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A Random Walk From the Landau to the Riemann Hypothesis

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Abstract
The Riemann hypothesis is arguably the most important unsolved problem in mathematics. It is even difficult to state to beginning students since it requires a knowledge of the zeta function for complex values of the argument. However, there are at least 23 equivalent statements of this hypothesis that are much easier to state. In this paper we examine one such idea called the Landau hypothesis. This Landau hypothesis has a very simple interpretation in terms of the prime factorization of integers. Surprisingly this hypothesis also has a useful description in terms of a one dimensional random walk. We show by means of a known, but not often seen, integral representation for any Dirichlet series that the Landau hypothesis implies the Riemann hypothesis.

Keywords: Riemann hypothesis, Landau hypothesis, random walk, number theory

1. Prologue: The Easy Equivalent to the Riemann Hypothesis

There is a very intuitive way of describing an equivalent statement to the Riemann hypothesis. Let $\omega(n)$ equal the number of prime factors of the positive integer n counting multiplicity. For example $\omega(6)=2$ while $\omega(12)=3$. Now the Riemann Hypothesis is roughly equivalent to the statement that $\omega(n)$ is equally likely to be even or odd. (For the precise meaning of this statement, see the statement of the Landau Hypothesis below.)

Read on if you want to understand the fascinating mathematics behind this equivalent statement of the Riemann hypothesis.

2. Introduction

This paper is concerned with understanding one aspect of the Riemann Hypothesis (RH) . The RH is a mathematically advanced idea that is even difficult to state since it requires at least an elementary familiarity with the zeta function and complex variables. However there is a much simpler hypothesis equivalent to the RH, due to Edmund Landau (1877 – 1938), the meaning of which can be illustrated in a very elementary probabilistic fashion. This Landau hypothesis (LH) is the main subject of this paper, and we will give an elementary proof that the Landau Hypothesis implies the Riemann Hypothesis. Surprisingly, we find that the behavior of a simple one-dimensional random walk is the key to intuitively understanding why LH, and thus RH, might be true. We investigate why LH is related to RH by means of an integral representation for any Dirichlet series. Our presentation is designed for a wide audience of non-experts.

3. Preliminary Ideas

We begin with the definition of the zeta function [1, 2, 3]

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \tag{3.1}$$

which in this series representation is defined only for $\mathrm{Re}(s)>1$. (It is common to use $s=\sigma+it$ rather than $z=x+iy$ for the complex variable since this was the notation used by Riemann.) Another representation (valid in the same region) is the product due to Euler

$$\zeta(s)=\prod_{n=1}^{\infty}\left(1-\frac{1}{p(n)^s}\right)^{-1}$$

where $p(n)$ is the *n*th prime number. The reader might want to expand this product using the geometric series to obtain

$$\zeta(s)=\prod_{n=1}^{\infty}\left(1+\frac{1}{p(n)^s}+\frac{1}{p(n)^{2s}}+\frac{1}{p(n)^{3s}}+\ldots\right)$$

and thereby show that the product representation is equivalent to the series.

It is possible to extend $\zeta(s)$ as an analytic function valid for all *s* except $s=1$, where the function has a simple pole. The most common form of RH is:

Riemann hypothesis: The complex zeroes of the zeta function all have real part equal to ½.

Many say that this is the most important unsolved problem in mathematics. It has existed since 1859 when Bernard Riemann wrote a seven page paper in which he showed that the prime counting function $\pi(x)$, which equals the number of prime numbers less than or equal to *x*, can be expressed *exactly* in terms of an infinite series summed over the complex roots of the zeta function. See the Appendix in [1] for a translation of this paper.

Landau's statement equivalent to the RH

There are at least 23 statements equivalent to the RH. See [4] and [5]. That is, the RH is true if and only if the equivalent statement is true. Some of these equivalent statements have the feature that they remove mention of complex variables and are easier to relate to the distribution of prime numbers. In this paper we will be interested in one such equivalent statement, which is attributed to Edmund Landau. We now describe it.

