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Elementary Uniform Matrix Symmetrization

R. Purushothaman Nair(rtd.), Advanced Technology Vehicles and Sounding Rockets Project(ATVP), Vikram Sarabhai Space Centre(VSSC), ISRO P.O, Thiruvananthapuram-695022, India,FAX:91-471-2562028, rpnprasadam@gmail.com

ABSTRACT. This paper introduces a symmetrization process for a given matrix $A \in R^{n\times n}$ using elementary column(row) operations. Transformed symmetric matrix $S \in R^{n\times n}$, $S = (s_{ij})$ has a structure $S = (s_{ij}); i, j = 1 : n, s_{ik} = s_{kj} = s_{kk}; k = 1 : n$ for all $i, j > k$. This process is applicable to any matrix $A \in R^{n\times n}$ in a generalized way. Existing equivalence symmetrization of A in the literature is derived from it, providing identical result. Classical Cholesky factorization in the literature is revisited in the context of this symmetrization process. Elementary matrices apply equal scaling quantities with opposite signs in resultant matrices so that column(row) entries are identical with the corresponding diagonal entries. Because of this uniformity in scaling as well as matrix S, it may be called *elementary uniform matrix symmetrization*.

Keywords: matrix symmetrization, equivalence symmetrization, matrix factorization, linear systems

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1. Introduction

Symmetrizing associated asymmetric matrices of linear systems will be convenient as it facilitates application of well developed analytical tools for handling symmetric systems[1]. Symmetrization of matrices has a long history. F. G. Frobenius [5] and J. Marty [7] were the pioneers who worked in the area of symmetrizing matrices. In 1910, Marty introduced symmetrization of linear integral operators. Contemporaneously, Frobenius proved that any matrix $A \in R^{n\times n}$ can be decomposed as product of two symmetric matrices as $A = S_1S_2$, $S_1 = S_1^T, S_2 = S_2^T, S_1, S_2 \in R^{n\times n}$. One of these can be nonsingular so that there exists a nonsingular symmetrizer matrix $S = S^T, S \in R^{n\times n}$ and product SA is symmetric [3].

Linear system solution finds application in many areas of science and technology, economy and sociology. For example, positive linear system solution is associated with network of reservoirs, industrial process involving chemical reactors, heat exchangers, distillation columns, transport and accumulation phenomena of substances in human control systems, signal processing, spectral analysis, storage systems of memory and space, environmental pollution models, social network analysis etc [4]. With pertinent influence among such vast and dynamic areas, linear system solution plays a vital role. However, real world linear systems may appear with asymmetric coefficient matrices in many situations. Adhikari [1] gives references to typical examples such as gyroscopic and circulatory systems, aircraft flutter, ship motion in sea water, actively controlled systems, constrained multi-body systems etc that give way to asymmetric non conservative systems. Symmetrizing associated asymmetric matrices of these linear systems will be convenient as it facilitates extending the application of well developed analytical tools for symmetric systems into such cases.

Frobenious result is known as Tauskky's theorem [3]. Taussky and Zassenhaus [10], and Taussky [9] contributed to strengthen further the fundamentals of symmetrizing a given matrix. Desautels[2] master's thesis discusses basic aspects of symmetrizers of $A \in R^{n\times n}$. Such theoretical works were followed by many attempts to numerically compute symmetrizers of a given matrix in 1960s and early part of 1970s. Those previous attempts based on similarity transformation of matrices were proved to be unstable and abandoned [3]. It is reported in [3] that first successful numerical computation of symmetrizers was in 2013 by Frank Uhlig [11,12]. In Uhlig [11] some of those previous unfruitful attempts are discussed. Thus the history of symmetrizing matrices spread across centuries. This underlines the role of other type of equivalence and congruence transformations in this area. Present work is an attempt in this direction.

Adhikari [1] introduced equivalence symmetrization for asymmetric systems in year 2000. Adhikari [1] gives examples of matrices which cannot be symmetrized by Taussky's conditions [3] but is symmetrized by equivalence symmetrization. The motivation behind [1] is that equivalence transformations are the most general class of nonsingular linear transformations. Much generality can be achieved using equivalence symmetrization compared to similarity transformations. In line with this observation of Adhikari, this work is an attempt in symmetrizing a matrix by elementary row and column operations.

Here for a given nonsingular matrix $A \in R^{n\times n}$ an equivalent symmetric matrix $S \in R^{n\times n}$ is derived by applying elementary lower(upper) unit triangular matrices. The matrix $S = (s_{ij}), i, j = 1 : n$ has a structure, $s_{ik} = s_{kj} = s_{kk}, k = 1 : n$ for all $i, j > k$. Entries of lower and upper triangular blocks are equalized with respective diagonal entries in resultant equivalent matrices. It is proved here that only one of the blocks need be processed to derive S and an equivalent diagonal matrix D of A. Generally equivalence and congruence transformations require both left and right matrix multiplications as in [1]. Hence this feature of the process is an advantage by reducing processing steps and computations by half of the theoretical requirements for completion. Elementary matrices apply equal scaling quantities with opposite signs. Because of this uniformity in scaling as well as in the structure of S, it may be called *elementary uniform matrix symmetrization.* Zero is also a valid scaling quantity of the process and accordingly a pivoting strategy is presented. Equivalence symmetrization in the literature of symmetrizing A into equivalent symmetric matrix $\tilde{A} \in R^{n\times n}$ is presented in a more generalized way.

When entries of D are positive, transforming A into S and its decomposition are shown to be stable. This is a generalized application of Cholesky factors [6] of symmetric positive definite matrices. When given matrix A is singular, these singularities are presented with visual patterns. For singular matrix $A \in R^{n\times n}$, it is shown that a corresponding symmetric matrix $S \in R^{n\times n}$ can be derived. Thus it is generalized that A can be always symmetrized into an equivalent symmetric matrix.

The paper is organized as follows. A briefing on classification of symmetrization [1] is presented. Construction of operator triangular matrices, properties of scaling factors, elementary symmetrization process and matrix S are narrated. When A is singular, how the process treats those singularities are discussed. Finally numerical illustration and conclusions are presented.

