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Weak Linear Independence of Vector Spaces

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Abstract

The problem of generation and oneness considered for expressing about an element is very important and has a big effect in mathematics in general and in algebra in special for example in vector spaces, every element from this space is expressed in a unique way as a linear combination of elements of its base.

In this paper, we introduce and study new concepts in vector space over a field, to express every element from this space in a unique way called weak linear combination.

Keywords: Weak linear combination, Weak generation, Weak linear independence, Full linear dependence, Weak base, Independent weak base.

1. Introduction

It is important to appreciate at outset that the idea of a vector space in the algebraic abstraction and generalization of the Cartesian coordinate system introduced into the Euclidean plane, that is, a generalization of analytic geometry. Therefore, a number of interesting papers have been published on the concepts of generating sets and linearly independence.

In 2014, Michal Hrbek [5] introduced the concept of weak independence as a generalization of independence, to modules over associative rings with an identity element, where a subset X of a left R −module M is called weakly independent if for any distinct elements $x_1, x_2, \dots, x_n$ from X such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$, then none of $\alpha_1, \alpha_2, \dots, \alpha_n$ is invertible in R. Equivalent, a subset X of M is weakly independent if $x \notin Span(X \setminus \{x\})$, i.e., x is not in the submodule of M generated by $X \setminus \{x\}$. In addition, he studied a weak base, where a weakly independent generating set X of a module M is called a weak base. He proved that weakly independent generating sets are exactly generating sets minimal with respect to inclusion.

In 2016, Daniel Herden [2] studied another generalization of independence for modules as following, let M be an R −module and N be a submodule of M, a subset X of M is weakly independent over N provided that $x \notin N + Span(X\backslash\{x\})$ for all $x \in X$. Also, a subset X of M is weakly independent if it is weakly independent over the zero submodule.

Weakly based Abelian groups were studied in [6] and [7]. In [6], the authors obtained their full characterization in terms of dimensions of certain residual vector spaces.

It is known that a vector space over a field is a special case of a module over a ring [1]. Thus, if X is a weakly independent subset of a vector space V over afield F, then for any distinct elements $x_1, x_2, \dots, x_n$ from X such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$ all $\alpha_1, \alpha_2, \dots, \alpha_n$ zero in F, i.e., X is independent.

The purpose of this paper is to generalize the concept of linear independence, to vector spaces over a field, where a subset X of a vector space V over a field F is weakly independent if for any distinct elements $x_1, x_2, \dots, x_n$ from X and any elements $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0$, then $\alpha_1, \alpha_2, \dots, \alpha_n$ all zero in F.

In section 2 of this paper, we study the concept of weak generation of vector space V over a field F, we proved that if $\langle X \rangle_W$ is the subspace of V weakly generated by X, then $\langle X \rangle_W \subseteq \langle X \rangle$, and $X \subseteq \langle X \rangle_W$ if and only if $\langle X \rangle = \langle X \rangle_W$. Moreover, if $X \nsubseteq \langle X \rangle_W$, then $\langle X \rangle_W$ is a maximal subspace of $\langle X \rangle$.

In section 3 of this paper, we study the concept of weak linear independence which is considered a generalization of linear independence. We show that if X is a maximal weakly independent subset of V, then X generates V weakly.

In section 4 of this paper, we study the concept of a weak base of vector space and its properties. We show that all weak bases of vector space are equipotent. We proved that a subset X of vector space is weak base if and only if X is a minimal weakly generated set. Also, we proved that every weakly generated subset of vector space contains a weak base of this space. We proved that every weak independent subset of vector space can be extended to a weak base of this space.

In section 5 of this paper, we study the concept of an independent weak base of a proper subspace. We show that if X is an independent weak base of a proper subspace U, then $X \nsubseteq U$. In addition to that, we study the relationship between the bases of $\langle X \rangle$ and the independent weak bases of $\langle X \rangle_W$. Also, we proved many important and interesting properties of an independent weak bases of a proper subspace.

In section 6 of this paper, we show the geometric interpretation of weak liner independence in the vector space $\mathbb{R}^n; n = 1,2,3$ over $\mathbb{R}$ where $\mathbb{R}$ is the field of real numbers.

Throughout this paper, all vector spaces V are left over a field F as in [4], a subset X of a vector space V over F is called a base of V [1], if it is generated of V and linearly independent. If V is a vector space and X is a subset of V, then X is a base of V if and only if X is maximal linearly independent if and only if X is minimal generating of V [1].

2. Weak Generation of Vector Spaces

In this section, we study the concept of weak linear combination which is a special case of linear combinations of elements of a non-empty subset of a vector space over a field and the concept of weak generation of vector spaces. We start with the following definition.

Definition 2.1. Let V be a vector space over a field F and X be a non-empty subset of V. For any distinct elements $v_1, v_2, \dots, v_n$ of X we say that every linear combination has the form $\sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i = 0$ is a *weak linear combination* of elements of X. We say that $v \in V$ is expressed as a weak linear combination of elements of X if there exist distinct elements $u_1, u_2, \dots, u_m$ of X and elements $\beta_1, \beta_2, \dots, \beta_m \in F$ such that $v = \sum_{i=1}^{m} \beta_i u_i$ and $\sum_{i=1}^{m} \beta_i = 0$.

Corollary 2.2. Let V be a vector space over a field F and X be a non-empty subset of V. For any distinct elements $v_1, v_2, \dots, v_n$ of X, then with $\alpha_1 = \alpha_2 = \cdots \alpha_n = 0 \in F$, we notice that $0 = \sum_{i=1}^{n} \alpha_i v_i$ and $\sum_{i=1}^{n} \alpha_i = 0$, i.e., the zero element of V is expressed as a weak linear combination of elements of any non-empty subset of V.

Lemma 2.3. *Let V be a vector space over a field F and X be a non-empty subset of V. Suppose that $\langle X \rangle_W$ is the set of all weak linear combinations of X, then $\langle X \rangle_W$ is a subspace of V.*

proof. Obvious.

According to the last Lemma, we can form the following definition.

Definition 2.4. Let V be a vector space over a field F and X be a non-empty subset of V. We call the subspace $\langle X \rangle_W$ a *weakly generated subspace* by X. If there exists a non-empty subset Z of V such that $V = \langle Z \rangle_W$, then we say that Z *generates* V *weakly*, i.e., any element $v \in V$ is expressed as a weak linear combination of elements of Z.

Example. With $\mathbb{R}$ as the field of real numbers, let $X = \{(1,0), (0,1), (2,3)\}$ be a subset of the vector space $\mathbb{R}^2$ over $\mathbb{R}$. It is easy to show that any $(x, y) \in \mathbb{R}^2$ is expressed as a weak linear combination of X by the form:

$$(x, y) = \frac{x - y}{2}(1{,}0) + \frac{y - 3x}{4}(0{,}1) + \frac{x + y}{4}(2{,}3)$$

Thus, X generates $\mathbb{R}^2$ weakly.

Lemma 2.5. *Let V be a vector space over a field F. The following hold:*

i. $\langle\{v\}\rangle_W = \{0\}$ *for every* $v \in V$.

ii. For any non-empty subset X of V, then $\langle X\rangle_W \neq \{0\}$ if and only if $Card\ X \geq 2$.

iii. For any non-empty subset X of V, then $\langle X\rangle_W \subseteq \langle X\rangle$.

proof. Obvious.

Theorem 2.6. *Let V be a vector space over a field F and X be a non-empty subset of V. Then, the following are equivalent:*

i. $X \subseteq \langle X\rangle_W$.

ii. $\langle X\rangle = \langle X\rangle_W$.

proof. $(i) \Longrightarrow (ii)$. Suppose that $X \subseteq \langle X\rangle_W$. Since $\langle X\rangle$ is the smallest subspace in V containing X, we have $\langle X\rangle \subseteq \langle X\rangle_W$. On the other hand, $\langle X\rangle \subseteq \langle X\rangle_W$ by Lemma 2.5. Thus, $\langle X\rangle = \langle X\rangle_W$.

$(ii) \Longrightarrow (i)$. Suppose that $\langle X\rangle = \langle X\rangle_W$. Since $X \subseteq \langle X\rangle$, then $X \subseteq \langle X\rangle_W$.

According to the last theorem, we can form the following corollary.

Corollary 2.7. Let V be a vector space over a field F and X be a non-empty subset of V. The following hold:

i. If $V = \langle X\rangle_W$, then $V = \langle X\rangle$.

ii. $\langle X\rangle_W$ is the smallest subspace in V containing X if and only if $\langle X\rangle = \langle X\rangle_W$.

iii. $\langle X\rangle_W \subsetneq \langle X\rangle$ if and only if $X \nsubseteq \langle X\rangle_W$.

Example. With $\mathbb{R}$ as the field of real numbers, let $X = \{(1{,}0), (0{,}1)\}$ be a subset of the vector space $\mathbb{R}^2$ over $\mathbb{R}$. It is clear that $\langle X\rangle_W \neq \langle X\rangle$ and $X \nsubseteq \langle X\rangle_W$ where $\langle X\rangle_W = \{(x, -x); x \in \mathbb{R}\}$ and $\langle X\rangle = \mathbb{R}^2$.