Let $\omega(n)$ equal the number of prime factors of the positive integer *n* counting multiplicity. For example $\omega(10)=2$ and $\omega(12)=3$. We define the Liouville function as $\lambda(n)=(-1)^{\omega(n)}$. This simply converts the previous function into a function taking the two values plus and minus one. Now we can state Landau's equivalent hypothesis: (See [4, page 46].)

The Landau hypothesis (LH): $\lim_{N\to\infty}\frac{\sum_{n=1}^{N}\lambda(n)}{N^{\frac{1}{2}+\varepsilon}}=0$ *where* ε *is any positive number.*

In this paper we will mostly be concerned with the only if part of the following:

Theorem. LH if and only if RH. (See [4, page 46].)

We call $L(N)=\sum_{n=1}^{N}\lambda(n)$ the summatory Liouville function.

We will show in Section 8 that the Liouville function can be generated by the series [3, page 6]

$$\frac{\zeta(2s)}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}=\sum_{n=1}^{\infty}\frac{(-1)^{\omega(n)}}{n^s}$$

which is valid for $\operatorname{Re}s>1$. We will show later that the LH implies that this representation is valid for $\operatorname{Re}s>1/2$,and that this implies RH. But next we will discuss the connection of the behavior of $\lambda(n)$ to a uniform random walk in one dimension, and thus uncover how the Landau Hypothesis is related to an intuitively plausible probabilistic statement about the distribution of the primes.

4. A Random Walk in One Dimension

To understand LH, we must take an unexpected detour. We will examine the uniform random walk in one dimension. It is known that if x_i, for $i=1,2,3,...,n$ is a uniform random variable, taking on the values +1 and -1 only, then the sum $|x_1+x_2+x_{3.}...+x_n|$ (for large *n*) has the expected value $\sqrt{\frac{2}{\pi}}\sqrt{n}$ which is about $0.8\sqrt{n}$. See the Appendix of this paper for a proof.

We caution the reader that this result is not at all intuitive. At first glance the reader might think that the expected value is zero. This would be true if the sum under consideration was, (without absolute values), $x_1 + x_2 + x_{3.} ... + x_n$.

We will see that this theorem is of importance in studying the asymptotic behavior of the summatory Liouville function $\sum_{n=1}^{N} \lambda(n)$.

5. The Random Walk, an Assumption, and the Summatory Liouville Function

Let *n* be a random integer such that such that $2 \le n$. Now what can we say about $\lambda(n) = (-1)^{\omega(n)}$? Should not $\lambda(n)$ act like a random variable taking on the values 1 or -1? In other words, select a positive integer *n* at random, what is the probability that the number of prime factors in *n* is even? Is it not reasonable to assume that this probability is 0.5? It is like flipping a fair coin! **Assuming** that $\lambda(n)$ is such a "random like" variable, then the summatory Liouville function $L(N) = \sum_{n=1}^{N} \lambda(n)$ might behave, for large *N*, like a random one dimensional walk and from the previous statement we expect $\left| \sum_{n=1}^{N} \lambda(n) \right| \sim \sqrt{\frac{2}{\pi}} \sqrt{N}$. Using this result it is easy to see why LH could be true:

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N} \lambda(n)}{N^{\frac{1}{2}+\varepsilon}} = \lim_{N \to \infty} \frac{\pm \sqrt{\frac{2}{\pi}} \sqrt{N}}{N^{\frac{1}{2}+\varepsilon}} = \lim_{N \to \infty} \frac{\pm \sqrt{\frac{2}{\pi}}}{N^{\varepsilon}} = 0.$$

Here the notation $f(x) \sim g(x)$ means that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$ and we say that $f(x)$ is asymptotic to $g(x)$. Thinking loosely, it means that the two functions are approximately equal for large *x*.