2. **Notations and Definitions.**

The upper bidiagonal matrix $B = (b_{ij}); b_{ii} = 1, b_{i-1,i} = -1$; b_{ij}=0 elsewhere, $i, j = 1 : n$ is denoted by bd(1,-1). Its transpose is denoted as bd(-1,1). Its inverse matrix $U(1)$ denotes an upper triangular matrix $U = (u_{ij}); u_{ij} = 1$ for $i \leq j; i, j = 1 : n$. Transpose of $U(1)$ is denoted as $L(1)$. An equivalent matrix of $A \in R^{n\times n}$ is represented as $\widetilde{A}$ and will be given by $\widetilde{A} = M_1^T AM_2$ for nonsingular matrices M_1, M_2 of $R^{n\times n}$. This transformation is called equivalence transformation. When $M_1 = M_2$ it is a congruence transformation. When $M_1^T = M_2^{-1}$ the transformation is called similarity transformation. A square matrix $K \in R^{n\times n}$ is called an elementary matrix if it is obtained by applying exactly one elementary row(column) operation to the identity matrix, I_n. Uniformly symmetric matrix $S \in R^{n\times n}$ is defined as $S = (s_{ij}); i, j = 1 : n, s_{ik} = s_{kj} = s_{kk}, k = 1 : n$ for all $i, j > k$. It is denoted as $uniform(s_1, s_2,...,s_n); s_i = s_{ii}$. First supra diagonal entries of $A = (a_{ij})$ are entries $a_{i-1,i}; i$=2,3,...,n and first infra diagonal entries are $a_{j,j-1}; j$=2,3,...,n. Resultant matrix of step-r is denoted as $A_r = (a_{ij}(r)); r = 1 : n-1, ..., 2n-2; i, j = 1 : n$. Unit lower and upper triangular operator matrices are denoted as $L_r = (l_{ij}(r))$ and $U_r = (u_{ij}(r))$ respectively; r=1:n-1; $i, j = 1 : n$. We call entries of $L_r(U_r)$ as scaling factors derived out of entries of column(row)-r of $A_{r-1}; r = 1 : n - 1$.

3. **Classification of Matrix Symmetrization**

Taussky and Zassenhaus introduced the concept of symmetrizability of an asymmetric matrix[10]. According to that, for every $A \in R^{n\times n}$ there is a nonsingular symmetric matrix $S \in R^{n\times n}$ transforming it into A^T. In her later paper Taussky[9] introduced the necessary and sufficient conditions for the symmetrization of $A \in R^{n\times n}$ as follows: A matrix A is symmetrizable if and only if any one of the following holds:
i) A is the product of two symmetric matrices, one of which is positive definite.
ii) A is similar to a symmetric matrix.
iii)$A^T = S^{-1}AS$ with $S = S^T \succ 0$.
iv) A has real characteristic roots and a full set of characteristic vectors.

Taussky's conditions were derived on the basis of similarity transformation of A and are discussed in[1,3,7]. But in general, symmetrization of A can also be achieved by other type of transformations such as equivalent[1], $(A + A^T)/2, AA^T, A^T A$ etc. Adhikari[1] classified the similarity transformation based symmetrization as first type. He introduced the equivalence transformation based symmetrization and termed it as second type. The second kind of symmetrization is defined as follows:

v) A matrix A is symmetrizable of the second kind if and only if there exist two nonsingular matrices L, R such that $\widetilde{A} = L^T AR$ is symmetric.

This is a more general classification. It includes the first type as a special case, $L^T = R^{-1} = R^{-T}$. This elementary symmetrization also is of the second type. Matrices L, R are nonsingular triangular matrices and are product of elementary matrices.

Later we will prove that this elementary symmetrization is applicable to the whole matrices of $R^{n\times n}$ in a more generalized way. We relax this requirement for the existence of matrices L and R for equivalence symmetrization by Adhikari[1] by deducing it from elementary symmetrization of A.

4. **Construction of Operating Triangular Matrices**

Time being assume $A \in R^{n\times n}$ be nonsingular. At step-r, operating matrix L_r is derived from $A_r = L_r A_{r-1}$ as a linear system solution for applying into the same system to obtain resultant matrix A_r of step-r.

Consider step-1. We have $A_{r-1} = A$. Now to derive A_1, we have to construct L_1. For this we proceed as follows.

$$A = (a_{ij}); i, j = 1 : n. \tag{4.1}$$

$$m_1 = a_{k1}; |a_{k1}| = max(|(a_{i1})|)); i = 1 : n. \tag{4.2}$$

Exchange rows in A given by (4.1), appropriately such that $m_1 \neq 0$ in (4.2) and we have

$$a_{11} = m_1 \tag{4.3}$$

Compute the $n-1$ scaling factors $k_{i1}(1); i = 2 : n$ from the equation

$$k_{i1}(1)m_1 + a_{i1} = m_1; i = 2 : n. \tag{4.4}$$

so that from (4.3) and (4.4), we have

$$k_{i1}(1) = (a_{11} - a_{i1})/a_{11}; i = 2 : n. \tag{4.5}$$

Equations (4.2) and (4.3) represent a pivoting strategy to satisfy $k_{i1}(1) \in [-1, 1]; i = 2 : n$. Unit lower triangular matrix L_1 derived from unit identity matrix $I_n = (e_{ij}), e_{ii} = 1; e_{ij} = 0, i \neq j; i, j = 1 : n$ can be now defined as below.

$$L_1 = (l_{ij}(1)) = e_{ij}; i, j = 1 : n. \tag{4.6}$$

$$l_{i1}(1) = k_{i1}(1), i = 2 : n; l_{ij}(1) = e_{ij}, elsewhere; i, j = 1 : n. \tag{4.7}$$

With this construction of operating matrix L_1 in (4.6) and (4.7) by applying scaling factors (4.5), we can derive,

$$A_1 = L_1 A. \tag{4.8}$$

Thus resultant matrix $A_1 = (a_{ij}(1))$; $a_{i1}(1) = m_1, i = 1 : n$ is possible as in (4.8). Consider step-2. Let,

$$m_2 = max(a_{12}(1), a_{22}(1), a_{32}(1), a_{42}(1), ..., a_{n2}(1)) \tag{4.9}$$

$$l_2 = min(a_{12}(1), a_{22}(1), a_{32}(1), a_{42}(1), ..., a_{n2}(1)) \tag{4.10}$$

It should be such that $l_2 \neq m_2$ in (4.9),(4.10). Make appropriate row exchanges in A_1 among rows, row-1, row-2, ..., row-n to have

$$a_{12}(1) = l_2; a_{22}(1) = m_2 \tag{4.11}$$

(4.11) is the pivoting for step-2. To derive the $(n-2)$ scaling factors $k_{i2}(2); i = 3 : n$ for step-2, consider the linear systems of equations for solving.

$$k_{i1}(2)m_1 + k_{i2}(2)m_1 + m_1 = m_1; i = 3 : n. \tag{4.12}$$

$$k_{i1}(2)a_{12}(1) + k_{i2}(2)m_2 + a_{i1}(1) = m_2; i = 3:n, a_{12}(1) = l_2, a_{22}(1) = m_2. \quad (4.13)$$

Solving system of equations (4.12) and (4.13) we get

$$k_{i1}(2) = -k_{i2}(2), i = 3:n. \quad (4.14)$$

$$k_{i2}(2) = (a_{22}(1) - a_{i2}(1))/(a_{22}(1) - a_{12}(1)); i = 3:n. \quad (4.15)$$

Equations (4.9), (4.10) and (4.11) represent a pivoting strategy for step-2 so that $|k_{i2}(2)| \in [0,1]; i = 3:n$. Now the required scaling factors for step-2 are computed as in (4.14) and (4.15) using (4.12) and (4.13). We can proceed to construct operating matrix $L_2 = (l_{ij}(2))$. It is first initialized to I_n as $l_{ij}(2) = e_{ij}; i,j = 1,2,...,n.$. Now proceed as below.