Theorem 2.8. *Let V be a vector space over a field F and X be a non-empty subset of V. Then, the following are equivalent:*

i. $X \nsubseteq \langle X\rangle_W$.

ii. There exists an element $v_1 \in X$ such that for any subset $\{v_1, v_2, \dots, v_n\}$ of X and elements $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ for which $v_1 = \sum_{i=1}^{n} \alpha_i v_i$ yields that $\sum_{i=1}^{n} \alpha_i = 1$.

iii. $\langle X\rangle \neq \langle X\rangle_W$.

proof. $(i) \Longrightarrow (ii)$. Suppose that $X \nsubseteq \langle X\rangle_W$, then there exists an element $v_1 \in X$ such that $v_1 \notin \langle X\rangle_W$, i.e., for any subset $\{v_1, v_2, \dots, v_n\}$ of X and elements $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ for which $v_1 = \sum_{i=1}^{n} \alpha_i v_i$

yields that $b = \sum_{i=1}^{n} \alpha_i \neq 0$. We suppose that $b \neq 1$, i.e., $1 - b \neq 0$, then

$$v_1 - bv_1 = \sum_{i=1}^{n} \alpha_i v_i - bv_1 = \sum_{i=1}^{n} \alpha_i c_i$$

where $c_1, c_2, \dots, c_n \in F$, with $c_1 = \alpha_1 - b$ and $c_i = \alpha_i$ for $2 \leq i \leq n$. Therefore:

$$v_1 = \sum_{i=1}^{n} [(1-b)^{-1} c_i] v_i$$

and

$$\sum_{i=1}^{n} (1-b)^{-1} c_i = (1-b)^{-1} \sum_{i=1}^{n} c_i = (1-b)^{-1} [\sum_{i=1}^{n} \alpha_i - b] = 0$$

which means that $v_1 \in \langle X \rangle_W$, a contradiction. Therefore, $b = \sum_{i=1}^{n} \alpha_i = 1$.

$(ii) \Longrightarrow (i)$. Obvious.

$(i) \Leftrightarrow (iii)$. Direct by Theorem 2.6.

According to the last theorem, we can form the following corollary.

Corollary 2.9. Let V be a vector space over a field F and X be a non-empty subset of V. If X is an independent subset of V, then $X \nsubseteq \langle X \rangle_W$ and $\langle X \rangle \neq \langle X \rangle_W$, i.e., $\langle X \rangle_W \subsetneq \langle X \rangle$.

Theorem 2.10. *Let V be a vector space over a field F and X be a non-empty subset of V where $0 \notin X$. Then, the following are equivalent:*

i. $V = \langle X \rangle$.

ii. $V = \langle X \cup \{0\} \rangle_W$.

proof. $(i) \Longrightarrow (ii)$. Suppose that $V = \langle X \rangle$, then for any element $v \in V$ there exist distinct elements $v_1, v_2, \dots, v_n$ of X and elements $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_i v_i$. Suppose that $v_{n+1} = 0$ and $\alpha_{n+1} = -\sum_{i=1}^{n} \alpha_i$, then $v = \sum_{i=1}^{n+1} \alpha_i v_i$ and $\sum_{i=1}^{n+1} \alpha_i = 0$. Therefore, $V = \langle X \cup \{0\} \rangle_W$.

$(ii) \Longrightarrow (i)$. Suppose that $V = \langle X \cup \{0\} \rangle_W$, then $V = \langle X \cup \{0\} \rangle$ by Corollary 2.7. Therefore, $V = \langle X \rangle$.

Corollary 2.11. Let V be a vector space over a field F and X be a non-empty subset of V. If $0 \in X$, then the following are equivalent:

i. $V = \langle X \rangle$.

ii. $V = \langle X \rangle_W$.

Lemma 2.12. *Let V be a vector space over a field F and $X = \{v_j\}_{j \in J}$ be a non-empty subset of V. The following hold:*

i. $\langle X \rangle_W = \langle \{v_j - u\}_{j \in J} \rangle_W$ *for any element* $u \in V$.

ii. If $u \in V$ can be expressed as a linear combination of elements of X by the form $u = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i = 1$, then $\{v_j - u\}_{j \in J} \subseteq \langle X \rangle_W$.

proof.

i. Let $u \in V$ and let $v \in \langle X \rangle_W$, then there exist distinct elements $v_1, v_2, \dots, v_n$ of X and elements $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_i v_i$ and $\sum_{i=1}^{n} \alpha_i = 0$. Then,

$$v = \sum_{i=1}^{n} \alpha_i v_i - 0.u = \sum_{i=1}^{n} \alpha_i (v_i - u) \ .$$

Thus, $\langle X \rangle_W \subseteq \langle \{v_j - u\}_{j \in J} \rangle_W$.

Let $w \in \langle \{v_j - u\}_{j \in J} \rangle_W$ then there exist distinct elements $v_{i1} - u, v_{i2} - u, \dots, v_{im} - u$ of $\{v_j - u\}_{j \in J}$ and elements $\beta_1, \beta_2, \dots, \beta_m \in F$ such that $w = \sum_{j=1}^{m} \beta_j (v_{ij} - u)$ and $\sum_{j=1}^{m} \beta_j = 0$. Then,

$$w = w = \sum_{j=1}^{m} \beta_j (v_{ij} - u) = \sum_{j=1}^{m} \beta_j v_{ij} - (\sum_{j=1}^{m} \beta_j).u = \sum_{j=1}^{m} \beta_j v_{ij}$$

Thus, $\langle \{v_j - u\}_{j \in J} \rangle_W \subseteq \langle X \rangle_W$. Therefore, $\langle X \rangle_W = \langle \{v_j - u\}_{j \in J} \rangle_W$.

ii. Obvious.

Lemma 2.13. *Let V be a vector space over a field F and X, Y are non-empty subsets of V. The following hold:*

i. If $X \subseteq Y$, then $\langle X \rangle_W \subseteq \langle Y \rangle_W$.

ii. If $X \subseteq \langle Y \rangle_W$, then $\langle X \rangle_W \subseteq \langle Y \rangle_W$ and $\langle X \rangle \subseteq \langle Y \rangle_W$.

iii. If $X \subseteq \langle Y \rangle_W$ and $Y \subseteq \langle X \rangle_W$, then $\langle X \rangle_W = \langle Y \rangle_W$ and $\langle X \rangle = \langle Y \rangle$.

proof. Obvious.

Lemma 2.14. *Let V be a vector space over a field F and U be a proper subspace of V. Suppose that $X = \{v_j\}_{j \in J}$ is a subset of V such that $X \nsubseteq U$ and $U = \langle X \rangle_W$. The following hold:*

i. $v_j \notin U$ for any $j \in J$.

ii. If $Y \subseteq X$ such that $U = \langle Y \rangle_W$, then $Y \nsubseteq U$.

iii. U is a maximal subspace of $\langle X \rangle$.

proof.

i. Suppose that there exists $v \in X$ such that $v \in U$. Then, $u_j = v_j - v \in U$ for any $j \in J$ by Lemma 2.12. Thus, $v_j = u_j + v \in U$ for any $j \in J$, a contradiction. Therefore, $v_j \notin U$ for any $j \in J$.

ii. Direct by (*i*).

iii. Suppose that $X \nsubseteq U$ and $U = \langle X \rangle_W$, then $U \subsetneq \langle X \rangle$ by Corollary 2.7, i.e., U is a proper subspace of $\langle X \rangle$. We Suppose that U is not a maximal subspace of $\langle X \rangle$, then there exists a proper subspace W of $\langle X \rangle$ such that $U \subsetneq W \subsetneq \langle X \rangle$. Since W is a proper subspace of $\langle X \rangle$, there exists an element $v \in \langle X \rangle$ such that $v \notin W$. Since $U \subsetneq W$, the element v is expressed as a linear combination of elements of X by the form $v = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $\alpha = \sum_{i=1}^{n} \alpha_i \neq 0$. Then, $\sum_{i=1}^{n} \alpha^{-1} \alpha_i = 1$. On other hand, since $U \subsetneq W$, there exists an element $w \in W$ such that $w \notin U$. Since $W \subsetneq \langle X \rangle$, the element w is expressed as a linear combination of elements of X by the form $w = \sum_{i=1}^{m} \beta_i w_i$ where $\beta_1, \beta_2, \dots, \beta_m \in F$ such that $\beta = \sum_{i=1}^{m} \beta_i \neq 0$. Then, $\sum_{i=1}^{m} \beta^{-1}\beta_i = 1$. Obviously,

$$u = \alpha^{-1}v - \beta^{-1}w = \sum_{i=1}^{n}(\alpha^{-1}\alpha_i)v_i - \sum_{i=1}^{m}(\beta^{-1}\beta_i)w_i \in U \subsetneq W.$$

Then, $v = \alpha u + \alpha\beta^{-1}w \in W$, a contradiction. Therefore, U is a maximal subspace of $\langle X \rangle$.

Lemma 2.15. *Let V be a vector space over a field F and U be a proper subspace of V. Suppose that $X = \{v_j\}_{j\in J}$ is a subset of V such that $X \nsubseteq U$ and $\langle X \rangle_W \subseteq U$. The following hold:*

i. $X \nsubseteq \langle X \rangle_W$.

ii. $v_j \notin U$ *for any* $j \in J$.

proof.

i. Suppose that $X \subseteq \langle X \rangle_W$, then $X \subseteq U$ since $\langle X \rangle_W \subseteq U$, a contradiction. Therefore, $X \nsubseteq \langle X \rangle_W$.

ii. Suppose that there exists $v \in X$ such that $v \in U$. Since $u_j = v_j - v \in \langle X \rangle_W$ for any $j \in J$ by Lemma 2.12 and $\langle X \rangle_W \subseteq U$, $u_j = v_j - v \in U$. Thus, $v_j = u_j + v \in U$ for any $j \in J$, a contradiction. Therefore, $v_j \notin U$ for any $j \in J$.

A minimal weakly generated set of vector space is one of the important subsets of vector space as we will show later, and it is defined as the following.

