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The Bifurcation and Stability of the Solutions of the Boussinesq Equations

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Abstract

The study determined the bifurcation and stability of the solutions of the Boussinesq equations as well as the onset of the Rayleigh-Benard convection. The article established the nonlinear theory for this problem using a new notion of bifurcation known as attractor bifurcation. This article considered the theory that comprises the following three perspectives. We initially deal with the problem that bifurcates from the trivial solution an attractor AR while the Rayleigh number R intersects the first critical Rayleigh number Rc for all physically boundary conditions, despite the multiplicity of the eigenvalue Rc for the linear problem. Hence, secondly, the study considered the bifurcated attractor AR as asymptotically stable. Lastly, the bifurcated solutions are also structurally stable when the spatial dimension is two, and are classified as a bifurcated solution as well. Furthermore, the technical method explained here provides a means, which can be adopted for many different problems in bifurcation and other pattern formation that are related.

Keywords: Rayleigh-Benard Convection, Dynamic bifurcation, Boussinesq equation, Rayleigh number, Attractor bifurcation

Introduction

There are general concerned and a fully understanding in the concept of matter and its compositions. It leads with the fully understood of something flows from a hot bodies/objects to a cold bodies/objects (Kirchgassner, K, 2004). The phenomenon of flows is known as “heat.” During the eighteenth and the early nineteenth centuries many scientific approach revealed that all bodies consists of an invisible fluid within it known to be caloric. Hence this caloric has a variety of properties some of which proved to be inconsistent with nature, for instance it has weight and cannot be created nor destroyed. However, it flows from hot bodies to the cold bodies and this was considered as among the most important feature of it. Therefore, it is important to consider heat as a valuable aspect of live.

In a nutshell thermal convection refers to a specific type of convection phenomena where temperature differences drive a fluid flow. More precisely temperature variations induce an unstable fluid stratification which cause the transition of the fluid from a state of rest to a state of motion (Foias, C., et al. 1987). The fluid flow may undergo much successive instability, which reduce the spatial coherence and the level of predictability of the details of movement progressively. In this case, the flow is called turbulent. Few examples of (turbulent) thermal convection are air circulation, solar granulation, oceanic currents and convective flows in the earth’s mantle and stars. Transport properties of turbulent convective flow are the object of interest and investigation in many field ranging from physical sciences like geophysics, astrophysics, meteorology, and oceanography to engineering and industrial applications (Foias, C., et al.

1987). In this thesis, we are interested in deriving mathematically rigorous bounds for the heat transport when the flow is turbulent. For this purpose, consider a fluid enclosed between two rigid parallel and infinitely extended plates separated by a vertical distance h and held at different temperatures $T = Tbottom$ and $T = Ttop$ at height 0 and h respectively, with $Tbottom > Ttop$. This model of thermal convection goes under the name of Rayleigh-Benard convection.

A fluid heated from the bottom and cooled at the surface in a cylindrical container will cause convection if the temperature difference (ΔT) between the surface and the base plates is at least has a critical temperature difference (ΔTc). The phenomenon above is called Rayleigh-Benard convection, or in short form as RBC. However, convection does not occur in the fluid when $\Delta T < \Delta Tc$, due to viscous and thermal dissipation and will settled in what is called the "conducting" or "uniform" solution. Therefore, whenever ΔT large enough, convection is will occur as the thermal driving force is significant enough to overcome the dissipative effects of thermal conduction and viscosity (Yudovich, V.I. 2013). Convection will only happen when the dimensionless control parameter, the Rayleigh number

$$Ra = \frac{\propto gH^3}{vk}\Delta T \tag{1}$$

reaches a critical value Rac, where α is the thermal-expansion coefficient, g is the acceleration of gravity, H is the fluid layer thickness, ΔT is the temperature difference, κ is the thermal diffusivity, and ν is the kinematic viscosity. Fundamentally the Rayleigh number characterizes the ratio of the undermining buoyancy force $\rho\alpha g\Delta T$ to the steadying dissipative force $\nu\kappa\rho/H3$. We also stated that

$$\in = \frac{Ra - Rac}{Rac} \tag{2}$$

To normalize the degree above threshold; a specific Rayleigh number is for a specific aspect ratio. The dimensionless Prandtl number

$$\mathrm{Pr} = \frac{v}{k} \tag{3}$$

Characterizes the fluid as well as the dimensionless aspect ratio

$$\Gamma \equiv \frac{D}{H} \tag{4}$$

Where D is the diameter and H is the depth of the cylinder, characterizes the geometry.