$$l_{i1}(2) = -k_{i2}(2); i = 3,4,...,n. \quad (4.16)$$

$$l_{i2}(2) = k_{i2}(2); i = 3,4,...,n. \quad (4.17)$$

Thus column-1 and column-2 entries of L_2 are updated as in equations (4.16) and (4.17). Now we can complete step-2 by applying L_2 to derive resultant matrix $A_2 = (a_{ij}(2)); i,j = 1,2,...n$. We have in $A_2, a_{i1}(2) = m_1; i = 1,2,...n$ and $a_{i2}(2) = m_2; i = 2,3,4,...n$.

$$A_2 = L_2 A_1 \quad (4.18)$$

This completes step-2 with the construction of A_2 as in (4.18). This process of maintaining those scaling done with entries of previous columns 1,2,...,$r-1$ at a step-r and scaling entries of current column-r so that they become equal with the diagonal entry $a_{rr}(r-1)$ of resultant matrix A_{r-1}, we may call *uniformity of entries* at step-r. Continuing with this uniformity of entries for step-3, step-4,..., we shall now consider uniformity of entries at step-r. In this step-r also we shall do the row exchanges as in previous steps so that $|(k_{ir}(r))| \in [0,1]$, $i = r+1 : r+n$.

$$l_r = min(a_{r-1,r}(r-1), a_{rr}(r-1), ..., a_{n,r}(r-1)) \quad (4.19)$$

$$m_r = max(a_{r-1,r}(r-1), a_{rr}(r-1), ..., a_{n,r}(r-1)) \quad (4.20)$$

It should be that $m_r \neq l_r$ in (4.19), (4.20). Exchange rows in A_{r-1} among row-r-1, row-r,....,row-n appropriately so that

$$a_{r-1,r}(r-1) = l_r \quad (4.21)$$

$$a_{rr}(r-1) = m_r \quad (4.22)$$

(4.21) and (4.22) represent the general pivoting strategy for a setp-r, $r = 2,3,...n-1$. For convenience, consider first three linear systems of equations at step-r derived out of equation $A_r = L_r A_{r-1}$.

$$\sum\nolimits_{j=1}^{r} k_{ij}(r)m_1 + m_1 = m_1; i = r+1, r+2, ..., n. \quad (4.23)$$

$$k_{i1}(r)a_{12}(r-1) + \sum\nolimits_{j=2}^{r} k_{ij}(r)m_2 + m_2 = m_2; i = r+1, r+2, ..., n. \quad (4.24)$$

$$k_{i1}(r)a_{13}(r-1) + k_{i2}(r)a_{23}(r-1) + \sum\nolimits_{j=3}^{r} k_{ij}(r)m_3 + m_3 = m_3; i = r+1, r+2, ..., n. \quad (4.25)$$

From system of equations (4.23), we get

$$\sum\nolimits_{j=1}^{r} k_{ij}(r) = 0; i = r+1, r+2, ..., n... \quad (4.26)$$

Using (4.26) we can substitute for $k_{i1}; i = r+1, r+2, ..., n$ in equation (4.24) to get the result that

$$k_{i1}(r) = 0; i = r+1, r+2, ..., n \quad (4.27)$$

Because of (4.27) we get

$$\sum\nolimits_{j=2}^{r} k_{ij}(r) = 0; i = r+1, r+2, ..., n. \tag{4.28}$$

Applying this results in (4.25) we can see that

$$k_{i2}(r) = 0; i = r+1, r+2, ..., n. \tag{4.29}$$

Proceeding in this manner, we can get the result

$$k_{ij}(r) = 0; i = r+1, r+2, ..., n; j = 1, 2, ..., r-2. \tag{4.30}$$

Thus for step-r as illustrated in (4.28), (4.29), (4.30), we need consider only multiplication of L_r with column-(r-1) and column-r of A_{r-1}. Linear system of equations corresponding to column-$(r-1)$ of A_{r-1} will be

$$k_{i,r-1}(r)m_{r-1} + k_{ir}(r)m_{r-1} + m_{r-1} = m_{r-1}; i = r+1, r+2, ..., n. \tag{4.31}$$

From (4.31), we have

$$k_{i,r-1}(r) = -k_{ir}(r); i = r+1, r+2, ..., n \tag{4.32}$$

(4.32) makes it clear that scaling quantities are coupled as equal in magnitude but opposite in sign. The last linear system of equations corresponding to column-r of A_{r-1} for computing the scaling factors $k_{ir}(r); i = r+1, r+2, ..., r+n$ will be

$$k_{i,r-1}(r)a_{r-1,r}(r-1) + k_{ir}(r)m_r + a_{ir}(r-1) = m_r; i = r+1, r+2, ..., n. \tag{4.33}$$

From equation (4.33) we get

$$\begin{aligned} &k_{ir}(r) = (m_r - a_{ir}(r-1))/(m_r - a_{r-1,r}(r-1)); i = r+1, r+2, ..., n; \\ &a_{r-1,r}(r-1) = l_r, a_{rr}(r-1) = m_r. \end{aligned} \tag{4.34}$$

(4.34) gives the required scaling quantities for step-r. Now we can construct the unit lower triangular operating matrix $L_r = (l_{ij}(r)) = e_{ij}; i, j = 1, 2, ..., n$ by updating its column-r-1 and column-r as below.

$$l_{i,r-1}(r) = -k_{ir}(r); i = r+1, r+2, ..., n \tag{4.35}$$

$$l_{ir}(r) = k_{ir}(r); i = r+1, r+2, ..., n. \tag{4.36}$$

The last step is to complete uniformity of entries at step-r by deriving the resultant matrix A_r as

$$A_r = L_r A_{r-1} \tag{4.37}$$

Equations (4.35), (4.36) and (4.37) compute the elementary operating matrix L_r for deriving the resultant matrix A_r of step-r. We may continue this process with steps, r+1, r+2, ...,n-1 to complete uniformity of entries for the lower triangular block as

$$A_{n-1} = L_{n-1} A_{n-2} \tag{4.38}$$

We may extend this process as in (4.35),(4.36),(4.37) and (4.38) to the upper triangular block also. We should not make any row or column exchanges at this upper block operations as this will affect the uniformity of entries. The pivoting we followed during lower triangular block is an optional strategy to derive absolute values of scaling factors from closed interval $[0, 1]$. For upper triangular block, we can construct unit upper triangular matrices $U_1, U_2, ..., U_{n-1}$ with respective similar structures as transposes of $L_1, L_2, ..., L_{n-1}$ so that

$$S = L_{n-1}L_{n-2}....L_2L_1AU_1U_2...U_{n-2}U_{n-1} \tag{4.39}$$

$$k_{ij}(n) = (a_{11}(n-1) - a_{1j}(n-1))/a_{11}(n-1); j = 2, 3, ..., n. \tag{4.40}$$

Scaling factors corresponding to operator matrix U_1 of (4.39) will be provided by (4.40).

$$\begin{aligned} &k_{rj}(n+r-1) = \\ &(a_{rr}(n+r-2) - a_{rj}(n+r-2))/(a_{rr}(n+r-2) - a_{r,r-1}(n+r-2)); \\ &j = r+1 : n \end{aligned} \tag{4.41}$$

Scaling factors corresponding to U_r of (4.39) $r = 2, 3, ..., n-1$ can be computed as in (4.41). The matrix $S = (s_{ij}); s_{ij} = s_{ji}; i, j = 1, 2, ..., n$, is symmetric. We may call S *uniformly symmetric* as it has the additional structural simplicity that $s_{ik} = s_{kj} = s_{kk}; k = 1 : n$, for all $i, j > k; i, j = 1 : n$. The matrix S may be denoted as $S = uniform(s_1, s_2, ..., s_n)$ where $s_i = s_{ii}; i = 1, 2, ..., n$.