Definition 2.16. Let V be a vector space over a field F and U be a subspace of V. Suppose that X is a non-empty subset of V. We say that X is a *minimal weakly generated set* of U if it satisfies the following:

i. $U = \langle X \rangle_W$.

ii. No proper subset of X generates U weakly.

More precisely, $U = \langle X \rangle_W$ and $U \neq \langle X \setminus \{v\} \rangle_W$ for all $v \in X$.

3. Weak Linear Independence and Full linear Dependence

In this section, we study a special type of non-empty subsets of a vector space over a field which is considered a generalization of linearly independent subsets. We start with the following definition.

Definition 3.1. Let V be a vector space over a field F and X be a non-empty subset of V. We say that X is *weakly linearly independent* or (*weakly independent* for short) if for any subset $\{v_1, v_2, \dots, v_n\}$ of X and any elements $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i v_i = 0$ and $\sum_{i=1}^{n} \alpha_i = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. If X is not weakly independent, then we say that X is *fully linearly dependent* or (*fully dependent* for short).

Example. With $\mathbb{R}$ as the field of real numbers, let $X = \{(1,0), (0,1), (2,3)\}$ and $Y = \{(1,0), (0,1), (2,-1)\}$ are subsets of the vector space $\mathbb{R}^2$ over $\mathbb{R}$. It is easy to show that X is weakly independent while Y is fully dependent.

Lemma 3.2. *Let V be a vector space over a field F and X be a non-empty subset of V. The following hold:*

i. X is weakly independent if and only if any subset of X is weakly independent.

ii. X is fully dependent if and only if there exists a fully dependent finite subset of X.

iii. If X is independent, then X is weakly independent.

iv. If X is independent, then for any $v \in V$ the set $Y = \{v_j - v\}_{j \in J}$ is weakly independent.

v. If X is fully dependent, then X is dependent.

vi. If X is dependent, then X is either weakly independent or fully dependent.

proof. Obvious.

Let V be a vector space over a field F and X be a non-empty subset of V. It is known that, if $0 \in X$, then X is dependent [1]. The following lemma shows the necessary and sufficient condition for X to be independent.

Lemma 3.3. *Let V be a vector space over a field F and X be a non-empty subset of V, such that $0 \notin X$. Then, the following are equivalent:*

i. X is independent.

ii. $X \cup \{0\}$ is weakly independent.

proof. $(i) \Longrightarrow (ii)$. Suppose that X is independent, then any finite subset Y of X is independent. Moreover, Y is weakly independent by Lemma 3.2. With $v_0 = 0 \in V$, let $\{v_1, v_2, \ldots, v_n\}$ be a subset of X and $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that $\sum_{i=0}^{n} \alpha_i v_i = 0$ and $\sum_{i=0}^{n} \alpha_i = 0$, then $\sum_{i=1}^{n} \alpha_i v_i = 0$ and $\sum_{i=1}^{n} \alpha_i = 0$. Since X is independent and $\alpha_0 = -\sum_{i=1}^{n} \alpha_i$, then $\alpha_0 = \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Therefore, $X \cup \{0\}$ is weakly independent.

$(ii) \Longrightarrow (i)$. Suppose that $X \cup \{0\}$ is weakly independent and let $Y = \{v_1, v_2, \ldots, v_n\}$ be a subset of $X \cup \{0\}$. We recognize two cases:

– $0 \notin Y$, then $Y \subseteq X$, and $Y, Y \cup \{0\}$ are weakly independent by Lemma 3.2. We suppose that Y is dependent, then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ not all zero in F such that $\sum_{i=1}^{n} \alpha_i v_i = 0$. With $v_0 = 0$ and $\alpha_0 = -\sum_{i=1}^{n} \alpha_i$ we find that $\sum_{i=0}^{n} \alpha_i = 0$ and $\sum_{i=0}^{n} \alpha_i v_i = 0$. Since $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n \in F$ not all zero in F, $Y \cup \{0\}$ is fully dependent, a contradiction. Thus, Y is independent.

– $0 \in Y$, then $Y \backslash \{0\} \subseteq X$, and $Y, Y \backslash \{0\}$ are weakly independent by Lemma 3.2. We suppose that $Y \backslash \{0\}$

is dependent, then with $v_1 = 0$ there exist $\alpha_2, \alpha_3, \dots, \alpha_n \in F$ not all zero in F such that $\sum_{i=2}^{n} \alpha_i v_i = 0$. Let $\alpha_1 = -\sum_{i=2}^{n} \alpha_i$, then $\sum_{i=1}^{n} \alpha_i v_i = 0$ and $\sum_{i=1}^{n} \alpha_i = 0$. Since $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ not all zero in F, Y is fully dependent, a contradiction. Thus, $Y \backslash \{0\}$ is independent.

Therefore, every finite subset of X is independent, i.e., X is independent.

According to the last lemma, we can formulate the following corollary.

Corollary 3.4. Let V be a vector space over a field F and X be a non-empty subset of V, such that $0 \notin X$. Then, the following are equivalent:

i. X is dependent.

ii. $X \cup \{0\}$ is fully dependent.

Lemma 3.5. *Let V be a vector space over a field F. The following hold:*

i. Any subset of V consisting of two elements is weakly independent.

ii. $\{0\}$ is weakly independent.

iii. If $v \in V$ is a non-zero element, then $\{0, v\}$ is weakly independent.

proof. Obvious.

Two equivalent conditions for weak independence are presented by the following theorem.

Theorem 3.6. *Let V be a vector space over a field F and X be a non-empty subset of V. Then, the following are equivalent:*

i. X is weakly independent.

ii. Zero of V is expressed in a unique way as a weak linear combination of distinct elements of X.

iii. Every element of $\langle X \rangle_W$ is expressed in a unique way as a weak linear combination of elements of X.

proof. $(i) \Longrightarrow (ii)$. Obvious.

$(ii) \Longrightarrow (iii)$. Let $v \in \langle X \rangle_W$, then there exist distinct elements $v_1, v_2, \dots, v_n$ of X and $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_i v_i$ and $\sum_{i=1}^{n} \alpha_i = 0$. Let $\beta_1, \beta_2, \dots, \beta_n \in F$ such that $v = \sum_{i=1}^{n} \beta_i v_i$ and $\sum_{i=1}^{n} \beta_i = 0$. Then,

$$\sum_{i=1}^{n} \alpha_i v_i - \sum_{i=1}^{n} \beta_i v_i = \sum_{i=1}^{n} (\alpha_i - \beta_i) v_i = 0$$

$$\sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} (\alpha_i - \beta_i) = 0$$

Thus, $\alpha_i - \beta_i = 0$ for every $1 \leq i \leq n$, by assumption. Therefore, $\alpha_i = \beta_i$ for every $1 \leq i \leq n$, i.e., v is expressed in a unique way as a weak linear combination of elements of X.

$(iii) \Longrightarrow (i)$. Let $v_1, v_2, \dots, v_n$ any distinct elements of X and $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_i v_i$ and $\sum_{i=1}^{n} \alpha_i = 0$. Since $0 \in \langle X \rangle_W$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ by assumption. Thus, X is weakly independent.

Theorem 3.7. *Let V be a vector space over a field F and X be a non-empty subset of V. The following hold:*

i. If X is weakly independent and non-independent, then $X \subseteq \langle X \rangle_W$ and $\langle X \rangle = \langle X \rangle_W$.

ii. If X is dependent such that $X \nsubseteq \langle X \rangle_W$, then X is fully dependent.

proof.

i. Suppose that X is weakly independent and non-independent. We recognize two cases:

– $0 \in X$. Then for every $v \in X \backslash \{0\}$ we have $v = -1.0 + 1.v$, i.e., $v \in \langle X \rangle_W$. Since $0 \in X$, $X \subseteq \langle X \rangle_W$. Thus, $\langle X \rangle = \langle X \rangle_W$ by Theorem 2.6.

– $0 \notin X$. Since X is non-independent, there exists a dependent finite subset of X let it be $Y = \{v_1, v_2, \ldots, v_n\}$, and by Lemma 3.2 Y is weakly independent. Since Y is dependent, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ not all zero in F such that $\sum_{i=1}^{n} \alpha_i v_i = 0$. $\beta = \sum_{i=1}^{n} \alpha_i \neq 0$ since Y is weakly independent. Hence, $\sum_{i=1}^{n} (-\beta)^{-1} \alpha_i = -1$ and $\sum_{i=1}^{n} (\beta)^{-1} \alpha_i v_i = 0$. Moreover, for any $v \in X$, we have $\sum_{i=1}^{n} (-\beta)^{-1} \alpha_i + 1 = 0$ and $v = \sum_{i=1}^{n} (\beta)^{-1} \alpha_i v_i + v$, i.e., $v \in \langle X \rangle_W$, then $X \subseteq \langle X \rangle_W$.Thus, $\langle X \rangle = \langle X \rangle_W$ by Theorem 2.6.

ii. Direct by (*i*).

Theorem 3.8. *Let V be a vector space over a field F and X be a weakly independent and non-independent subset of V. If there exists an element $v \in X$ can be expressed as a linear combination of elements of $X \backslash \{v\}$, then $X \backslash \{v\}$ is independent.*

proof. Suppose that X is weakly independent and non-independent, then $X \subseteq \langle X \rangle_W$ and $\langle X \rangle = \langle X \rangle_W$ by Theorem 3.7. Moreover, there exists an element $v \in X$ can be expressed as a linear combination of elements of $Y = X \backslash \{v\}$, then $v \in \langle Y \rangle$ and $\langle X \rangle = \langle Y \rangle$. On the other hand by Lemma 3.2, Y is weakly independent. We suppose that Y is dependent, then $Y \subseteq \langle Y \rangle_W$ and $\langle Y \rangle = \langle Y \rangle_W$ by Theorem 3.7. Thus, $\langle X \rangle_W = \langle Y \rangle_W$ and $X \subseteq \langle Y \rangle_W$. Let v_1 be another element of X, then $u = v - v_1 \in \langle X \rangle_W$. Since $\langle X \rangle_W = \langle Y \rangle_W$, u can be expressed as a weak linear combination of elements of Y, and v not one of these elements. Thus, the element u from $\langle X \rangle_W$ can be expressed as a weak linear combination of elements of X as two different ways, this is contradictory to Theorem 3.6. Therefore, $Y = X \backslash \{v\}$ is independent.