It is perfect to identify and noted that a complete nonlinear bifurcation and stability theory for this problem must at any rate contain the following aspects:

a) The bifurcation theorem while the Rayleigh number bisected the initial critical number for all the physically boundary conditions,

b) The asymptotic stability of bifurcated solutions, and lastly

c) The pattern or structure and their stability and transitions within the physical space.

The leading difficulties concerning such a complete theory are two-fold. Initially is due to the high nonlinearity of the problem as in other fluid problems, also secondly is due to the lack of an approach to handle bifurcation and stability at the eigenvalue of the linear problem has even multiplicity.

The main aim of this research is to reduce the bifurcation problems to the centre manifold together with an $S1$ attractor bifurcation theorem and structural stability theorem for 2D incompressible flows to achieve the following objectives:

a) To classify the solutions in the bifurcated attractor AR.
b) To study the structure and its transition of the solution of the Benard problem in the physical space.
c) To study the dynamic bifurcation and the structural stability of the bifurcated solutions of the 2-D Boussinesq equations related to the Rayleigh-Benard convection

Methodology

In this article, the method applied in attaining our objectives was outline. Here the study explained the main theorem related with attractor bifurcation states which has the control parameter crosses some specific critical value when there are $m + 1(m \geq 0)$ eigenvalues crossing the imaginary axis.

In the initial stage, the research shows that as the Rayleigh number R crosses the first critical value Rc, the Boussinesq equations bifurcate from the trivial solution an attractor AR, with the dimension between m and $m + 1$. In this regard, the first critical Rayleigh number Rc is stated to be the first eigenvalue of the linear eigenvalue problem, and $m + 1$ is the multiplicity of this eigenvalue Rc. In comparison with known results, the bifurcation theorem achieved in this article is for the whole cases with the multiplicity $m + 1$ ofthe critical eigenvalue Rc for the Benard problem under any set of physically boundary conditions. As the trivial solution becomes unstable as the Rayleigh number crosses the critical value Rc, AR does not contain this trivial solution.

Secondly, being an attractor, the bifurcated attractor AR becomes asymptotic stability meaning that it attracts the whole solutions with original data in the phase space outside of the stable manifold, with co-dimension $m + 1$, of the trivial solution. As Kirchgassner showed that in his paper titled "Bifurcation in nonlinear hydrodynamic stability," an ideal stability theorem would include all physically significant perturbations and establish the local stability of a selected class of stable solutions, and today we are yet far away from this purpose. Moreover, fluid flows are usually time-dependent. Consequently, bifurcation analysis for steady state problems general gives only partial answers to the problem and this is not enough for solving the stability problem. Therefore it appears that the right notion of asymptotic stability after the first bifurcation should be best described by the attractor near, but excluding the trivial state. It is one of our principal motivations for proposing attractor bifurcation, and we hope that the stability of the bifurcated attractor achieved in this article will contribute to an ideal stability theorem. Thirdly, another critical perspective of a complete nonlinear theory for the Rayleigh-Benard convection is to classify the structure or pattern of the solutions after the bifurcation. A standard tool to attack problem mentioned above is the structural or trend stability of the solutions in the physical space. Many kinds of literature have made an extensive study approaching this goal, and set a systematic theory on structural stability and bifurcation of 2-D divergence-free vector fields; as recommended in a survey article titled Topology of 2-D incompressible flows and applications to geophysical fluid dynamics by Tian Ma and Shouhong Wang therein. Applying, in precise, the structural stability theorem explained in (Pazy, A. 1983) This article shows that in the two-dimensional case, for any initial data outside of the stable manifold of the trivial solution, the solution of the Boussinesq equations will have the roll structure as t is adequately large.

In the real sense, the above results for the Rayleigh-Benard convection are achieved using a new notion of dynamic bifurcation, called attractor bifurcation, introduced recently by the Tian Ma and Shouhong Wang therein. The main theorem associated with attractor bifurcation states that as the control parameter crosses a specific critical value when there $are\, m + 1(m \geq 0)$ eigenvalues intersecting the imaginary axis, the system bifurcates from a trivial steady state solution to an attractor with dimension between m and $m + 1$, provided the critical state is asymptotically stable. This new bifurcation theory concludes the aforementioned known bifurcation theories. There are a few relevant features of attractor bifurcation. Initially, the bifurcation attractor does not constitute the trivial steady state, and is stable; hence it is physically meaningful. Secondly, the attractor includes a collection of solutions of the evolution equation, including perhaps steady states, periodical orbits, as well as homoclinic and heteroclinic orbits. Thirdly, it gives a unified point of view on dynamic bifurcation and can be applied to many problems in physics and mechanics. Fourth, from the application point of view, the Krasnoselskii-Rabinowitz theorem needs the number of eigenvalues $m + 1$ crossing the imaginary axis to be an odd integer, and the Hopf bifurcation is for the case where $m + 1 = 2$. Though, the new attractor bifurcation theorem obtained in this article can be employed in cases for all $m \geq 0$.