5. **Properties of Scaling Factors**

The matrices considered here are to be assumed from $R^{n\times n}$, if not exclusively stated. Some simple results are presented below, but are relevant in highlighting salient aspects of the scaling factors. We shall, for convenience, recall here equation (4.34) of generating scaling factors at step-r ;
$k_{ir}(r) = (m_r - a_{ir}(r-1))/(m_r - a_{r-1,r}(r-1)); i = r+1, r+2, ..., n; a_{r-1,r}(r) = l_r, a_{rr}(r) = m_r$.

Lemma 5.1. *If an entry $a_{ir}(r-1) = a_{r-1,r}(r-1)$ then*

$$k_{ir}(r) = 1. \tag{5.1}$$

Proof. (5.1) follows from equation (4.34). □

Lemma 5.2. *Elementary uniform symmetrization of a nonsingular diagonal matrix $D \in R^{n\times n}$ gives scaling factors $k_{ir}(r)$ =1 ; $i = r+1, r+2, ..., n; r = 1, 2, ..., n-1$. Product of lower(upper) operating matrices will be $L(1)(U(1))$.*

Proof. From Lemma 5.1, it follows that scaling factors, $k_{ir}(r)$ =1 ; $i = r+1, r+2, ..., n; r = 1, 2, ..., n-1$. In Operating matrix L_r, entries of column-r-1 and column-r are -1 and 1 respectively for r=2,3,...,n-1. In matrix L_1, its entries of column-1 will be 1. Product of these will be $L(1)$. Similarly product of upper triangular operating matrices will be $U(1)$. □

Lemma 5.3. *If uniformity of entries of lower triangular block is considered first for a given upper triangular matrix $U \in R^{n\times n}$, $k_{ir}(r)$ =1 ; $i = r+1, r+2, ..., n; r = 1, 2, ..., n-1$. Similarly is the case with uniformity of entries of upper triangular block of a nonsingular lower triangular matrix $L \in R^{n\times n}$.*

Proof. In both these situations, Lemma 5.1 will be applicable. Entries of a column(row) of the block being considered will be same as respective first supra(infra) entry of the column(row) Hence the result. □

Lemma 5.4. *If in A_{r-1}, an entry $a_{ir}(r-1) = a_{rr}(r-1)$ then in K_r, corresponding sclaing factor $k_{ir} = 0$.*

Proof. This is a direct result derived from equation (4.34). Since an entry of a column(row) is already equal to the corresponding diagonal entry, it requires only zero scaling. □

Lemma 5.4 implies that for this symmetrizing process, zero is a valid scaling quantity. It also implies how to pivot with this symmetrization process. It suggests that for minimizing perturbations, we should minimize the absolute value of scaling factor $k_{ir}; i = r+1 : n$ in [0,1] ; $r = 1 : n-1$.

Lemma 5.5. *If nonsingular matrix A is uniformly symmetric, then $k_{ir} = 0; i = r+1 : n; r = 1 : n$ then $L_i; i = 1 : n-1$ will be the identity matrix I_n.*

Proof. This follows from Lemma 5.4. □

Lemma 5.6. *If $A = const(c)$ is a constant matrix of a non-zero constant c, then the symmetrization cannot proceed beyond step-1.*

Proof. This is also a special case of Lemma 5.4. For step-1, we can compute scaling factors $k_{1r} = 0; i = 2 : n$. However, from step-2 onwards, $a_{r,r-1}(r) = a_{rr}(r) = c; r = 2 : n$ and so equation (4.34) breaks down. Also no column or row exchange will change the situation. □

While dealing with symmetrizing a given singular matrix, this will be again discussed.

There are exceptions with triangular and diagonal matrices. If A is a nonsingular triangular matrix with all non-zero entries a non-zero constant, or when A is a nonsingular diagonal matrix with all diagonal entries as a non-zero constant, then the symmetrization can be completed. Identity matrix is an example of this situation.

It follows from equation (4.34) that if two or more entries of a column-r of resultant matrix A_{r-1}, $a_{ir}(r-1) = a_{i+m,r}(r-1)$ then corresponding scaling factors $k_{ir}(r) = k_{i+m,r}(r); i = r+1; m = 1, 2, ...(n-r-1); r = 1, 2, ..., n-1$. In L_r, if we interchange $l_{ir}(r) = -k_{ir}(r)$ and $l_{i,r-1}(r) = k_{ir}(r)$, we get L_r^{-1}. We may interchange the role of l_r, m_r so that $l_r, r = 1 : n-1$ are at pivot positions, the entries in resultant matrices $A_r, r = 1 : n-1$ may be minimized as per requirements of situations.

6. General Properties of Elementary Uniform Symmetrization

We shall introduce here some features, advantages etc. of this symmetrization process of a given nonsingular matrix $A \in R^{n \times n}$.

Lemma 6.1. *Let $A \in R^{n \times n}$ be uniformly symmetrized into $S \in R^{n \times n}$. Then diagonalization of $S = (s_{ij}); i, j = 1, 2, ..., n$ is by a universal congruence transformation given by $D = bd(1, -1)^T S\, bd(1, -1)$.*

Proof. The structure of $S = (s_{ij})$ is $s_{ik} = s_{kj} = s_{kk}, k = 1 : n$ for all $i, j > k; i, j = 1 : n$. As $bd(1, -1)$ and its transpose $bd(-1, 1)$ reduce columns and rows, the result follows. Corresponding diagonal matrix will be given by $D = diag(s_{11}, s_{22} - s_{11}, ..., s_{nn} - s_{n-1,n-1})$. □

Corollary 6.1 From Lemma 6.1 it follows that

$$S = L(1)DU(1).$$

Lemma 6.2. *Given a nonsingular matrix $A \in R^{n \times n}$, equivalent uniformly symmetric matrix and diagonal matrix $S, D \in R^{n \times n}$ respectively can be derived in $n-1$ steps.*

Proof. We have

$$bd(1,-1)^T A_{n-1} = U \tag{6.1}$$

In equation (6.1), U is an upper triangular matrix and equivalent to A. The quantities, $a_{11}(n-1), a_{22}(n-1)-a_{12}(n-1), ..., a_{nn}(n-1)-a_{n-1,n}(n-1)$ will constitute its diagonal. During the scaling process of A_{n-1} using $U_1, U_2, ..., U_{n-1}$ in (4.39), lower triangular block columns will be still identically equidistant from coordinate axes. So in A_{n-1} and $A_{2n-2} = S$, $a_{ii}(n-1)-a_{i-1,i}(n-1) = a_{ii}(2n-2)-a_{i-1,i}(2n-2); i = 2:n-1; a_{11}(n-1) = a_{11}(2n-2)$. We have $D = bd(-1,1)Sbd(1,-1)$ and hence $A_{2n-2} = L(1)DU(1)$, where D is derivable from A_{n-1} itself at the midway of the process. Hence the result. □

This makes it an easier way of symmetrizing a square matrix A compared to Adhikari[1] or by diagonalization methods where both left and right matrix multiplications are involved. This convenience of selective bypassing of operations with a triangular block minimizes computation, memory and time by half of that theoretically required in symmetrizing A into S. Also no eliminations are required. These advantages are due to the scaling process.