Now, we state the basic properties of the fully dependent set, we start with the following theorem.

Theorem 3.9. *Let V be a vector space over a field F and X be a non-empty subset of V. Then, the following are equivalent:*

i. X *is fully dependent.*

ii. There exists an element $v \in X$ *can be expressed as a linear combination of elements of* $X\backslash\{v\}$ *by the form* $v = \sum_{i=1}^{n} \alpha_i v_i$ *where* $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ *such that* $\sum_{i=1}^{n} \alpha_i = 1$.

proof. $(i) \Longrightarrow (ii)$. Suppose that X is fully dependent, then there exists a fully dependent finite subset of X, let it be $Y = \{v_1, v_2, \dots, v_n\}$. Since Y is fully dependent, there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ not all zero in F such that $\sum_{i=1}^{n} \alpha_i v_i = 0$ and $\sum_{i=1}^{n} \alpha_i = 0$. Suppose that $\alpha_1 \neq 0$, then $v_1 = \sum_{i=2}^{n} (-{\alpha_1}^{-1} \alpha_i)\, v_i$ and $\sum_{i=2}^{n} (-{\alpha_1}^{-1} \alpha_i) = 1$.

$(i) \Longrightarrow (ii)$. Obvious.

According to the last theorem, we can form the following corollary.

Corollary 3.10. Let V be a vector space over a field F and X be a non-empty subset of V. Then, the following are equivalent:

i. X is weakly independent and non-independent.

ii. For each element v from X that is expressed as a linear combination of elements of $X\backslash\{v\}$ by the form $v = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$, then $\sum_{i=1}^{n} \alpha_i \neq 1$.

Theorem 3.11. *Let* V *be a vector space over a field* F *and* X *be a fully dependent subset of* V. *If* X *generates* V *weakly, then there exists an element* $u \in X$ *such that* $X\backslash\{u\}$ *generates* V *weakly.*

proof. Since X is fully dependent. Then, by Theorem 3.9 there exists an element $u \in X$ can be expressed as a linear combination of elements of $X\backslash\{u\}$ by the form $u = \sum_{i=1}^{m} \beta_i u_i$ where $\beta_1, \beta_2, \dots, \beta_m \in F$ such that $\sum_{i=1}^{m} \beta_i = 1$. Suppose that $V = \langle X \rangle_W$, then for every $v \in V$ there exist a subset $Y = \{v_1, v_2, \dots, v_n\}$ of X and $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_i v_i$ and $\sum_{i=1}^{n} \alpha_i = 0$. We recognize two cases:

– $u \notin Y$. Then, $v \in \langle X\backslash\{u\} \rangle_W$.

– $u \in Y$. Suppose that $u = v_1$. Then, $v = \alpha_1 \sum_{i=1}^{m} \beta_i u_i + \sum_{i=2}^{n} \alpha_i v_i$. Since $\sum_{i=1}^{m} \beta_i = 1$ and $\sum_{i=1}^{n} \alpha_i = 0$, $\alpha_1 \sum_{i=1}^{m} \beta_i + \sum_{i=2}^{n} \alpha_i = 0$.Thus, $v \in \langle X\backslash\{u\} \rangle_W$.

Therefore, $V = \langle X\backslash\{u\} \rangle_W$.

We state a special type of weakly independent sets and its properties, we start with the following definition.

Definition 3.12. Let V be a vector space over a field F and X be a weakly independent subset of V. We say that X is *maximal weakly linearly independent* or (*maximal weakly independent* for short) if $X \cup \{v\}$ is fully dependent for any $v \in V\backslash X$.

Lemma 3.13. *Let* V *be a vector space over a field* F *and* X *be a maximal weakly independent subset of* V. *The following hold:*

i. X *is dependent.*

ii. $X \subseteq \langle X \rangle_W$ *and* $\langle X \rangle = \langle X \rangle_W$.

proof.

i. Suppose that X is maximal weakly independent. We suppose that X is independent, then $0 \notin X$. Thus, $X \cup \{0\}$ is weakly independent by Lemma 3.3, a contradiction. Therefore, X is dependent.

ii. Since X is maximal weakly independent, X is weakly independent and non-independent by (i). Thus, $X \subseteq \langle X \rangle_W$ and $\langle X \rangle = \langle X \rangle_W$ by Theorem 3.7.

Let V be a vector space over a field F and X be a non-empty subset of V. It is known that, if X is maximal independent, then $X \cup \{v\}$ is dependent for any $v \in V \backslash X$ [1]. The following lemma shows the necessary and sufficient condition for X to be maximal independent.

Lemma 3.14. *Let* V *be a vector space over a field* F *and* X *be a non-empty subset of* V, *such that* $0 \notin X$. *Then, the following are equivalent:*

i. X *is maximal independent.*

ii. $X \cup \{0\}$ *is maximal weakly independent.*

proof. $(i) \Longrightarrow (ii)$. Suppose that X is maximal independent, then X is independent. Thus, $X \cup \{0\}$ is weakly independent by Lemma 3.3. On the other hand, since X is maximal independent, $X \cup \{v\}$ is dependent for any $v \in V \backslash (X \cup \{0\})$. Thus, $X \cup \{v, 0\}$ is fully dependent by Corollary 3.4. Therefore, $X \cup \{0\}$ is maximal weakly independent.

$(ii) \Longrightarrow (i)$. Suppose that $X \cup \{0\}$ is maximal weakly independent, then $X \cup \{0\}$ is weakly independent. Thus, X is independent by Lemma 3.3. We suppose that X is not maximal independent, then there exists a non-zero element $v \in V \backslash X$ such that $X \cup \{v\}$ is independent, then $X \cup \{v, 0\}$ is weakly independent by Lemma 3.3, a contradiction. Thus, X is maximal independent.

Theorem 3.15. *Let* V *be a vector space over a field* F *and* X *be a maximal weakly independent subset of* V. *The following hold:*

i. There exists an element $v \in X$ *such that* $X \backslash \{v\}$ *is maximal independent.*

ii. $V = \langle X \rangle$ *and* $V = \langle X \rangle_W$.

proof.

i. Suppose that X is maximal weakly independent, then X is weakly independent. Moreover, X is dependent by Lemma 3.13. Then, there exists an element $u_0 \in X$ can be expressed as a linear combination of elements of $Y = X\backslash\{u_0\}$, i.e., there exist distinct $v_1, v_2, \ldots, v_n$ of Y and $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that $u_0 = \sum_{i=1}^{n} \alpha_i v_i$. In addition, Y is independent by Theorem 3.8. We suppose that Y is not maximal independent, then there exists an element $u_1 \in V\backslash X$ such that $Y \cup \{u_1\}$ is independent. On other hand, since X is maximal weakly independent, $X \cup \{u_1\}$ is fully dependent, then there exists a fully dependent finite subset of $X \cup \{u_1\}$ by Lemma 3.2, let it be Z. It is clear that $u_0, u_1 \in Z$, because if $u_1 \notin Z$, then Z will be a subset of X and it will be weakly independent by Lemma 3.2, and if $u_0 \notin Z$, then Z will be a subset of $Y \cup \{u_1\}$ and it will be independent. Suppose that $Z = \{u_0, u_1, u_2, \ldots, u_m\}$. Since Z is fully dependent, there exist $\beta_0, \beta_1, \beta_2, \ldots, \beta_m \in F$ not all zero in F such that $\sum_{i=0}^{m} \beta_i u_i = 0$ and $\sum_{i=0}^{m} \beta_i = 0$. $\beta_0 \neq 0$, $\beta_1 \neq 0$ since $Y \cup \{u_1\}$ is independent, X is weakly independent and $\beta_0, \beta_1, \beta_2, \ldots, \beta_m \in F$ not all zero in F. Thus, $u_1 = \sum_{i=2}^{m}(-\beta_1^{-1}\beta_i)u_i + \sum_{i=1}^{n}(-\beta_1^{-1}\beta_0\alpha_i)v_i$, i.e., u_1 can be expressed as a linear combination of elements of Y, a contradiction. Therefore, $X\backslash\{u_0\}$ is maximal independent.

ii. Suppose that X is maximal weakly independent, then there exists an element $v \in X$ such that $Y = X\backslash\{v\}$ is maximal independent by (*i*). Since Y is maximal independent, $V = \langle Y \rangle$. On the other hand, $V = \langle X \rangle$ since $Y \subset X$. Thus, $V = \langle X \rangle_W$ by Lemma 3.13.