6. Expected Random Walk – Simplified Argument

We now give a simple argument that will only suggests the truth of our random walk theorem. (For a proof, see the Appendix.) Let x_k be a sequence of numbers randomly taking on the values +1 or -1. We wish to study $d(n) = \left| \sum_{k=1}^{n} x_k \right|$ and give some simple explanation for the:

Random walk theorem:

$$d(n) \sim \sqrt{\frac{2}{\pi}}\sqrt{n} \approx 0.798\sqrt{n} \,.$$

Let us look at $d(n)^2 = \left(\sum_{k=1}^{n} x_k\right)^2$ .This is equal to $d(n)^2 = x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 + (other\, terms) = n + (other\, terms)$. Since the "*other terms*" can be both positive and negative, they tend to cancel one another. This leads us to conclude that $d(n) \sim c\sqrt{n}$ for some constant *c*. This completes our simple heuristic argument. A rigorous analysis is given in [6] and in the appendix at the end of this paper.

7. Example of the Random Walk

Consider two baseball teams, denoted by *A* and *B*, of exactly equal strength. That is, on any given day, either *A* or *B* stand the same chance of winning. Suppose the baseball season for a given year consists of just these two teams playing 100 games. Suppose this year *A* wins 55 games and thus looses 45. The net number of games won by *A* is 10. Obviously then *B has a net* loss of 10 games. We will say that the win-loss score for this year is 10. Now we ask the question, after playing for many seasons, what is the expected average win-loss score? Our random walk theorem gives us the answer 8. For we can describe the season score for team *A* as a random walk -1 +1 -1 + … +1, where our team has lost the first game, won the second, lost the third, and finally won the one hundredth game. Our random walk theorem tells us that the expected *absolute value* of this sum (the win-loss score), after a season of 100 games is $\sqrt{\frac{2}{\pi}}\sqrt{N} \approx .8\sqrt{100} = 8$. An interesting, non-intuitive conclusion! See the appendix for a proof of this theorem.

8. Liouville Function and Dirichlet Series

Since $\zeta(s) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{p(n)^s}\right)^{-1}$ where $p(n)$ is the *n* th prime number, we have

$$\frac{\zeta(2s)}{\zeta(s)} = \frac{\prod_{n=1}^{\infty}\left(1 - \frac{1}{p(n)^{2s}}\right)^{-1}}{\prod_{n=1}^{\infty}\left(1 - \frac{1}{p(n)^{s}}\right)^{-1}} = \frac{\prod_{n=1}^{\infty}\left(1 - \frac{1}{p(n)^{s}}\right)^{-1}\left(1 + \frac{1}{p(n)^{s}}\right)^{-1}}{\prod_{n=1}^{\infty}\left(1 - \frac{1}{p(n)^{s}}\right)^{-1}} \,.$$

So we get

$$\frac{\zeta(2s)}{\zeta(s)}=\prod_{n=1}^{\infty}\left(1+\frac{1}{p(n)^s}\right)^{-1}=\prod_{n=1}^{\infty}\left(1-\frac{1}{p(n)^s}+\frac{1}{p(n)^{2s}}-\frac{1}{p(n)^{3s}}+\ldots\right)$$

and displaying the first few terms of this product, we see

$$\frac{\zeta(2s)}{\zeta(s)}=\left(1-\frac{1}{2^s}+\frac{1}{2^{2s}}-\frac{1}{2^{3s}}+\frac{1}{2^{4s}}-\ldots\right)$$
$$\left(1-\frac{1}{3^s}+\frac{1}{3^{2s}}-\frac{1}{3^{3s}}+\frac{1}{3^{4s}}-\ldots\right)$$
$$\left(1-\frac{1}{5^s}+\frac{1}{5^{2s}}-\frac{1}{5^{3s}}+\frac{1}{5^{4s}}-\ldots\right)$$
$$\ldots$$