Corollary 6.2 Matrix D^{-1} is equivalent to A^{-1}.

Proof.

$$D^{-1} = U(1)S^{-1}L(1) \tag{6.2}$$

As S^{-1} is equivalent to A^{-1}, from (6.2), the result follows. □

Lemma 6.3. *Given a linear system of equations where $A \in R^{n\times n}$ is the nonsingular coefficient matrix, then it can be solved using the resultant matrix A_{n-1} of step-$n-1$ of elementary symmetrization of A.*

Proof. Consider the resultant matrix at step-$n-1$

$$L_{n-1}L_{n-2}...L_1A. \tag{6.3}$$

In (6.3), let

$$M = L_{n-1}L_{n-2}...L_1. \tag{6.4}$$

From (6.3) and (6.4)

$$A_{n-1} = MA \tag{6.5}$$

(6.5) can be presented as

$$U = bd(1,-1)A_{n-1} \tag{6.6}$$

In (6.6), U is an upper triangular matrix and is equivalent to A. Let $Ax = b$ be the given linear system. Then equivalent linear system will be given by

$$Ux = bd(1,-1)^T Mb \tag{6.7}$$

Backward substitution in equation (6.7) will give the unknown solution vector x. □

Lemma 6.4. *We can derive LU decomposition of $A \in R^{n\times n}$ from elementary symmetrization of it.*

Proof. Consider matrix M in (6.4). It is a unit lower triangular matrix. Hence from (6.5) we get

$$A = M^{-1}bd(1,-1)^{-T}U \tag{6.8}$$

Now applying (6.6) in (6.8), we get

$$L = M^{-1}bd(1,-1)^{-T} = M^{-1}L(1) \tag{6.9}$$

Since L is a unit lower triangular matrix, we have from (6.9)

$$A = LU. \tag{6.10}$$

□

(6.10) is the decomposition from elementary symmetrization of A.

Lemma 6.5. *If $P \in R^{n\times n}$ is a matrix that transform $A \in R^{n\times n}$ into uniformly symmetric matrix $S \in R^{n\times n}$ then P is idempotent.*

Proof. Recall from Lemma(5.5) that when A is a uniformly symmetric matrix, all operating matrices will be the identity matrix I_n. So we get

$$P(A) = A; P(PA) = P(A) = A; P^2 = P. \tag{6.11}$$

(6.11) shows that P is idempotent. □

Lemma 6.6. *Given a nonsingular matrix $A \in R^{n\times n}$ such that $LAU = S; S = L(1)DU(1)$ be its elementary symmetrization where $L, U \in R^{n\times n}$ are unit lower and upper triangular matrices and $D \in R^{n\times n}$ is diagonal matrix. Then S will be positive definite if entries of D are positive. The symmetrization will be extremely stable.*

Proof. From resultant matrix $A_{n-1} = (a_{ij}(n-1))$ of step-$n-1$ of elementary symmetrization of A, consider equivalent diagonal matrix

$$\begin{aligned} &D = diag(d_1, d_2, ..., d_n) = diag(a_{11}(n-1), a_{22}(n-1) - a_{12}(n-1), ..., \\ &a_{nn}(n-1) - a_{n-1,n}(n-1). \end{aligned} \tag{6.12}$$

Computed D in (6.12) from A_{n-1} will be same as that from S of (4.39) and recalling from (4.39), we have

$$L_{n-1}L_{n-2}....L_2L_1AU_1U_2...U_{n-2}U_{n-1} = S = L(1)DU(1). \tag{6.13}$$

In (6.13), as $d_1, d_2, ..., d_n > 0$, Let $Q = L(1)diag(d_1^{1/2}, d_2^{1/2}, ..., d_n^{1/2})$. Then $S = QQ^T$ and recalling from [6], S is a positive definite matrix.

$$L_{n-1}L_{n-2}....L_2L_1AU_1U_2...U_{n-2}U_{n-1} = QQ^T. \tag{6.14}$$

Also from [6], in line with the classical Cholesky decomposition of symmetric positive definite matrices, symmetrization (6.14) is an extremely stable process. □

Equation (6.14) is a generalized extension of Cholesky decomposition into the subspace of all non-singular matrices of $R^{n\times n}$. It proves that any nonsingular matrix $A \in R^{n\times n}$ which has all positive entries in its equivalent diagonal matrix, has an equivalent symmetric positive definite matrix S in $R^{n\times n}$. The matrix Q in (6.14) has row(column)wise constant entries and has same structure of matrix factors, Nair[8] derived from his non-unit bidiagonal decomposition of matrix A.

Lemma 6.7. *The matrix $W \in R^{n\times n}$*

$$W = U(1)DL(1) \tag{6.15}$$

is a symmetric matrix, where $D = diag(d_1, d_2, ..., d_n)$.

Proof. Let $W = (w_{ij})$. In (6.15) as $U(1) = L(1)^T$, $w_{ij} = w_{ji}; i, j = 1 : n$. □

Matrix W is in fact uniformly persymmetric matrix in the sense that $w_{ik} = w_{kj} = w_{kk}, k = 1:n$ for $i, j < k; i, j = 1:n$.

