Value is the same, but it is coalitional rational. Recently, a similar problem has been solved for other two efficient values, the Egalitarian Allocation and the Egalitarian Nonseparable Contribution. In this paper, by using very similar ideas, we considered the problem: for a given Shapley Value of a cooperative TU game, find out a new game in which the Shapley Value is the same, but the game is convex. The procedure for finding such a game has been derived, depending on a parameter, and it was proved that in the new game, which is convex, the Shapley Value is unchanged and coalitional rational. As a corollary, it followed that in this new game the Egalitarian Nonseparable Contribution is unchanged and coalitional rational. Numerical examples illustrate the procedure.

Keywords: Inverse Set, Almost Null Family of Game, Convexity Threshold, Egalitarian Nonseparable Contribution.

1. Introduction

For the natural number $n > 1$, In an earlier work, we introduced and solved the Inverse Problem, relative to the Shapley Value: for a given game (N, v), let $SH(N, v) = L$; find out the set of all games for which the Shapley Value is the same.

This set, called the Inverse Set, is expressed by the formula

$$w = \sum_{|S| \le n-2} c_S W_S + c_N (W_N + \sum_{j \in N} W_{N-\{j\}}) - \sum_{j \in N} L_j W_{N-\{j\}}, \tag{1}$$

where we have the basis $W = \{W_S : S \subseteq N\}$, called a potential basis, and the coefficients are arbitrary constants (see [1] and [2]). Further, we introduced and solved a connected problem: find out, in the Inverse Set a game in which the Shapley Value is coalitional rational, that is belongs to the Core of the game. The main idea was to look for a solution in the so called Family of Almost Null Games, (see [3]). This family has been represented as the subset of the Inverse Set given by formula

$$w = c_N (W_N + \sum_{j \in N} W_{N-\{j\}}) - \sum_{j \in N} L_j W_{N-\{j\}}, \tag{2}$$

depending on only one arbitrary constant. Taking into account the basic vectors occurring in (1) and (2), shown in [3], we have the scalar form of the games in the family of Almost Null Games given by:

$$w_a(N-\{j\}) = (n-1)(a-L_j), \forall j \in N, \quad w_a(N) = \sum_{j \in N} L_j, \tag{3}$$

with $a = c_N$, and null values for any other coalition. These formulas were proved to be good also for other efficient values.

Of course, our problem arises in case that the Shapley Value of the initial game is not coalitional rational, or pehaps its Core is empty.

Now, to get a solution, we imposed to the games (3), in the Almost Null Family, the coalitional rationality conditions:

$$\sum_{h \in N} L_h - L_k \geq w_a(N-\{k\}), \forall k \in N, \tag{4}$$

or from (3):

$$a \leq \frac{1}{n-1} \sum_{k \in N} L_k + \frac{n-2}{n-1} MIN_k L_k = \alpha. \tag{5}$$

Hence, we have an infinite set of solutions, given by (3), where $a \in [0, \alpha]$.

The right end of the interval is the threshold of coalitional rationality.

In [4] and [5], the similar problem has been discussed for other values: the Egalitarian Value and the Egalitarian Nonseparable Contribution.

2. The Shapley Value and the Convexity

The convex games form a class of games with many interesting properties, some of them connected to the Shapley Value. This fact led us to consider in the present paper, the following problem: find out in the Inverse Set relative to the Shapley Value, some convex game. We apply the same ideas, used in the case of coalitional rationality, namely look for a solution in the Family of Almost Null Games of the Inverse Set. First, we discuss a convenient characterization of convex games taken from [6], where many characterizations of convex games may be found. For our problem we shall be using a characterization derived from the following: a game (N, v) is a convex game, if we have

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T), \forall S, T \subseteq N. \tag{6}$$

Recall the definition (2) of games (N, w) from the Family of Almost Null Games, which have the worth of characteristic function different of zero only for coalitions of sizes n and $n-1$. We would modify the above definition (6) for the case of such games. Note also that in the following we consider only games with nonnegative values of the characteristic function, and the fact that our definition (6) can be written in different ways, as shown by the discussion that follows now. If $S = N,$ then $S \cup T = N,$ and $S \cap T = T,$ so that (6) holds, hence both S and T should have the cardinality at most $n-1,$ in order to get some condition. If we have $\left|S\right| \le n-2,$ then $\left|S \cap T\right| \le n-2,$ so that (6) becomes $w(T) \le w(S \cup T),$ and this holds for $\left|T\right| \le n-2.$ It is clear that we have some condition only if at least one of the coalitions has the cardinality $n-1.$ Suppose that we have $S = N - \{i\}.$ Then, (6) becomes

$$w(N-\{i\}) + w(T) \le w[(N-\{i\}) \cup T] + w[(N-\{i\}) \cap T] \tag{7}$$

and if we have $T = N,$ or $T = N - \{i\},$ this inequality (7) is trivially holding.