The multiplication begins to show

$$\frac{\zeta(2s)}{\zeta(s)}=1-\frac{1}{2^s}-\frac{1}{3^s}+\frac{1}{4^s}-\frac{1}{5^s}+\frac{1}{6^s}-\frac{1}{7^s}+\ldots+\frac{(-1)^{\omega(n)}}{n^s}+\ldots$$

where every natural number *n* appears in the denominator just once and we recognize that the numerator, which we will write (as before) as $\lambda(n)=(-1)^{\omega(n)}$ is the Liouville function. Again the exponent $\omega(n)$ is the number of prime factors in *n* counting multiplicity. Thus we have

(8.1) $$\frac{\zeta(2s)}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}=\sum_{n=1}^{\infty}\frac{(-1)^{\omega(n)}}{n^s}$$. This is valid for $\operatorname{Re} s>1$.

Functions with a representation

$$f(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

are called Dirichlet series, (see [7]). Here the $a(n)$ are usually real numbers. It is known that if the series converges at $\sigma_0=\operatorname{Re}(s)$, then the series converges absolutely and uniformly for $\sigma_0+\varepsilon\le\operatorname{Re}(s)$, (ε any positive value), and thereby represents an analytic function in this region. See [2, page 30]. So we now have two important Dirichlet series, one that defines the zeta function (3.1), and this one (8.1) that is a generating function for the Liouville function.

9. A Remarkable Integral Representation for Any Dirichlet Series

We begin by showing that, under certain conditions on s and the rate of growth of the summatory function $A(x)$, the general Dirichlet series has the integral representation

(9.1) $$f(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=s\int_1^{\infty}\frac{A(x)}{x^{s+1}}dx,$$

where $A(x)$ is the summatory function

$$A(x)=\sum_{n=1}^{\lfloor x\rfloor}a(n),$$

and $\lfloor x\rfloor$ is the "floor of *x*" or the "greatest integer less than or equal to *x*".

To show this we first note that $a(n)=A(n)-A(n-1)$, so we have

$$f(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=\sum_{n=1}^{\infty}\left(\frac{A(n)}{n^s}-\frac{A(n-1)}{n^s}\right)=\sum_{n=1}^{\infty}\frac{A(n)}{n^s}-\sum_{n=1}^{\infty}\frac{A(n-1)}{n^s}.$$

This last step requires that $\sum_{n=1}^{\infty}\frac{A(n)}{n^s}$ converges. We will assume that this is true. See note at the end of this section.

Changing the index in this last summation gives us

$$f(s)=\sum_{n=1}^{\infty}\frac{A(n)}{n^s}-\sum_{n=0}^{\infty}\frac{A(n)}{(n+1)^s}$$

so we have, since $A(0)=0$

$$f(s)=\sum_{n=1}^{\infty}A(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right).$$

But $s\int_n^{n+1}\frac{dx}{x^{s+1}}=\frac{1}{n^s}-\frac{1}{(n+1)^s}$ and we have defined $A(x)=A(\lfloor x\rfloor)$, (a constant over the interval of integration). So we can write

$$f(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=\sum_{n=1}^{\infty}s\int_n^{n+1}A(x)\frac{dx}{x^{s+1}}.$$

Summing all the integrals we get our result

$$f(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=s\int_1^{\infty}\frac{A(x)}{x^{s+1}}dx.$$

Note: A more detailed analysis using the method of summation by parts shows that this integral representation is valid if s is such that:

1. $\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$ converges , and $\lim_{n\to\infty}\frac{A(n)}{n^s}=0$, or
2. $\lim_{n\to\infty}\frac{A(n)}{n^s}=0$ and $s\int_1^{\infty}\frac{A(x)}{x^{s+1}}dx$ converges.

10. Integral Representation Connects LH and RH

It is known that Dirichlet series converge in some half plane of the form $s_0 < \mathrm{Re}(s)$. In some important cases we can use our integral representation (9.1) to help us decide the value of s_0.

