"Science Stays True Here"
Journal of Mathematics and Statistical Science (ISSN 2411-2518, USA), Vol.6, issues 5, 182-186 | Science Signpost Publishing

Analytical Study of Hilbert Space and Algebra of Operators

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Abstract

This present paper deals with the study of Hilbert Space and Algebra of Operators. Here, we consider R as additive group of reals with discrete topology and several ways of constructing C* - algebras Canonically associated with R and π, The Universal representation of R on Hilbert Space H, it is proved in this paper that all C*- algebras homomorphism and representation will be * - preserving

Keywords: Hilbert space, Tensor product, C* - tensor norms, C* - algebras, Normal and Binormal norms, W*- algebras.

Introduction

E.G. Effros (1) and Kothe (3,4) are the pioneer workers of the present area. In fact, the present work is the extension of work done by Halub, J.R. (2), Kumar et al. (5), Kumar et al. (6), Kumar et al. (7), Srivastava et al. (8), Srivastava et al. (9), Srivastava et al. (10) and Srivastava et al. (11). In this paper, we have studied on Algebra of operators and Tensor products.

Mathematical Treatment of the Problem

Let, R = additive group of reals with discrete topology. There are several ways of constructing c*-algebras canonically associated with R. For example if π is the universal representation of R on the Hilbert space H, the group c*- algebra c*(R) is the c*- subalgebra of $\mathcal{L}$(H) genrated by the set $\{\pi(g): g \in R\}$. The left regular group c*- algebra $c_\ell^*(R)$ is the c*- subalgebra of $\mathcal{L}(\ell^2(R))$ generated by $\pi\ell(R)$, $\pi\ell$ being the left regular representation of R, when

$$\pi\ell(g)\, \xi(h) = \xi(g^{-1}(h)), (g, h \in R, \xi \in \ell^2(R))$$

The right regular group C*-algebra $C_r^*(R)$ is defined analogously using the right regular representation π_r of R, when

$$\pi_r(g)\,\xi(h) = \xi(hg),\ (g, h \in R,\ \xi \in \ell^2(R))$$

By its definition π contains π_ℓ and π_r, and there are natural homomorphism λ_ℓ and λ_r of c*(R) onto $c_\ell^*(R)$ and $c_r^*(R)$ resp.

(Note also that the representations π_ℓ and π_r are equivalent, so that $c_\ell^*(R)$ and $c_r^*(R)$ are infact isomorphic).

Let A and B be c*-algebras, with algebraic tensor product $A \odot B$. In general there are several distinct (usually incomplete).

c*- norms on $A \odot B$. Two such norms are of particular interest: the maximal norm ν of Guichander and the minimal (or spatial) norm α of Turumaru.

If π_1 and π_2 are representations of A and B, respectively, on the Hilbert space H, $\{\pi_1, \pi_2\}$ is said to be a committing pair of representations of A, B if

$$\pi_1(a)\,\pi_2(b) = \pi_2(b)\,\pi_1(a)\ (a \in A,\ b \in B)$$

The norm ν is defined by

$$\nu\left(\sum_i a_i \otimes b_i\right) = \sup\left(\left\|\sum_i \pi_1(a_i)\,\pi_2(b_i)\right\|\right)$$

the supremum being taken over all commuting pairs of representations of A, B. The norm α is defined as follows:

if $x \in A \odot B$, $\alpha(x)$ is the smallest non-negative real number $\mathcal{R}$ such that

$$< f \otimes g,\ a^*x^*\,x\,a > \ \leq\ \mathcal{R}^2 < f \otimes g,\ a^*a >$$

For all $a \in A \odot B$ and all satisfies f and g of A and B respectively. If for all $B\alpha = \nu$ on $A \odot B$, A is said to be nuclear (the terminology in due to Lance, which an introduction to the theory of c*- tensor product. For a discrete group R, $c_\ell^*(R) = c^*(R)$ iff G is amenable, and this is the case iff $c_\ell^*(G)$ is nuclear.

Let R be a discrete group and let be the representation of $c^*(R) \odot C^*(R)$ on $\ell^2(R)$ given by

$$\lambda\,(\Sigma\, a_i \otimes b_i) = \Sigma\, \lambda_i(a_i)\,\lambda_r(b_i)$$

It is natural to ask relative to which c*- norms η on $c^*(R) \odot c^*(R)$, λ is bounded. If $\eta =$, λ is already bounded with $\eta = \alpha$, the same is true if R is amenable (by the remark above). One of the main results of this section is essentially that $R = F_2$, the free group on two generators, λ is bounded when $\eta = \alpha$. A consequence of this fact is that, even in the separable case, it is not always possible to describe the ideal structure of the spatial tensor product of two c*-algebras in terms of the ideal structures of the individual algebras. It follows moreover that the spatial c*- norm is not in general preserved by quotients, (by quotients respect the maximal norm ν).