Lemma 6.8. *Given a nonsingular and symmetric matrix $A \in R^{n\times n}$, then from its elementary symmetrization, it can be derived $S = LAL^T$ where $L \in R^{n\times n}$ is a unit lower triangular matrix and S is the equivalent uniformly symmetric matrix of A. If A is also positive definite, then $A = GG^T$, the Cholesky triangular decomposition of A.*

Proof. If A is uniformly symmetric, then $L = U = I_n$, U being unit upper triangular component of A and the result is true. Now if A is symmetric, we have from (4.39), $L_{n-1}L_{n-2}....L_2L_1AU_1U_2...U_{n-2}U_{n-1} = S$. This can be written as $LAU = S, L = L_{n-1}L_{n-2}...L_2L_1; U = U_1U_2...U_{n-2}U_{n-1}$; Let $S = unifrom(s_1, s_2, ..., s_n)$. From consistency of linear systems for solving scaling factors to construct L, U, we have

$$LAU = (LAU)^T = U^TAL^T \Rightarrow L = U^T \Rightarrow LAL^T = S. \tag{6.16}$$

(6.16) is the decomposition of A.

$$A = L^{-1}L(1)DU(1)L^{-T}; D = diag(d_1, d_2, ..., d_n) = diag(s_1, s_2 - s_1, ..., s_n - s_{n-1}) \tag{6.17}$$

Now if A is also positive definite, then in (6.17), let $G = L^{-1}L(1)diag(d_1^{1/2}, d_2^{1/2}, ..., d_n^{1/2})$ so that

$$A = GG^T \tag{6.18}$$

In (6.18) lower triangular matrix G is the classical Cholesky Triangle derived from the elementary uniform symmetrization (6.16) which is extremely stable[6]. □

Lemma 6.9. *Given a nonsingular matrix $A \in R^{n\times n}$ such that $LAU = S$ is its elementary symmetrization, then $S = unifrom(s_1, s_2, ..., s_n)$ is nonsingular if and only if for any two consecutive entries, $s_i \neq s_{i+1}; i = 1 : n-1$.*

Proof. Let S be nonsingular. Then from resultant matrix A_{n-1}, we have $d_1 = a_{11}(n-1), d_i = a_{ii}(n-1) - a_{i-1,i}(n-1); i = 2, 3, ..., n-1$ are non-zeros. Consider their partial sums $s_i = d_1 + d_2 + ... + d_i; i = 1, 2, 3, ..., n$. It cannot be possible that $s_i = s_{i+1}$ for some $i; i = 1, 2, ..., n-1$. Conversely, if for some index $i \geq 2$, $s_i = s_{i-1}$, then $d_i = s_i - s_{i-1} = 0 = a_{ii}(n-1) - a_{i-1,i}(n-1)$. So equivalent matrix A will be a singular matrix. S is singular. This proves that when none of these entries $s_1, s_2, ..., s_n$, repeat consecutively, two adjacent entries will not become equal. Then matrix S will be nonsingular. □

Lemma 6.10. *Elementary symmetrization of a nonsingular matrix $A \in R^{n\times n}$ into a uniformly symmetric matrix $S \in R^{n\times n}$ can be transformed as an equivalence symmetrization.*

Proof. Recalling from equation (4.39), we have

$$L_{n-1}L_{n-2}....L_2L_1AU_1U_2...U_{n-2}U_{n-1} = S \tag{6.19}$$

Let $B \in R^{n\times n}$ be a nonsingular matrix. Then consider nonsingular matrices $S_1, S_2 \in R^{n\times n}$ deduced from (6.19) and given by

$$S_1 = (L_{n-1}L_{n-2}....L_2L_1)^TB \tag{6.20}$$

$$S_2 = (U_1U_2...U_{n-2}U_{n-1})B \tag{6.21}$$

(6.20) and (6.21) represent the factors that Adhikari [1] requires for equivalence symmetrization and so we have

$$\widetilde{A} = S_1^T A S_2 \tag{6.22}$$

$$\widetilde{A} = B^T S B \tag{6.23}$$

Equation (6.22) is an example of the equivalence symmetrization proposed by Adhikari[1]. Equation (6.23) is a congruence transformation which also gives the same symmetric matrix $\widetilde{A}$ derived from S. This is a generalization of the equivalence symmetrization by congruence transformation on the uniformly symmetric matrix S. This generalization also simplifies the process because of the easiness in deriving S from the elementary symmetrization of A as well as the easiness of the congruence transformation. In order to simplify the process further we can even consider $B \in R^{n\times n}$ in (6.20), (6,21) as a nonsingular diagonal matrix. □

Lemma 6.11. *Given matrix $A \in R^{n\times n}$, there exists a nonsingular matrix $B \in R^{n\times n}$ to derive an equivalent symmetric matrix $\widetilde{A} \in R^{n\times n}$ such that $\widetilde{A} = B^T S B$.*

Proof. If A is nonsingular, then by Lemma 6.10 the result is true. Suppose A is singular with rank $r, n > r > 0$. Then we can continue elementary symmetrization up to step-r. Correspondingly we can derive from resultant matrix $A_r(r)$ partial sums $s_1, s_2, ..., s_r$, and for the remaining $n - r$ entries, we may repeat s_r, $n - r$ times to obtain $S \in R^{n\times n}$ as $S = uniform(s_1, s_2, ...s_{r-1}, s_r, s_r, ..., s_r)$. Now for any nonsingular matrix $B \in R^{n\times n}$ we can derive $\widetilde{A} = B^T S B$. This equivalent matrix $\widetilde{A}$ is also symmetric. If A is zero matrix, then S also will be zero and the result holds for any nonsingular matrix B. □

In section-5, we discussed that if A is a constant matrix, elementary symmetrization procedure will fail after step-1. This is a special case of this lemma when rank $r = 1$. We see that symmetrizing constant matrix also into an equivalent symmetric matrix has no restriction in this manner. Thus we have shown that when it comes to elementary symmetrization, any given matrix $A \in R^{n\times n}$ can be symmetrized into an equivalent symmetric matrix in a more generalized way.

7. Singularities of a Matrix and its Elementary Symmetrization

Column(row) dependencies in elementary uniform symmetrization can be summarized in this way. At step-r, in column-r of resultant matrix A_{r-1}, it can be observed that $a_{r-1,r}(r-1) = a_{rr}(r-1)$; for some $r = 2 : n-1$. Hence scaling factors using equation (4.34) cannot be computed. The process breaks and such dependencies will be reflected as visual patterns in appropriate resultant matrices.

Lemma 7.1. *In matrix $A \in R^{n\times n}, A = (a_{ij}); i, j, = 1 : n$ if column-r is linearly dependent on columns l and m, $(n-1) \geq r > m > l \geq 1$ so that $a_{ir} = \alpha a_{il} + \beta a_{im}; i = 1 : n; \alpha, \beta \in R$ are scalars, then in the resultant matrices $A_m, A_{m+1}, ..., A_{r-1}$*

$$a_{ij}(j-1) = \alpha a_{ll}(l-1) + \beta a_{mm}(m-1); i = m, m+1, ..., n; j = m, m+1, ..., r-1, \tag{7.1}$$

are constant entries and the symmetrization process breaks at step-r.