If $i \in T,$ then the second term in (7) is null, and it remains to be seen

what happens if $T = N - \{j\},\ j \ne i.$ Of course, in this case (7) becomes

$$w(N-\{i\}) + w(N-\{j\}) \le w(N), \forall i, j \in N, i \ne j. \tag{8}$$

because the second term in (7) is null. Hence, we have the result:

Theorem 1: A cooperative game (N, w) from the Family of Almost Null Games of the Inverse Set, relative to the Shapley Value, is a convex game, if and only if the inequality (8) holds.

Suppose that we apply (8) to the games of the Family of Almost Null Games, given by the scalar form (3). We obtain:

$$(n-1)(2a - L_i - L_j) \le \sum_{k \in N} L_k, \forall i, j \in N, i \ne j. \tag{9}$$

or the inequality given by the formula:

$$a \le \frac{1}{2(n-1)} \sum_{h \in N} L_h + \frac{1}{2} MIN_{ij}(L_i + L_j) = \kappa. \tag{10}$$

Here the number κ will be the convexity threshold. Hence, we have a solution set obtained from (3), when a satisfies (10), that is we have $a \in [0, \kappa].$ In this way, we got the central result of the paper:

Theorem 2: A game $(N, v),$ with the Shapley Value $SH(N, v) = L,$ which is not convex, provides a convex game from the Family of Almost Null Games in the Inverse Set relative to the Shapley Value, if we use the formula (3) with $a \in [0, \kappa],$ where the threshold κ is given by (10).

Some numerical illustrations of the Theorems 1 and 2 will follow.

Example 1. Consider the constant sum game

$$v(1) = v(2) = v(3) = 0, v(1,2) = v(1,3) = v(2,3) = v(1,2,3) = 1. \tag{11}$$

Compute $SH(N,v) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}),$ and check that this is not coalitional rational, as expected because the Core of the game is empty. It is easy to check that the game is not convex, that is the convexity condition (6) dues not hold. Compute the threshold $\kappa = \frac{7}{12},$ by formula (10), and use the scalar form of the equations (3), for this maximal value of the parameter, to compute a solution to our problem, a convex game from the Family of Almost Null Games. We obtain the game:

$$w(1) = w(2) = w(3) = 0, w(1,2) = w(1,3) = w(2,3) = \frac{1}{2}, w(1,2,3) = 1. \tag{12}$$

It is easy to compute the Shapley Value, to show that this is unchanged.

Then, we see that (12) is convex, by using the definition (6); check also that the Shapley Value is coalitional rational. A different method could be to compute $\alpha = \frac{2}{3},$ to see that we have $[0, \frac{7}{12}] \subset [0, \frac{2}{3}].$ This illustrates the results stated in theorems 1 and 2. Another case is shown below.

Example 2. Consider the network game discussed in some earlier work (see [3]), devoted to the coalitional rationality property:

$$v(1) = v(2) = v(3) = 0, v(1,2) = 22, v(1,3) = v(2,3) = 18, v(1,2,3) = 25, \tag{13}$$

We computed $SH(N,v) = (9,9,7),$ and the threshold for coalitional rationality $\alpha = 16,$ given by formula (5). The game from the Family of Almost Null Games, obtained by taking for the parameter this last maximal value and using formula (3), has been given by formula:

$$w(1) = w(2) = w(3) = 0, w(1,2) = 18, w(1,3) = w(2,3) = 14, w(1,2,3) = 25. \tag{14}$$

We may check that the game is not convex, by using (8), even though the Shapley Value is coalitional rational in the new game. This is well justified by the fact that $\alpha = 16$ is not smaller than $\kappa = \frac{57}{4}.$ Now, if we take in (3) the maximal value offered by the convexity threshold, then we obtain a solution for our problem:

$$w(1) = w(2) = w(3) = 0, w(1,2) = \frac{29}{2}, w(1,3) = w(2,3) = \frac{21}{2}, w(1,2,3) = 25. \tag{15}$$

Indeed, this game is convex and the Shapley Value is unchanged and coalitional rational.

3. Convexity and Coalitional Rationality

Notice that in the two examples shown above, we have in both cases $\alpha \geq \kappa$, In the first example, $\alpha = \frac{2}{3}, \kappa = \frac{7}{12}$, and in the second example $\alpha = 16, \kappa = \frac{57}{4}$, so that we wonder whether, or not, this is the case in general. We can prove the following result:

Theorem 3. In the Family of Almost Null Games of the Inverse Set, relative to the Shapley Value, there is an infinite set of convex games in which the Shapley Value is unchanged and also coalitional rational, that is the thresholds are related by the inequality $\alpha \geq \kappa$.

Proof: From (5) and (10), we have:

$$\alpha - \kappa = \frac{1}{2(n-1)} w(N) + \frac{n-2}{n-1} MIN_k L_k - \frac{1}{2} MIN_{ij} (L_i + L_j). \tag{16}$$

Denote by L^* the smallest component of the Shapley Value and by L^{**} the second smallest component. Then, (16) becomes the equality:

$$\alpha - \kappa = \frac{1}{2(n-1)} w(N) + \frac{n-2}{n-1} L^* - \frac{1}{2}(L^* + L^{**}). \tag{17}$$

This may be rewritten as:

$$\alpha - \kappa = \frac{1}{2(n-1)} [w(N) - L^* - (n-1)L^{**}] + [\frac{1}{2(n-1)} + \frac{n-2}{n-1} - \frac{1}{2}] L^*. \tag{18}$$

The first bracket is nonnegative and the second, after operations, will give us from (16) the inequality:

$$\alpha - \kappa = \frac{1}{2(n-1)} [w(N) - L^* - (n-1)L^{**}] + \frac{n-2}{2(n-1)} L^* \geq 0. \tag{19}$$

The theorem has been proved.