Example 1: Suppose $|A(x)| \le Cx^{1/2+\varepsilon}$ for some constant *C*, then (9.1) gives us

(10.1) $$|f(s)| = \left|\sum_{n=1}^{\infty} \frac{a(n)}{n^s}\right| \le C|s|\int_1^{\infty} \frac{x^{1/2+\varepsilon}}{|x^{s+1}|}dx = C|s|\int_1^{\infty} \frac{x^{1/2+\varepsilon}}{x^{\sigma+1}}dx = \frac{C|s|}{\sigma - \left(\frac{1}{2}+\varepsilon\right)}.$$

Thus

(10.2) $f(s)$ is finite in the half plane $1/2+\varepsilon < \sigma = \mathrm{Re}(s)$, and is analytic there. This result gives information about the Riemann hypothesis as the next example shows.

Example 2: **Theorem. LH implies RH.** Recall the Liouville function $\lambda(n) = (-1)^{\omega(n)}$ where $\omega(n)$ equals the number of prime divisors of *n* (counting multiplicity) and $\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{(-1)^{\omega(n)}}{n^s} = \sum_{n=2}^{\infty} \frac{\lambda(n)}{n^s}$.

This series is known to converge only for $1 < \mathrm{Re}(s)$. See [3, page 6] and Section 8 above. Now examine the function $\frac{\zeta(2s)}{\zeta(s)}$ for its singularities. In the numerator, the zeta function having only a simple pole when the argument is 1, contributes a simple pole at $s = 1/2$. But the denominator will produce singularities wherever the zeta function is zero. (Note: the numerator cannot be zero if $\mathrm{Re}(s) > 1/2$. See Note below.) These singularities (in $0 < \mathrm{Re}(s)$) will all be complex numbers with real part ½ if RH is true.

If the LH is true, then the summatory Liouville function $\sum_{k=2}^{n} \lambda(k) = o(n^{1/2+\varepsilon})$ for any positive ε. By (10.1) and (10.2) and the properties of analytic continuation, we see that $\sum_{n=2}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$ has no singularities in the half plane $1/2+\varepsilon < \mathrm{Re}(s)$ (for any positive ε however small). It is well known that the set of complex roots of the zeta function that are not on the line with real part ½ is symmetric with respect to that line. Thus if RH is not true, and zeta has a complex root at $1/2-\delta+i\rho$, then it also has a complex root at $1/2+\delta+i\rho$. This follows easily from the functional equation for the zeta function [5, 10, 16]. Thus (10.2) allows us to conclude that $\zeta(s)$ has no zeros for $s > 1/2$, and we have finally shown that LH implies RH.

Note: $\frac{\zeta(2s)}{\zeta(s)}$ could possibly be finite if $\zeta(s)=0=\zeta(2s)$. But since we are talking about Re(s)>1/2, Re(2s)>1 and it is known (see the Prime Number Theorem) that $\zeta(s)$ has no roots for $\operatorname{Re} s \geq 1$. So if $\zeta(2s)/\zeta(s)$ is finite in Re(s)>1/2, then since $\zeta(2s)\neq 0$, it must be that $\zeta(s)\neq 0$.

This completes our study. We end with a few comments.

11. Comments on the Summatory Liouville Function

The summatory Liouville function $L(N)=\sum_{n=1}^{N}\lambda(n)$ has an interesting history.

In 1913 Polya [8] conjectured that $L(N)$ was never positive. If this were true, it would have troubled our notion than $\lambda(N)$ acts like a random variable taking on values plus and minus one. However in 1958 Haselgrove [9] disproved this conjecture. In 1980 Tanaka [10] showed that $N=906{,}150{,}257$ is the smallest value for which $L(N)=1$. In [11], Borwein, Ferguson, and Mossinghoff ,using results from Lehman [12] showed that $L(n)>0.061867\sqrt{n}$ infinitely often. In [13] Humphries shows that $L(n)<-1.3892783\sqrt{n}$ infinitely often. The paper [11] has a detailed history of this topic.