Theorem 3.16. *Let V be a vector space over a field F and X be a non-empty subset of V. Suppose that the element $u_0 \in \langle X \rangle\backslash X$ can be expressed as a linear combination of elements of X by the form $u_0 = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i \neq 1$. The following hold:*

i. If X is independent, then $X \cup \{u_0\}$ is weakly independent.

ii. If X is maximal independent, then $X \cup \{u_0\}$ is maximal weakly independent.

proof.

i. Suppose that X is independent and the element $u_0 \in \langle X \rangle\backslash X$ can be expressed as a linear combination of elements of X by the form $u_0 = \sum_{i=1}^{n} \alpha_i v_i$, where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i \neq 1$. Since X is independent, this expression is unique. We supposed that $X \cup \{u_0\}$ is not weakly independent, i.e., $X \cup \{u_0\}$ is fully dependent, then there exists a fully dependent finite subset of $X \cup \{u_0\}$ by Lemma 3.2, let it be Y. It is clear that $u_0 \in Y$, because if $u_0 \notin Y$, then Y will be a subset of X and it will be independent. Suppose that $Y = \{u_0, u_1, u_2, \ldots, u_m\}$. Since Y is fully dependent, there exist $\beta_0, \beta_1, \beta_2, \ldots, \beta_m \in F$ not all zero in F such that $\sum_{i=0}^{m} \beta_i u_i = 0$ and $\sum_{i=0}^{m} \beta_i = 0$. $\beta_0 \neq 0$ since X is independent and $\beta_0, \beta_1, \beta_2, \ldots, \beta_m$ not all zero in F. Hence, $u_0 = \sum_{i=1}^{m}[\,(-\beta_0)^{-1}\beta_i\,]u_i$ and $\sum_{i=1}^{m}(-\beta_0)^{-1}\beta_i = 1$, a contradiction. Therefore, $X \cup \{u_0\}$ is weakly independent.

ii. Since X is maximal independent, $V = \langle X \rangle$ and X is independent. Thus, $X \cup \{u_0\}$ is weakly independent by (*i*). We supposed that $X \cup \{u_0\}$ is not maximal weakly independent, then there exists an element v_{n+1} from V such that $v_{n+1} \notin X \cup \{u_0\}$ and $X \cup \{u_0, v_{n+1}\}$ is weakly independent. Since u_0 can be expressed as a linear combination of elements of X, $u_0 = \sum_{i=1}^{n+1} \alpha_i v_i$ where $\alpha_{n+1} = 0$. Thus, u_0 can be expressed as a linear combination of elements of $X \cup \{v_{n+1}\}$, then $X \cup \{v_{n+1}\}$ is independent by Theorem 3.8, a contradiction. Therefore, $X \cup \{u_0\}$ is maximal weakly independent.

Let V be a vector space over a field F and X be a non-empty subset of V. It is known that, X is a base of V if and only if X is maximal dependent [1]. Moreover, all bases of V are equipotent [1]. Accordingly, we can prove the following theorem.

Theorem 3.17. *Let V be a vector space over a field F, then all maximal weakly independent subsets of V are equipotent.*

proof. Suppose that X, Y are maximal weakly independent subsets of V, then there exist elements $v \in X$ and $u \in Y$ such that $X' = X \backslash \{v\}, Y' = Y \backslash \{u\}$ are maximal independent by Theorem 3.15. Thus, X', Y' are bases of V and they are equipotent. Therefore, X, Y are equipotent.

Corollary 3.18. Let V be a vector space over a field F and X be a maximal weakly independent subset of V, then $dim_F V = n$ if and only if $Card\ X = n + 1$.

4. Weak Base and Weak Dimension

In this section, we study the concept of a weak base of a vector space over a field and its basic properties. We start with the following definition.

Definition 4.1. Let V be a vector space over a field F and X be a non-empty subset of V. We say that X is a *weak base* of V if it satisfies the following:

i. $V = \langle X \rangle_W$.

ii. X is weakly independent.

If X is a weak base of V, then we call $Card\ X$ a *weak dimension* of V over F, and we denote it as $w.dim_F V$. We say that V is a *finite weak dimensional vector space* if X is finite.

Example. With $\mathbb{R}$ as the field of real numbers, let $X = \{(1,0), (0,1), (2,3)\}$ be a subset of the vector space $\mathbb{R}^2$ over $\mathbb{R}$. It is easy to show that X is a weak base of $\mathbb{R}^2$. Thus, $w.dim_{\mathbb{R}} \mathbb{R}^2 = 3$.

Example. With $\mathbb{R}$ as the field of real numbers, let $X = \{-1, 1, x, x^2, \ldots, x^n, \ldots\}$ be a subset of the vector space of polynomials $\mathbb{R}[x]$ over $\mathbb{R}$. It is easy to show that X is a weak base of $\mathbb{R}[x]$. Thus, $\mathbb{R}[x]$ is an infinite weak dimensional vector space.

Lemma 4.2. *Let V be a vector space over a field F. The following hold:*

i. $\{0\}$ *is a weak base of the zero subspace and* $w.dim_F\{0\} = 1$.

ii. If X *is a weak base of* V, *then* X *is dependent and* $V = \langle X \rangle$.

iii. If $V \neq \{0\}$ *and* X *is a weak base of* V, *then* $V = \langle X \rangle$ *and* $w.dim_F V \geq 2$.

proof. By Lemma 2.5, Corollary 2.9 and Lemma 3.5.

Now, we state the basic properties of the weak base, we start with the following theorem.

Theorem 4.3. *Let V be a vector space over a field F and X be a non-empty subset of V. Then, the following are equivalent:*

i. X *is a weak base of* V.

ii. X *is a minimal weakly generated set of* V.

iii. X *is maximal weakly independent.*

iv. Every element of V *is expressed in a unique way as a weak linear combination of elements of* X.

proof. $(i) \Longrightarrow (ii)$. Suppose that X is a weak base of V, then X is weakly independent. We suppose that X is not a minimal weakly generated set of V, then there exists $v_0 \in X$ such that $V = \langle X \backslash \{v_0\} \rangle_W$. And since $V = \langle X \backslash \{v_0\} \rangle$ by Theorem 2.6, then v_0 can be expressed as a linear combination of elements of $X \backslash \{v_0\}$. Thus, $X \backslash \{v_0\}$ is independent by Theorem 3.8. Moreover, $X \backslash \{v_0\} \nsubseteq V$ by Corollary 2.9, a contradiction. Therefore, X is a minimal weakly generated set of V.

$(ii) \Longrightarrow (iii)$. Suppose that X is a minimal weakly generated set of V. Since $X \subseteq V$ and $V = \langle X \rangle_W$, X is dependent by Corollary 2.9. Moreover, $V = \langle X \rangle$ by Theorem 2.6. We suppose that X is fully dependent, then there exists an element $u \in X$ such that $X \backslash \{u\}$ generates V weakly by Theorem 3.11, a contradiction. Thus, X is weakly independent. We suppose that X is not maximal weakly independent, then there exists an element $v \in V \backslash X$ such that $X \cup \{v\}$ weakly independent. Since $v \in \langle X \rangle$, v can be expressed as a linear combination of elements of X. Thus, X is independent by Theorem 3.8, a contradiction. Therefore, X is maximal weakly independent.

$(iii) \Longrightarrow (iv)$. By Theorem 3.15 and Theorem 3.6.

$(iv) \Longrightarrow (i)$. It is clear that $V = \langle X \rangle_W$ by assumption. Moreover, X is weakly independent by Theorem 3.6. Thus, X is a weak base of V.

It is known that every independent subset of vector space V over a field F can be expanded to a base of V [1]. Moreover, every vector space over afield has a base [1]. Now, we will show that every vector space over afield has a weak base. We start with the following theorem, which shows that every weakly independent subset of V can be expanded to a weak base of V.

Theorem 4.4. *Let V be a non-zero vector space over a field F. The following hold:*

i. If X is a base of V and $v \in V \setminus X$ such that v can be expressed as a linear combination of elements of X by the form $v = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i \neq 1$, then $X \cup \{v\}$ is a weak base of V, i.e., every base of V can be expanded to a weak base of V.

ii. If X is a weak base of V, then there exists an element $v \in X$ such that $X \setminus \{v\}$ is a base of V.

iii. All weak bases of V are equipotent.

iv. Every weakly independent subset of V can be expanded to a weak base of V.

v. If K is a subspace of V, then any weak base of K can be expanded to a weak base of V.

proof.

i. Direct by Theorem 3.16 and Theorem 4.3.

ii. Direct by Theorem 3.15 and Theorem 4.3, since every base of V is maximal independent.

iii. Direct by Theorem 3.17 and Theorem 4.3.

iv. Let X be a weakly independent subset of V, then we recognize two cases:

– X is independent, then X can be expanded to a base of V, let it be Y. On other hand, since Y is a base of V, it can be expanded to a weak base of V by (*i*), i.e., X can be expanded to a weak base of V.

– X is weakly independent and non-independent, then there exists an element $v \in X$ can be expressed as a linear combination of elements of $X \setminus \{v\}$. Thus, there exist distinct elements $v_1, v_2, \dots, v_n$ of $X \setminus \{v\}$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_i v_i$. Then, $\sum_{i=1}^{n} \alpha_i \neq 1$ by Corollary 3.10. Moreover, $X \setminus \{v\}$ is independent by Theorem 3.8. Thus, $X \setminus \{v\}$ can be expanded to a base of V, let it be Y. Since Y is a base of V and $X \setminus \{v\} \subseteq Y$, then Y is maximal independent and $v \notin Y$. Thus, $Y \cup \{v\}$ is maximal weakly independent by Theorem 3.16. Therefore, $Y \cup \{v\}$ is a weak base of V by Theorem 4.3 and $X \subseteq Y \cup \{v\}$, i.e., X can be expanded to a weak base of V.

v. Direct by (*iv*) since any weak base of K is a weakly independent subset of V.

According to the last theorem, we can formulate the following corollary:

Corollary 4.5. Every vector space over a field has a weak base. Moreover, If V is vector space over a field F, then $dim_F V = n$ if and only if $w.dim_F V = n + 1$.

Theorem 4.6. *Let V be a vector space over a field F and X be a non-empty subset of V such that* $0 \notin X$. *Then, the following are equivalent:*

i. X is a base of V.

ii. $X \cup \{0\}$ is a weak base of V.

proof. Direct by Lemma 3.14 and Theorem 4.3.