A few new remarks may lead to an interesting result. In [4] and [5], we considered other two efficient values, the Egalitarian Allocation and the Egalitarian Nonseparable Contribution, which are equal to the initial one, like the Shapley Value, for the games found in the Family of Almost Null Games. However, their coalitional rationality thresholds denoted by β and γ, respectively, have been different.

Recall that in the same papers it has been proved that they were related by the inequalities $\gamma \geq \alpha \geq \beta.$ So, a good research topic will be to see how will be situated these coalitional rationality thresholds relative to the convexity threshold $\kappa,$ introduced in this paper.

Recall that in the case of the second value, that is for the Egalitarian Nonseparable Contribution, we have the thresholds $\gamma = \frac{2}{3},$ in the first example, and $\gamma = \frac{50}{3},$ in the second example, hence in both cases, the coalitional rationality threshold is greater than $\kappa = \frac{7}{12},$ and $\kappa = \frac{57}{4},$ respectively. So, the convexity will imply the coalitional rationality in both cases, for these values. In theorem 3, this was proved to be true, only for the Shapley Value and the Egalitarian Nonseparable Contribution, as for the second value, the coalitional rationality threshold is in general greater than, or equal to, the convexity threshold of the Shapley Value. Indeed, this follows from the inequality of the coalitional rationality thresholds mentioned above and our new theorem 3. The situation for the coalitional rationality threshold of the Egalitarian Allocations perhaps is different. This may be a new interesting topic of research.

4. Remarks on Computations

Consider a cooperative TU game (N,v) for which the Shapley Value has been computed, $SH(N,v) = L,$ and we determined by using the definition (6) that the game is not convex. The procedure for finding a new game in which the Shapley Value is unchanged and the new game is convex, has the following steps:

a) Compute the convexity threshold by using formula (10).

b) Choose a value of the parameter $a,$ in the interval $[0,\kappa].$

c) If the chosen value is $a^*,$ then compute the game (N, w^*) in the Family of Almost Null Games given by formulas (3).

Now, to avoid mistakes during the computation procedure just described, compute the Shapley Value, before and after using the procedure, to see whether, or not, this value is unchanged in the new game and check by (6), whether or not the new game is convex.

Let us make a few final remarks and give one numerical illustration.

Example 3. As the characteristic functions for the games (N,w), from the Family of Almost Null Games have non zero values only for the coalitions of sizes $n-1$ and $n,$ perhaps the computation of the Shapley Value is easier done by using our Average per Capita formula (see [7]):

$$SH_i(N,w) = \frac{w_{n-1} - w_{n-1}^i}{n-1} + \frac{w_n}{n}, \forall i \in N, \tag{20}$$

where

$$w_{n-1} = \binom{n}{n-1}^{-1} \sum_{j \in N} w(N - \{j\}), \quad w_{n-1}^i = w(N - \{i\}), \forall i \in N, \quad w_n = w(N). \tag{21}$$

For example, in the case of the game (15), which is a 3-person game:

$$w_2 = \frac{71}{6}, w_2^1 = w_2^2 = \frac{21}{2}, w_2^3 = \frac{29}{2}, w_3 = \frac{25}{3}, \tag{22}$$

so that, by substituting into (20), we obtain $SH(N,w) = (9,9,7)$, the initial value; the convexity of the new game (15) easily follows.

Another remark is that we can compute also a game which is convex, for the Egalitarian Nonseparable Contribution, because its coalitional rationality threshold is higher than the convexity threshold for the Shapley Value. For the $ENSC$ value, suppose that we use a procedure similar to the one summarized above. Initially, for game (15) we have

$$ENSC(N,v) = (\frac{29}{3}, \frac{29}{3}, \frac{17}{3}), \tag{23}$$

and the convexity threshold is $\kappa = \frac{57}{4}$. Now, choose in (3) the parameter $a = \kappa = \frac{57}{4}$. After computation, we obtain the new game:

$$w(1,2) = \frac{29}{2}, w(1,3) = w(2,3) = \frac{21}{2}, w(1,2,3) = 25, \tag{24}$$

and the other values equal zero, that is the same as (15). Now, we can compute our Egalitarian Nonseparable Contribution, which will be the same as initially; check the convexity, and the coalitional rationality, which hold.

Perhaps, the situation is not the same for the Egalitarian Allocations, where further research should be done. Such a new research seems justified by the fact that in the two examples considered above, in both cases, the inequality $\beta \geq \kappa$, with β the coalitional rationality threshold for the Egalitarian Allocations, still holds.

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