Proof. Column dependency $a_{ir} = \alpha a_{il} + \beta a_{im}; i = 1 : n$ will be preserved from step-1 to step-$(l-1)$ in resultant matrices $A_1, A_2, ..., A_{l-1}$. At step-l, in the matrix multiplication, $A_l = L_l A_{l-1}$, column-r entries will be scaled as

$$a_{ir}(l) = \alpha a_{ll}(l) + \beta a_{im}(l); i = l+1, l+2, ..., n. \tag{7.2}$$

At step-m, in the matrix multiplication $A_m = L_m A_{m-1}$ the scaling of entries in column-r will be

$$a_{ir}(m) = \alpha a_{ll}(l) + \beta a_{mm}(m); i = m+1, m+2, ..., n. \tag{7.3}$$

These are constant entries in column-r during the processing of uniformity of entries at a previous step-m. This out of turn uniformity of column-r extends to beyond the lower triangular block to equalize upper block entries $a_{mm}(m), a_{m-1,m}(m), ...a_{r+1,m}(m)$ with its diagonal entry $a_{rr}(m) = \alpha a_{ll}(l) + \beta a_{mm}(m)$. This will be then a visual strip of constant numbers which is an out of turn, isolated and extended to upper triangular block uniformity of entries. This strip of constant entries of column-r will be persistent through step-$m+1$, step-$m+2$,...,step-$r-1$ because of the scaling by $\pm k_{ij}(j); i = j+1, j+2, ..., n; j = m+1, m+2, ..., r-1$. While computing scaling factors $k_{ir}; i = r+1, r+2, ..., n$ at step-r, by applying equation (4.34), since $a_{r-1,r}(r-1) = a_{rr}(r-1) = \alpha a_{ll}(l) + \beta a_{mm}(m)$, the equation fails and the process breaks. □

Suppose that the column dependency is limited to $a_{ir} = \alpha a_{il} + \beta a_{im}; i = 1, 2, ..r$ the leading principal submatrix of dimension $r \times r$. Then also the above out of turn uniformity of entries, isolated and extending to upper block can be observed, but trimmed to diagonal entry $a_{rr}(r)$. Also if a leading column $1 \leq (r) < l < m < n$ is dependent on succeeding column-l and column-m, at step-m this uniformity of entries, extending to upper triangular block can be observed. The process will be breaking at step-m.

Recalling from [6], rank of $n \times n$ matrix A is defined as the maximum number of linearly independent rows(columns) of A. Suppose rank of A is r. Let it be symmetrized up to step-r, by appropriate row(column) exchanges whenever an out of turn uniformity of current column is encountered. Then remaining $n-r$ columns will form a block of zig-zag pattern of strips of constant entries in these columns.

The dependency of rows is handled in a different way. A dependent row is replaced by independent rows on turn to turn basis. This is like approaching closer to the dependent row. Finally an independent row become adjacent to the dependent row. In this way two adjacent rows become identical and the process breaks.

Lemma 7.2. *In matrix $A \in R^{n \times n}, A = (a_{ij}); i, j, = 1 : n$ if row -r is linearly dependent on row-l and row-m, $(n-1) \geq r > m > l \geq 1$ so that $a_{rj} = \alpha a_{lj} + \beta a_{mj}; j = 1, 2, ..., n; \alpha, \beta \in R$ are scalars, then*

- $a_{rj}(j) = \alpha a_{lj}(j) + \beta a_{rj}(j); j = 1, 2, ...l, l+1, ..., m-1.$
- *At Step-m :* $a_{rj}(m) = a_{ij}(m); j = 1 : n; k_{r+1,m}(m) = 1$
- *At Step-m+1 :* $a_{rj}(m+1) = a_{ij}(m+1); j = 1, 2, ..., n; k_{r+1,m+1}(m+1) = 1$
- *At Step-m+2 :* $a_{rj}(m+2) = a_{ij}(m+2); j = 1, 2, ..., n; k_{r+1,m+2}(m+2) = 1$
-
-

- *At Step-$r-1$:* $a_{r,j}(r-1) = a_{r-1,j}(r-1); j = 1:n; k_{rr}(r) = 1$

The process breaks at step-r.

Proof. At step-l, the entry $a_{rl}(l) = \alpha a_{ll}(l) + \beta a_{ml}(l)$. This is because of uniformity of entries at step-l. Other entries of row-r will be linear combination of entries of these two rows. This will continue up to step-$m-1$. At step-m the entry $a_{rm}(m) = a_{mm}(m)$. This is because of the uniformity of entries of column-m. Because of uniformity of entries of previous columns, the entries $a_{r,j}(m) = a_{m,j}(m); j = 1, 2, ...m-1$. Since row-$r$ depends both on row-l and row-$m, l < m$ and as uniformity of entries of both the corresponding columns l and m are completed, all remaining columns $m+1, m+2, ...n$ of row-r are such that $a_{rj}(m) = a_{mj}(m); j = m+1, ..., n$. In the next step-$m+1$, as $a_{r,m}(m) = a_{m,m+1}(m)$;we get $k_{r+1,m+1}(m+1) = 1; k_{r+1,m}(m+1) = -1$. Then in the matrix multiplication, $A_{m+1} = L_{m+1}A_m$ we get

$$a_{rj}(m+1) = -a_{rj}(m) + a_{mj}(m) + a_{m+1,j}; j = 1:n; \tag{7.4}$$

Since $a_{rj}(m) = a_{mj}(m); j = 1:n$, the row $a_{rj}(m+1)$ of A_{m+1} will be replaced with the row-$m+1$ of A_m. Thus

$$a_{rj}(m+1) = a_{m+1,j}(m)j = 1:n. \tag{7.5}$$

Again when we compute the scaling factors from column-$(m+1)$ of A_{m+1}, as row entries $a_{rj}(m+1) = a_{m+1,j}(m+1); j = 1:n$, the scaling factors $k_{r,m+2}(m+2) = 1; k_{r,m+1}(m+1) = -1$. Due to this, in resultant matrix A_{m+2}, row-r will be replaced with row-$m+2$. This turn by turn replacement of row-r in resultant matrices will be continuing till step-$(r-1)$. At step-r, as row-$r-1$ and row-r are identical in A_{r-1}, scaling factors cannot be computed and the process will break. □

Above result will be true for the leading principal submatrix of dimension $r \times r$ also when it is singular as stated above but trimmed to column-r. The difference is that in the submatrix level column(row) dependency, we will be able to complete the process by suitable row exchanges among row-r through row-n. In the full column or row dependency also we may be able to appropriately exchange columns(rows) and continue the process for row-r. But we will not be able to complete the process through the required $n-1$ steps.

Suppose all remaining rows, $r+1, r+2, ..n$ are dependent rows of some of the previous rows $1, 2, ..., r$. Because of Lemma-7.2, all these rows, row-$r+1$, row-$r+2$, ..., row-n, will be identical with last processed independent row-$r-1$. Thus a patch of column-wise constant entries will be visible across these remaining rows in the resultant matrix A_r. This highlights that A is of rank-$r-1$. Summarizing these results we have

Lemma 7.3. *If all n leading principal sub-matrices of nonsingular matrix $A \in R^{n\times n}$ are nonsingular then uniformity of entries of lower triangular block can be completed without any column or row exchanges in $n-1$ steps.*

Had we started with the upper triangular block, the role will be exchanged and whatever results stated for columns will be applicable to rows instead.

8. **Numerical Illustration of Elementary Symmetrization**

Hilbert matrix $H = (h_{ij}); h_{ij} = 1/(i + j - 1)$ of dimension 4×4 is considered to illustrate the symmetrization process. It is ill-conditioned and incurable[10,16]. H is symmetric and totally positive. For H, by default, scaling factors will be in closed interval $[0, 1]$.