Theorem 4.7. *Let V be a vector space over a field F and $X = \{v_j\}_{j\in J}$ be a non-empty subset of V. Then, the following are equivalent:*

i. $V = \langle X\rangle_W$.

ii. X contains a weak base of V.

proof. $(i) \Longrightarrow (ii)$. Suppose that $V = \langle X\rangle_W$. If X is weakly independent, then X is a weak base of V. Suppose that X is not weakly independent, and let $u \in X$, then $X' = \{v_j - u\}_{j\in J}$ generates V weakly by Lemma 2.12. Since $0 \in X'$, $V = \langle X'\backslash\{0\}\rangle$ by Theorem 2.10. Thus, $X'\backslash\{0\}$ contains a base of V, let it be $Y' = \{v_i - u\}_{i\in I}$ where $I \subset J$.

Then, for any element $v \in V$ there exist distinct elements $v_1 - u, v_2 - u, \dots, v_n - u$ of Y' and $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $v = \sum_{i=1}^{n} \alpha_i(v_i - u)$. Since $v = \sum_{i=1}^{n} \alpha_i v_i - (\sum_{i=1}^{n} \alpha_i)u$ and $\alpha + \sum_{i=1}^{n} \alpha_i = 0$ where $\alpha = -\sum_{i=1}^{n} \alpha_i$, then $Y = \{v_i\}_{i\in I} \cup \{u\}$ generates V weakly. Let $Y_1 = \{v_{i1}, v_{i2}, \dots, v_{im}\}$ be a subset of Y, we recognize two cases:

– $u \in Y_1$.

With $u = v_{i1}$, let $\beta_1, \beta_2, \dots, \beta_m \in F$ such that $\sum_{j=1}^{m} \beta_j v_{ij} = 0$ and $\sum_{j=1}^{m} \beta_j = 0$, then $\beta_1 u + \sum_{j=2}^{m} \beta_j v_{ij} = 0$ and $\beta_1 = -\sum_{j=2}^{m} \beta_j$. Hence, $\sum_{j=2}^{m} \beta_j(v_{ij} - u) = 0$. Since Y' is a base of V, $\beta_2 = \beta_3 = \dots = \beta_m = 0$. Then, $\beta_1 = \beta_2 = \dots = \beta_m = 0$.

– $u \notin Y_1$.

Let $\lambda_1, \lambda_2, \dots, \lambda_m \in F$ such that $\sum_{j=1}^{m} \lambda_j v_{ij} = 0$ and $\sum_{j=1}^{m} \lambda_j = 0$, then $0.u + \sum_{j=1}^{m} \lambda_j v_{ij} = 0$, i.e., $\sum_{j=1}^{m} \lambda_j(v_{ij} - u) = 0$. Since Y' is a base of V, $\lambda_1 = \lambda_2 = \dots = \lambda_m = 0$.

Thus, Y is weakly independent. Therefore, Y is a weakly base of V, and $Y \subset X$.

$(ii) \Longrightarrow (i)$. Suppose that Y is a weak base of V such that $Y \subseteq X$. Then, $V \subseteq \langle X\rangle_W$ by Lemma 2.13. Therefore, $V = \langle X\rangle_W$.

Theorem 4.8. *Let V be an n-weak dimensional vector space over a field F. The following hold:*

i. Any subset of V consisting of m elements where $m > n$, is fully dependent.

ii. If $X = \{v_1, v_2, \dots, v_n\}$ is a weakly independent subset of V, then X is a weak base of V.

iii. If $X = \{v_1, v_2, \dots, v_n\}$ is a subset of V such that $V = \langle X\rangle_W$, then X is a weak base of V.

proof.

i. Suppose that Y is a weak base of V, then $Card\, Y = n$ since all weak bases of V are equipotent by Theorem 4.4. Since Y is maximal weakly independent by Theorem 4.3, any maximal weakly independent subset of V consists of n elements by Theorem 3.17. Therefore, any subset of V consisting of m elements where $m > n$ is fully dependent.

ii. Suppose that X is weakly independent. Since $w.dim_F V = n$, then any subset of V consisting of $n+1$ elements is fully dependent by (*i*)**.** Thus, X is maximal weakly independent. Therefore, X is a weak base of V by Theorem 4.3.

iii. Suppose that $V = \langle X \rangle_W$. Then, X contains a weak base of V by Theorem 4.7, and let it be S. Since $S \subseteq X$, then $Card\ S \leq Card\ X$ where both S and X are finite sets. We suppose that $Card\ S \neq Card\ X$, then $Card\ S < n$, i.e., $w.dim_F V < n$, a contradiction. Thus, $Card\ S = Card\ X$. Since $S \subseteq X$, then $S = X$. Therefore, X is a weak base of V.

Theorem 4.9. *Let V be an n-dimensional vector space over a field F and U be a subspace of V. The following hold:*

i. $w.dim_F K \leq n$.

ii. $K = V$ *if and only if* $w.dim_F K = n$.

proof.

i. Suppose that $w.dim_F V = n$, then $dim_F V = n-1$ by Corollary 4.5. Since K is a subspace of V, $dim_F K \leq n-1$. Hence, $1 + dim_F K \leq n$. Thus, $w.dim_F K \leq n$ by Corollary 4.5.

ii. Obvious.

5. Independent Weak Base of a Proper Subspace

Let V be a vector space over a field F and X be an independent subset of V. Then, X is weakly independent by Lemma 3.2. Moreover, $X \nsubseteq \langle X \rangle_W$ by Corollary 2.9. Thus, X is not a weak base of $\langle X \rangle_W$. In this section, we study a weakly generated proper subspace U of V by an independent subset of V. We start with the following definition.

Definition 5.1. Let V be a vector space over a field F and U be a proper subspace of V. We say that the non-empty subset X of V is an *independent weak base* of V if it satisfies the following:

i. $U = \langle X \rangle_W$.

ii. X is independent.

Example. With $\mathbb{R}$ as the field of real numbers, let $\mathbb{R}_2[x]$ is the vector space of polynomials over $\mathbb{R}$ of degrees at most 2 in unknown x, and let $\mathbb{R}_1[x]$ the vector space of polynomials over $\mathbb{R}$ of degrees at most 1 in unknown x. $\mathbb{R}_1[x]$ is a subspace of $\mathbb{R}_2[x]$. The subset $X = \{-1 + x^2, 1 + x^2, x + x^2\}$ of

$\mathbb{R}_2[x]$ is an independent weak base of $\mathbb{R}_1[x]$. Where X is independent, and

$$ax+b=\frac{-a-b}{2}(-1+x^2)+\frac{-a+b}{2}(1+x^2)+a(x+x^2)$$

for every $ax+b\in\mathbb{R}_1[x]$; $a,b\in\mathbb{R}$., i.e., $\mathbb{R}_1[x]=\langle X\rangle_W$.

Example. With $\mathbb{R}$ as the field of real numbers, let $\mathbb{R}[x,y]$ is the vector space of polynomials over $\mathbb{R}$ in unknowns x,y, and let $\mathbb{R}[x]$ is the vector space of polynomials over $\mathbb{R}$ in unknowns x. $\mathbb{R}[x]$ is a subspace of $\mathbb{R}[x,y]$. The subset $X=\{-1+y,1+y,x+y,x^2+y,\dots,x^n+y,\dots\}$ of $\mathbb{R}[x,y]$ is an independent weak base of $\mathbb{R}[x]$. Where X is independent, and

$$p(x)=\sum_{i=0}^{m}\alpha_i x^i=\frac{\alpha_0-c}{2}(1+y)+\frac{-\alpha_0-c}{2}(-1+y)+\sum_{i=1}^{m}\alpha_i x^i$$

for every $p(x)\in\mathbb{R}[x]$; $\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{R}$ and $c=\sum_{i=1}^{m}\alpha_i$, i.e., $\mathbb{R}[x]=\langle X\rangle_W$.

Corollary 5.2. Let V be a vector space over a field F and $v\in V$ be a non-zero element. Then, $\{v\}$ is an independent weak base of the zero subspace.

Now, we state the basic properties of an independent weak base of a proper subspace, with the following theorem.

Theorem 5.3. *Let V be a vector space over a field F and U be a proper subspace of V. Suppose that the subset $X=\{v_j\}_{j\in J}$ of V is an independent weak base of U. The following hold:*

i. $X\nsubseteq U$.

ii. Suppose that $v\in V$ such that v is expressed as a linear combination of elements of X by the form $v=\sum_{i=1}^{n}\alpha_i v_i$ where $\alpha_1,\alpha_2,\dots,\alpha_n\in F$ and $\sum_{i=1}^{n}\alpha_i=1$, then $Y=\{v_j-v\}_{j\in J}$ is a weak base of U.

iii. Every element of U is expressed in a unique way as a weak linear combination of elements of X.

iv. $w.dim_F U=Card\ X$.

v. X is a minimal weakly generated set of U.

vi. All independent weak bases of U are equipotent.

proof.

i. Direct by Corollary 2.9 and Definition 5.1.

ii. $U=\langle Y\rangle_W$ and $Y\subseteq U$ by Lemma 2.12. Moreover, Y is weakly independent by Lemma 3.2. Therefore, Y is a weak base of U.

iii. Since X is independent, X is weakly independent by Lemma 3.2. Therefore, every element $u\in U$ is expressed in a unique way as a weak linear combination of elements of X by Theorem 3.6.

iv. Direct by (*ii*) since $Card\ X=Card\ Y$.

v. $X \nsubseteq U$ by (*i*). Let $v_1, v_2 \in X$, then $v_1, v_2 \notin U$ by Lemma 2.14.Moreover, $u = v_1 - v_2 \in U$ is expressed in a unique way as a weak linear combination of elements of X by(*iii*). We suppose that $U = \langle X \backslash \{v_1\} \rangle_W$, then u can be expressed as a weak linear combination of elements of $X \backslash \{v_1\}$, a contradiction. Therefore, X is a minimal weakly generated set of U.

vi. Direct by (*ii*) and Theorem 4.4.