12. Comments on the References

The opening pages of [4] were the authors main motivation for exploring the ideas that lead to this paper.

The book [14] by Mazur and Stein is most remarkable. Unlike any other book on RH, this book is written at four mathematical levels. The first part is the most fascinating. Here Mazur and Stein assume that the reader has no calculus background! Yet the authors give a brilliant exposition of the sweeping saga of work on RH. Imagine explaining the Fourier transform to someone without a knowledge of calculus! This author has read this first part again and again to appreciate the poetry of the exposition as well as gain insights for improving his own teaching. The remaining three parts require successively stronger mathematical background. Everyone interested in RH will find this rewarding.

There are several good books for the non-expert on RH. Among these are [15] and [16]. The book by Havil [17] is not directly concerned with RH, but does have a section on it. This book is deeper mathematically than others, and is an excellent exposition.

Appendix: Proving the random walk theorem

Consider a uniform random one-dimensional walk in which we start on the *x* axis at the origin and make *N* random unit steps to the left and right with equal probability. (We prove the case where *N* is even.) The following table shows the probability of terminating at a give location.

Steps	-6	-5	-4	-3	-3	-1	0	1	2	3	4	5	6
0							1						
1						1/2		1/2					
2					1/4		2/4		1/4				
3				1/8		3/8		3/8		1/8			
4			1/16		4/16		6/16		4/16		1/16		
5		1/32		5/32		10/32		10/32		5/32		1/32	
6	1/64		6/64		15/64		20/64		15/64		6/64		1/64

Let $d(N)$ denote the expected value of the unsigned distance from the origin after *N* steps. As an example, consider the case of 4 steps. The calculation would be

$$d(4) = 2\left(0\left(\frac{6}{16}\right) + 2\left(\frac{4}{16}\right) + 4\left(\frac{1}{16}\right)\right) = 2\sum_{k=0}^{2}(2k)\frac{\binom{4}{2-k}}{2^4}.$$

The 2 on the left is needed because the probabilities must be doubled since both the same positive and negative distance count as one. The numbers 0, 2 and 4 are the moment arms for the corresponding probabilities. It is easy to see that the generalization of this in the case where *N* is the even number $N = 2n$ is

$$d(N) = d(2n) = 2\sum_{k=0}^{n}(2k)\frac{\binom{2n}{n-k}}{2^{2n}}.$$

We found the sum on the right with Mathematica as $\sum_{k=0}^{n}(2k)\binom{2n}{n-k} = (n+1)\binom{2n}{n-1}$.

But $(n+1)\binom{2n}{n-1} = (n+1)\frac{n}{n+1}\binom{2n}{n} = n\binom{2n}{n} = n\frac{(2n)!}{(n!)^2}$. We have

$$d(N) = d(2n) = \frac{n\ \ (2n)!}{2^{2n-1}(n!)^2}.$$

Now consider *n* approaching infinity. We can make the Stirling asymptotic approximation $x! \sim \sqrt{2\pi x}e^{-x}x^x$ for large *x* and get at once $n\frac{(2n)!}{(n!)^2} \sim 2^{2n}\sqrt{\frac{n}{2\pi}}$ and so since $n = N/2$ we finally have

$$d(N) \sim \sqrt{\frac{2}{\pi}}\sqrt{N} \approx 0.798\sqrt{N}.$$

(In the above, the statement $g(n) \sim f(n)$ means that $\lim_{n\to\infty}\frac{g(n)}{f(n)} = 1$ and we say that *g*(*n*) is asymptotic to *f*(*n*).)

This means that the expected distance from the origin after 1,000,000 random unit steps, each with the same probability of going right or left is about *d*(1,000,000) = 798. We checked this with Mathematica's random number generator. We calculated *d*(1,000,000) one thousand times and found the average to be 769 which is an error of less than 4 percent.

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