Example-1 Symmetrization of H-4

Step-1

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.666667 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1 \end{pmatrix}; A = \begin{pmatrix} 1 & 0.5 & 0.33 & 0.25 \\ 0.5 & 0.33 & 0.25 & 0.20 \\ 0.33 & 0.25 & 0.20 & 0.17 \\ 0.25 & 0.20 & 0.17 & 0.14 \end{pmatrix}$$

Step-2

$$L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1.33E-15 & -1.33E-15 & 1 & 0 \\ -0.1 & 0.1 & 0 & 1 \end{pmatrix}; A_1 = \begin{pmatrix} 1 & 0.5 & 0.3333 & 0.25 \\ 1 & 0.583333 & 0.416667 & 0.325 \\ 1 & 0.583333 & 0.422222 & 0.333333 \\ 1 & 0.575 & 0.416667 & 0.330357 \end{pmatrix}$$

Step-3

$$L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0.5 & -0.5 & 1 \end{pmatrix}; A_2 = \begin{pmatrix} 1 & 0.5 & 0.3333 & 0.25 \\ 1 & 0.583333 & 0.416667 & 0.325 \\ 1 & 0.583333 & 0.422222 & 0.333333 \\ 1 & 0.583333 & 0.425 & 0.337857 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 0.5 & 0.3333 & 0.25 \\ 1 & 0.583333 & 0.416667 & 0.325 \\ 1 & 0.583333 & 0.422222 & 0.333333 \\ 1 & 0.583333 & 0.422222 & 0.33369 \end{pmatrix}$$

Diagonal entries from $A_3 : D = diag(1, 0.083333, 0.005556, 0.000357)$

Symmetric matrix from $D : S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1.083333 & 1.083333 & 1.083333 \\ 1 & 1.083333 & 1.08889 & 1.08889 \\ 1 & 1.083333 & 1.08889 & 1.08889 \end{pmatrix}$

Since S is also positive definite, by equation (6.15) $S = QQ^T; Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0.28867 & 0 & 0 \\ 1 & 0.28867 & 0.07454 & 0 \\ 1 & 0.28867 & 0.07454 & 0.018894 \end{pmatrix}$.

Nair [8] derived matrices of type Q with column(row) wise constant entries as factors for a given matrix. Whole processes on A are completed with step-3 itself as against the total six steps. All diagonal entries are positive and so decomposition QQ^T for S will be stable by Lemma-6.8.

Example-2 : Out of turn uniformity of entries of dependent columns.

$$\begin{pmatrix} 1 & 0.50 & 2.00 & 2.50 \\ 0.5 & 0.33 & 1.00 & 1.67 \\ 0.33 & 0.25 & 0.67 & 1.25 \\ 0.25 & 0.20 & 0.50 & 1.00 \end{pmatrix} \begin{pmatrix} 1 & 0.50 & \mathbf{2} & 2.5 \\ 1 & 0.583333 & \mathbf{2} & 2.916667 \\ 1 & 0.583333 & \mathbf{2} & 2.916667 \\ 1 & 0.575 & \mathbf{2} & 2.875 \end{pmatrix} \begin{pmatrix} 1 & 0.5 & \mathbf{2} & 2.5 \\ 1 & 0.583333 & \mathbf{2} & \mathbf{2.916667} \\ 1 & 0.583333 & \mathbf{2} & \mathbf{2.916667} \\ 1 & 0.583333 & \mathbf{2} & \mathbf{2.916667} \end{pmatrix}$$

Step-1 Matrix Step-2 Resultant Matrix Step-3 Resultant Matrix.

Column-3, 4 are dependent on column-1,2 respectively. At step-2, uniformity of entries of column-3(Bold Font) is out of turn. At step-3, column-3 uniformity is preserved, out of turn uniformity of entries of column-4(bold font) is displayed.

Example-3 : Turn by turn replacement of dependent row by current independent rows. Row-3, 4 are dependent on Row-1,2 respectively.

Step-1 Matrix Step-2 Resultant Matrix Step-3 Resultant Matrix.

$$\begin{pmatrix} 1 & 0.50 & 0.33 & 0.25 \\ 0.50 & 0.33 & 0.25 & 0.20 \\ 2 & 1.00 & 0.67 & 1.25 \\ 0.75 & 0.50 & 0.38 & 0.30 \end{pmatrix} \begin{pmatrix} 1 & 0.50 & 0.333333 & 0.25 \\ 1 & 0.583333 & 0.416667 & 0.325 \\ \mathbf{1} & \mathbf{0.50} & \mathbf{0.333333} & \mathbf{0.25} \\ 1 & 0.625 & 0.458333 & 0.3625 \end{pmatrix} \begin{pmatrix} 1 & 0.5 & 0.333333 & 0.325 \\ 1 & 0.583333 & 0.41667 & 0.325 \\ \mathbf{1} & \mathbf{0.583333} & \mathbf{0.41667} & \mathbf{0.325} \\ \mathbf{1} & \mathbf{0.583333} & \mathbf{0.41667} & \mathbf{0.325} \end{pmatrix}$$

At step-2, dependent row-3(bold font) is replaced with independent row-1. At step-3, both dependent row-3 and row-4(bold font) are replaced with independent row-2.

9. **Conclusions**

Symmetrization of a given nonsingular square matrix $A \in R^{n\times n}$ using elementary column(row) operations is introduced. The procedure progresses in a similar way as classical Gauss elimination. Contrary to eliminating entries, here for symmetry, column(row)wise lower and upper triangular blocks are scaled to become identical with corresponding diagonal entries. It is proved that in this way only one triangular block, either lower or upper block need be processed for symmetry. Entries of equivalent diagonal matrix D can be derived as differences of first supra(infra) entries from diagonal entries of resultant matrix A_{n-1} of step-$n-1$. It thus reduces the processing requirements to just half of the total $2(n-1)$ steps. As the symmetric matrix S is provided by a universal congruence transformation $S = L(1)DU(1)$, eliminations of row or column entries also can be avoided. In this way $S = (s_{ij})$ has a simpler structure compared to symmetric matrices. Its structure is $s_{ik} = s_{kj} = s_{kk}, k = 1 : n$, for all $i, j > k, i, j = 1 : n$. General properties of the symmetrization process and matrix S are discussed. Scaling factors applied by operator matrices are coupled as equal in magnitude but opposite in signs. Because of this property, it is called elementary uniform symmetrization. Zero is a valid scaling quantity and based on this feature, a pivoting strategy is presented. It is shown that this symmetrization is applicable to any given square matrix $A \in R^{n\times n}$ for deriving its equivalent symmetric matrix S in a generalized way. When entries of D are positive, classical Cholesky decomposition can be applied to such matrices for deriving equivalent Cholesky triangular matrix components. The symmetrization is extremely stable in such cases. A generalized extension of Cholesky decomposition to the subspace of all nonsingular square matrices is presented. The type-2 equivalence symmetrization proposed by Adhikari[1] is also generalized in a more simplified way as a congruence transformation. It is shown that dependencies of columns(rows) are presented as visual patterns in appropriate resultant matrices of the process.

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