According to the last theorem, we can form the following corollary.

Corollary 5.4. Let V be a vector space over a field F and X be a weakly independent subset of V. Then $w.dim_F \langle X \rangle_W = Card\, X$, whether X is independent or not.

Corollary 5.5. Let V be a vector space over a field F and U be a proper subspace of V. Suppose that X is a non-empty subset of V such that $X \nsubseteq U$ and $U = \langle X \rangle_W$. If X is not an independent weak base of U, then X is fully dependent by Theorem 3.7.

Theorem 5.6. *Let V be a vector space over a field F and U be a proper subspace of V. Suppose that the subset $X = \{v_j\}_{j \in J}$ of V is a weak base of U. The following hold:*

i. For any $v \in V \backslash U$ the set $Y = \{v_j + v\}_{j \in J}$ is an independent weak base of U.

ii. Every proper subspace of V has an independent weak base contained in V.

proof.

i. Since $X = \{v_j\}_{j \in J}$ is a weak base of U, $U = \langle X \rangle_W$. Let $v \in V \backslash U$ and $u = -v$, then $U = \langle Y \rangle_W$ by Lemma 2.12. We suppose that Y is dependent, i.e., there exists a dependent finite subset of Y let it be $Y' = \{v_{t1} + v, v_{t2} + v, \dots, v_{tr} + v\}$, then there exist $\gamma_1, \gamma_2, \dots, \gamma_r \in F$ not all zero in F such that $\sum_{i=1}^{r} \gamma_i (v_{ti} + v) = 0$. Since X is a weak base of U, and

$$\sum_{i=1}^{r} \gamma_i (v_{ti} + v) = \sum_{i=1}^{r} \gamma_i v_{ti} + \left(\sum_{i=1}^{r} \gamma_i\right) v = 0$$

then $\gamma = \sum_{i=1}^{r} \gamma_i \neq 0$. Thus, $v = \sum_{i=1}^{r} (-\gamma^{-1} \gamma_i) v_{ti} \in \langle X \rangle$. Since $U = \langle X \rangle$ by Lemma 4.2, $v \in U$, a contradiction. Therefore, Y is independent, i.e., Y is an independent weak base of U.

ii. Direct by (*i*) since every vector space has a base by Corollary 4.5.

Theorem 5.7. *Let V be a vector space over a field F and U be a proper subspace of V. Let $X = \{v_j\}_{j \in J}$ be a non-empty subset of V such that $X \nsubseteq U$. If $U = \langle X \rangle_W$, then X contains an independent weak base of U.*

proof. Suppose that $U = \langle X \rangle_W$. If X is independent, then X is an independent weak base of U. Suppose that X is not independent. Since $X \nsubseteq U$ by assumption, $v_j \notin U$ for any $j \in J$ by Lemma 2.14. Let $v \in X$, then $X' = \{v_j - v\}_{j \in J} \subseteq U$ and $U = \langle X' \rangle_W$ by Lemma 2.12. Thus, X' contains a weak base of

U by Theorem 4.7, let it be $Y' = \{v_i - v\}_{i\in I}$. Therefore, $Y = \{v_i\}_{i\in I} \subseteq X$ is an independent weak base of U by Theorem 5.6.

Let V be a vector space over a field F and U be a proper subspace of V. Suppose that X is a non-empty subset of V such that $X \nsubseteq U$. If X contains an independent weak base of U, then it is not necessary that $U = \langle X \rangle_W$. This is shown in the following example.

Example. With $\mathbb{R}$ as the field of real numbers, let $U = \{(x, -x, 0); x \in \mathbb{R}\}$ be a subspace of the vector space $\mathbb{R}^3$ over $\mathbb{R}$. It is easy to show that $X = \{(1,0,0), (0,1,0), (0,0,1)\} \nsubseteq U$ and $Y = \{(1,0,0), (0,1,0)\} \subset X$ is an independent weak base of U, but $U \neq \langle X \rangle_W$.

Let V be a vector space over a field F and U be a proper subspace of V. Suppose that the non-empty subset X of V contains an independent weak base of U. The following theorem shows the necessary and sufficient condition for X to generate U weakly.

Theorem 5.8. *Let V be a vector space over a field F and U be a proper subspace of V. Suppose that the subset $Y = \{v_i\}_{i\in I}$ of V is an independent weak base of U and $v \in V$ can be expressed as a linear combination of elements of Y by the form $v = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i =$ 1. Let $X = \{v_j\}_{j\in J}$ be a non-empty subset of U such that $X \nsubseteq U$ and $Y \subseteq X$. Then, the following are equivalent:*

i. $U = \langle X \rangle_W$.

ii. $X_1 = \{v_j - v\}_{j\in J} \subseteq U$.

proof. $(i) \Longrightarrow (ii)$. Direct by Lemma 2.12.

$(ii) \Longrightarrow (i)$. Since $Y = \{v_i\}_{i\in I}$ is an independent weak base of U, $Y_1 = \{v_i - v\}_{i\in I}$ is a weak base of U by Theorem 5.3. Since $Y_1 \subseteq X_1$, $U = \langle X_1 \rangle_W$ by theorem 4.7. For $u = -v$, then $U = \langle \{v_j - v - (-v)\}_{j\in J} \rangle_W$ by Lemma 2.12. Therefore, $U = \langle X \rangle_W$.

Theorem 5.9. *Let V be a vector space over a field F and U be an n-weak dimensional proper subspace of V. The following hold:*

i. Any subset Y of V consisting of m elements where $m > n$ such that $Y \nsubseteq U$ and $U = \langle Y \rangle_W$ is fully dependent.

ii. Let $X = \{v_1, v_2, \dots, v_n\}$ be a subset of V such that $X \nsubseteq U$ and $U = \langle X \rangle_W$, then X is an independent weak base of U.

proof.

i. Let $Y = \{u_1, u_2, \dots, u_m\}$ where $m > n$ be a subset of V such that $Y \nsubseteq U$ and $U = \langle Y \rangle_W$, then $Y_1 = \{0, u_2 - u_1, \dots, u_m - u_1\} \subseteq U$ by Lemma 2.12. Since $m > n$, then Y_1 is fully dependent by Theorem 4.8. Thus, there exist elements $\alpha_1, \alpha_2, \dots, \alpha_m$ not all zero in F such that $\sum_{i=1}^{m} \alpha_i (u_i - u_1) = 0$ and $\sum_{i=1}^{m} \alpha_i = 0$. Moreover, we have $\sum_{i=1}^{m} \alpha_i (u_i - u_1) = \sum_{i=1}^{m} \alpha_i u_i - (\sum_{i=1}^{m} \alpha_i) u_1 = \sum_{i=1}^{m} \alpha_i u_i = 0$. Then, $\sum_{i=1}^{m} \alpha_i u_i = 0$ and $\sum_{i=1}^{m} \alpha_i = 0$. Since the elements $\alpha_1, \alpha_2, \dots, \alpha_m$ not all zero in F, Y is fully dependent.

ii. Suppose that $X \nsubseteq U$ and $U = \langle X \rangle_W$, then X contains an independent weak base of U by Theorem 5.7, let it be S. Since, $S \subseteq X$, $Card\, S \leq Card\, X$. We suppose that $Card\, S \neq Card\, X$. Since S is an independent weak base of U, $w.dim_F U = Card\, S$ by Theorem 5.3. Thus, $w.dim_F U < n$, a contradiction. Thus, $Card\, S = Card\, X$. Since $S \subseteq X$ and X is finite, $S = X$. Therefore, X is an independent weak base of U.

Let V be a vector space over a field F, and X be a subset of V such that $X \nsubseteq \langle X \rangle_W$. If the subset Y of V is a base of $\langle X \rangle$, then it is not necessarily that Y is an independent weak base of $\langle X \rangle_W$. Moreover, if the subset Z of V is an independent weak base of $\langle X \rangle_W$, then it is not necessarily that Z is a base of $\langle X \rangle$. This is shown in the following example.

Example. With $\mathbb{R}$ as the field of real numbers, let $X = \{(1,0,0), (0,1,0)\}$ be a subset of the vector space $\mathbb{R}^3$ over $\mathbb{R}$. Suppose that $V = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $U = \{(x, -x, 0) : x \in \mathbb{R}\}$.
It is clear that $Y = \{(0,0,1), (1,-1,1)\}$ is an independent weak base of U, but it is not a base of V. Moreover, $Y_1 = \{(1,1,0), (0,1,0)\}$ is a base of V, but it is not an independent weak base of U.

Let V be a vector space over a field F, and X be a subset of V such that $X \nsubseteq \langle X \rangle_W$. Lemma 2.14 shows that $\langle X \rangle_W$ is a maximal subspace of $\langle X \rangle$. Now, we study the relationship between the bases of $\langle X \rangle$ and the independent weak bases of $\langle X \rangle_W$. We start with the following theorem.