Effros and Lance have recently introduced two new c*- tensor norms, the normal and binormal norms. Let A and B be c*-algebras.

If A is a W*- algebra, the left normal norm ν_ℓ on A ⊙ B is defined by

$$\nu_\ell(\Sigma\, a_i \otimes b_i) = \sup\left(\| \Sigma\, \pi_1(a_i)\, \pi_2(b_i) \| \right)$$

the supremum being taken over all commuting pairs of representations $\{\pi_1, \pi_2\}$ of A, B with π_1 normal. If B is a W*-algebra, the right normal norm ν_r is defined analogously with π_2 rather π_1 required to be normal. If A and B are both W*-algebras, the binormal norm β is given by a similar expression, the supremum being taken this time over all commuting pairs of normal representations of A, B. The notation and terminology used here follow those of for the most part.

If A and B are c*- algebras and η is a c*- norm on A⊙B, then η- completion of A ⊙ B will be denoted $A \otimes_\eta B$, the α- completion will be denoted simply by $A \otimes B$. If B is a c*- subalgebra of A, a linear map $\rho : A \to B$ is a refraction if it is a projection of norm 1, i.e. if $\|\rho\| = 1$ and $\rho(x) = x$ for $x \in B$. Finally, all c*- algebras homomorphism and representations will be assumed to be *-preserving.

1. We recall that a W*- algebra M is said to have the extension property (or to be injective(iii)) if it satisfies the following equivalent condition:

   (i) For some faithful normal representation π of M on the Hilbert space H there is retraction

   $$\rho : \mathcal{L}(H) \to \pi(M).$$

   (ii) such a ρ exists for every normal representation π of M.

   (iii) For some faithful normal representation π of M on H there is a retraction,

   $$\rho' : \mathcal{L}(H) \to \pi(M)'.$$

A c*- algebra A is of type E if for every representation π of A, the weak closure π (A) has the extension property (i.e. if A** is injective).

(2) It is not known whether a c*- subalgebra of a c*-algebra of type E is automatically of type E also. For certain types of subalgebra this is the case, which is the Main Result as

Let A be a c*-algebra of type E. If B is a unital c*-subalgebra of A and there is a retraction

$\rho : A \to B$ then B is of type E.

Proof: Let A** non-degenerately as a Von Neumann algebra on the Hilbert space H. The map $\rho^{**}: A^{**} \to B^{**}$ is normal regarding B** as a W*- subalgebra of A** under its canonical embedding, ρ^{**} is a refraction. Let σ be a representation of B on the Hilbert space K, it is sufficient to assume that σ is non-degenerate. There is a central projection,

$F \in B^{**}$ such that $B^{**} F \cong \boldsymbol{\sigma}(B)$

By hypothesis there is a retraction,

$\tau \; : \; \mathcal{L}(H) \; \rightarrow A^{**}$

define W: $\mathcal{L}(FH) \; \rightarrow B^{**}$ by,

$W(T) = (\rho^{**} \odot \tau)\,(TF)\;(T \in \mathcal{L}(FH))$

It is clear that $||W|| \leq 1$ and by the nodule property of retractions,

$W(TF) = W(T)\,F$ for $T \in \mathcal{L}(FH)$

so that

$W(T) \; \in B^{**} F$ for $T \in \mathcal{L}(FH)$

and moreover,

$W(T) = T$ for $T \in B^{**} F$.

Thus, W is a retraction with image B** F. Thus, $\boldsymbol{\sigma}$ (B) has the extension property, so that B is of type E.

Hence the result.

Acknowledgment

The authors are thankful to Prof. (Dr.) S.N. Jha, Ex. Head, Prof. (Dr.) P.K.Sharan, Ex.Head, and Prof.(Dr.) B.P. Singh, present Head of the Deptt. Of Mathematics, B.R.A.B.U. Muzaffarpur, Bihar, India and Prof. (Dr.) T.N. Singh, Ex. Head, Ex. Dean (Science) and Ex. Chairman, Research Development Council, B.R.A.B.U. Muzaffarpur, Bihar, India for extending all facilities in the completion of the present research work.

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