Theorem 5.10. *Let V be a vector space over a field F and U be a proper subspace of V. Then, the following are equivalent:*

i. U is a maximal subspace of V.

ii. If $X \subseteq V$ is an independent weak base of U, then X is a base of V.

proof. $(i) \Longrightarrow (ii)$. Suppose that $X \subseteq V$ is an independent weak base of U. Since $X \nsubseteq U$ by Theorem 5.3, $U \subsetneq \langle X \rangle$ by Corollary 2.9 and U is a maximal subspace of $\langle X \rangle$ by Lemma 2.14. Moreover, U is a maximal subspace of V by assumption. Thus, $U \subsetneq \langle X \rangle \subseteq V$, i.e., $\langle X \rangle = V$. Therefore, X is a base of V.
$(ii) \Longrightarrow (i)$. Suppose that $X \subseteq V$ is an independent weak base of U, then X is a base of V by assumption, i.e., $V = \langle X \rangle$. Since $X \nsubseteq U$ by Theorem 5.3, $U \subsetneq \langle X \rangle$ by Corollary 2.9 and U is a

maximal subspace of $V = \langle X \rangle$ by Lemma 2.14.

Theorem 5.11. *Let V be a vector space over a field F and U be a maximal subspace of V. Suppose that the subset $X = \{v_j\}_{j \in J}$ of V is base of V and $v \in V$ can be expressed as a linear combination of elements of X by the form $v = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that $\sum_{i=1}^{n} \alpha_i = 1$. Then, the following are equivalent:*

i. X is an independent weak base of U.

ii. $X_1 = \{v_j - v\}_{j \in J}$ is a weak base of U.

proof. $(i) \Longrightarrow (ii)$. Direct by Theorem 5.3.

$(ii) \Longrightarrow (i)$. Direct by Theorem 5.6 where $v \notin U$.

Let V be a vector space over a field F, and X be a subset of V such that $X \nsubseteq \langle X \rangle_W$. If $V \neq \langle X \rangle$, then $\langle X \rangle_W$ is not a maximal subspace of V by Lemma 2.14. Now, we study the basic property of $\langle X \rangle_W$. We start with the following Lemma.

Lemma 5.12. *Let V be a vector space over a field F and W, U be subspaces of V such that $U \subseteq W$. If every independent weak base of U from V is a base of W, then $W = V$.*

proof. Suppose that $U \subseteq W$ and let $X = \{v_j\}_{j \in J}$ be a base of U, then $Y = X \cup \{0\}$ is a weak base of U by Theorem 4.6. Moreover, for every $v \in V \backslash U$, then $Z = \{v_j + v\}_{j \in J} \cup \{v\}$ is an independent weak base of U by Theorem 5.6. Since Z is a base of W by assumption, then $v \in W$, i.e., $V \subseteq W$. Since W is a subspace of V, then $W = V$.

Theorem 5.13. *Let V be a vector space over a field F and U be proper subspace of V. Suppose that $\mathfrak{I} = \{Y_i\}_{i \in I}$ is the family of all subsets of V such that $Y_i \nsubseteq U$ and $U = \langle Y_i \rangle_W$ for all $i \in I$. Then, the following are equivalent:*

i. $U = \bigcap_{i \in I} \langle Y_i \rangle$.

ii. U is not a maximal subspace of V.

proof. $(i) \Longrightarrow (ii)$. Let $Y_i \in \mathfrak{I}$. Since $Y_i \nsubseteq U$ and $U = \langle Y_i \rangle_W$, Y_i contains an independent weak base of U by Theorem 5.7, let it be X_i. We suppose that U is a maximal subspace of V, then X_i is a base of V by Theorem 5.10, i.e., $V = \langle X_i \rangle$. Thus, $V = \langle Y_i \rangle$ for all $i \in I$, then $V = \bigcap_{i \in I} \langle Y_i \rangle$, i.e., $V = U$, a contradiction. Therefore, U is not a maximal subspace of V.

$(ii) \Longrightarrow (i)$. Since $Y_i \nsubseteq U$ and $U = \langle Y_i \rangle_W$ for every $Y_i \in \mathfrak{I}$, then $U \subsetneq \langle Y_i \rangle$ by Corollary 2.7. Thus, $U \subseteq \bigcap_{i \in I} \langle Y_i \rangle$. Let $X \subseteq V$ is an independent weak base of U, then $X \in \mathfrak{I}$, and U is a maximal subspace of $\langle X \rangle$ by Lemma 2.14. Since U is not a maximal subspace of V, then X is not a base of V by Theorem 5.10, i.e., $V \neq \langle X \rangle$. Moreover, $U \subseteq \bigcap_{i \in I} \langle Y_i \rangle \subseteq \langle X \rangle$. We suppose that $U \neq \bigcap_{i \in I} \langle Y_i \rangle$. Since U is a maximal subspace of $\langle X\rangle$, then $\bigcap_{i\in I}\langle Y_i\rangle = \langle X\rangle$. Thus, every independent weak base of U is a base of $\bigcap_{i\in I}\langle Y_i\rangle$.

Then, $V = \bigcap_{i\in I}\langle Y_i\rangle$ by Lemma 5.12, i.e., $V = \langle Y_i\rangle$ for all $i \in I$. Thus, U is a maximal subspace of V, a contradiction. Therefore, $U = \bigcap_{i\in I}\langle Y_i\rangle$.

Theorem 5.14. *Let V be a vector space over a field F and U be proper subspace of V. Suppose that $X = \{v_j\}_{j\in J}$ is an independent subset of V such that $X \nsubseteq U$. Then, the following are equivalent:*

i. $\langle X\rangle_W \subseteq U$.

ii. X *can be expanded to an independent weak base of* U.

proof. $(i) \Longrightarrow (ii)$. Since X is independent, $X \nsubseteq \langle X\rangle_W$ by Corollary 2.9 and X is an independent weak base of $\langle X\rangle_W$. Then, $X' = \{v_j - v\}_{j\in J}$ where $v \in X$ is a weak base of $\langle X\rangle_W$ by Theorem 5.3. Since $\langle X\rangle_W \subseteq U$, X' can be expanded to a weak base of U by Theorem 4.4, let it be $Y' = \{v_j - v\}_{j\in J} \cup \{u_i\}_{i\in I}$. Since $X \nsubseteq U$, $v \notin U$ by Lemma 2.15. Thus, $Y' = \{v_j\}_{j\in J} \cup \{u_i + v\}_{i\in I}$ is an independent weak base of U by Theorem 5.6. Therefore, X can be expanded to an independent weak base of U.

$(ii) \Longrightarrow (i)$. Obvious.

6. Geometric Interpretation of Weak Linear Independence

With $\mathbb{R}$ as the field of real numbers, it is known that for $n = 1,2$ or 3 the vector space $\mathbb{R}^n$ over $\mathbb{R}$ has a useful geometric interpretation in which a vector is identified with the directed line segment [3–4].

For $n = 1$, the non-zero vector $v = (x)$ is identified with the directed line segment on the real line that has initial point at the origin and its terminal point at x. Any subset $A = \{v_1, v_2\}$ of $\mathbb{R}$ is dependent.

For $n = 2$ or 3, the non-zero vector $v = (x, y)$ or $v = (x, y, z)$ is identified with the directed line segment that has initial point at the origin and its terminal point with rectangular coordinates given by the components of the vector. The subset $A = \{v_1, v_2\}$ of $\mathbb{R}^2$ or $\mathbb{R}^3$ is independent if and only if v_1 and v_2 are not collinear.

In making identifications of vectors with direct line segment, we shall follow the convention that any line segment with the same direction and the same length as the one we have described may be used to represent the same vector. If two vectors represent the same vector, then they are said to be equivalent.

Thus, in $\mathbb{R}$ for any points A, B, C, D of the real line, if the vectors $\overrightarrow{AB}, \overrightarrow{CD}$ are not equivalent, then they are dependent . In $\mathbb{R}^2$ or $\mathbb{R}^3$ for any points A, B, C, D of the coordinate plane or coordinate space, the vectors $\overrightarrow{AB}, \overrightarrow{CD}$ are independent if and only if $\overrightarrow{AB} \nparallel \overrightarrow{CD}$, dependent if and only if $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and equivalent if and only if $\overrightarrow{AB} \equiv \overrightarrow{CD}$.

By Lemma 3.2 and Lemma 3.5, if the subset $A = \{v_1, v_2\}$ of $\mathbb{R}^n$; $n = 1,2,3$ is dependent, then A is weakly independent. Moreover, if $A = \{v_1, v_2\}$ is independent, then A is weakly independent by Lemma 3.2.

According to the above, we can formulate the following corollaries, which show us the geometric interpretation of weak linear independence in the vector space vector space $\mathbb{R}^n; n = 1,2$ or 3.

Corollary 6.1. Let A, B, C, D any points of the real line. The following hold:

i. The vectors $\overrightarrow{AB}, \overrightarrow{CD}$ are weakly independent if and only if they are not equivalent ($\overrightarrow{AB} \not\equiv \overrightarrow{CD}$).

ii. The vectors $\overrightarrow{AB}, \overrightarrow{CD}$ are fully dependent if and only if they are equivalent ($\overrightarrow{AB} \equiv \overrightarrow{CD}$).

Corollary 6.2. Let A, B, C, D any points of the coordinate plane or coordinate space. The following hold:

i. The vectors $\overrightarrow{AB}, \overrightarrow{CD}$ are weakly independent if and only if they are not equivalent ($\overrightarrow{AB} \not\equiv \overrightarrow{CD}$).

ii. If $\overrightarrow{AB} \not\equiv \overrightarrow{CD}$, then the vectors $\overrightarrow{AB}, \overrightarrow{CD}$ are weakly independent and non-independent if and only if $\overrightarrow{AB} \parallel \overrightarrow{CD}$.

iii. The vectors $\overrightarrow{AB}, \overrightarrow{CD}$ are fully dependent if and only if they are equivalent ($\overrightarrow{AB} \equiv \overrightarrow{CD}$).

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