Also, the bifurcated attractor, as mentioned earlier, is stable, which is another subtle issue for other known bifurcation theorems. Of course, here what we do is the verification of the asymptotic stability of the crucial state, an adjunct to this analysis required for the eigenvalues problems in the linearized problem. The study uses the Theorem 2 to presents a method of obtaining asymptotic stability of the crucial state in problems with symmetric linearized equations. This theorem is great; the asymptotic stability of the trivial solution to the Rayleigh-Benard problem is simply established. This article recommended this theorem as a useful for solving problems in many issues of mathematical physics with symmetric linearized equations. Here organized the research as follows. Firstly, the study recalls of the Boussinesq equations and their mathematical setting, and also identifies some known existence and uniqueness results of the solutions. The study was summaries in the next section where the main attractor bifurcation theory from as discussed in the literature entitled "Dynamic Bifurcation of Nonlinear Evolution Equations" in (Tian Ma and Shouhong Wang, 2004), and a theorem, in the next section this article will use Theorem 2, for the asymptotic stability of the critical state for problems for an evolution system with symmetric linearized equations. States and proves the main attractor bifurcation results from the Raleigh-Benard convection. In the second to the last section, this article considered Examples and topological structure of the bifurcated solutions. Hence the latter part presented the corresponding results for the two-dimensional problem. In the Appendix in Section 7 and the concept and main findings on the structural stability of 2-D divergence-free vector fields are recalled.

Preliminary Results
(Boussinesq Equations and Their Mathematical Setting)

Boussinesq Equations

The Boussinesq equations model is the large scale atmospheric and oceanic flows that are responsible for cold fronts and the jet stream.

$$\frac{\partial u}{\partial t} + (u\,.\nabla)\,u - \nu\Delta u + \rho_0{}^{-1}\nabla p = -gk[\,1 - \alpha\,(T - T_0)] \qquad (5)$$

$$\frac{\partial T}{\partial t} + (u\,.\nabla)\,T - k\Delta T = 0 \tag{6}$$

$$div\,u = 0 \tag{7}$$

Where v, k, α, g are all constant, and $u = (u_1, u_2, u_3)$ is the velocity field, p is the pressure function, T is the temperature function, T_0 is the constant representing the lower surface temperature at $x_3 = 0$ and $k = (0, 0, 1)$ where the unit vector is in the x-direction.

Therefore, to make the equations non-dimensional, we consider the following relations:

$$x = hx',$$

$$t = h^2 t'/k,$$

$$u = ku'/h,$$

$$T = \beta h\left(\frac{T'}{\sqrt{R}}\right) + T_0 - \beta h x_3',$$

$$p = \rho o k^2 p'/h^2 + po - gpo(hx_3' + \alpha\beta h^2 (x_3')^2/2,$$

$$p_r = v/k$$

At this point the Rayleigh number R is defined by equation (5), and $p_r = v/\kappa$ is the Prandtl number. Omitting the primes, the equations (6) to (8) can be modified as the format below

$$\frac{1}{p_r}\left[\frac{\partial u}{\partial t} + (u \cdot \nabla\,)u\ + \nabla p\right] - \Delta u - \sqrt{R}Tk = 0 \tag{8}$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla\,)T - \sqrt{Rv_3} - \Delta T = 0, \tag{9}$$

$$div\,v = 0. \tag{10}$$

Therefore, the non-dimensional domain in this regard is $\Omega = D \times (0,1) \subset R_3$, in which the relation $D \subset R_2$ is an open set. And the coordinate system is specified by $x = (x_1, x_2, x_3) \in R^3$.

The Boussinesq equations given in equations (8) to (10) are the basic equations to study the Rayleigh B´enard problem given in this paper. They are complemented with the subsequent initial value conditions:

$$(u, T) = (u_0, T_0) \qquad at\ t = 0 \tag{11}$$

Hence, boundary conditions are required both at the top and the bottom as well as the lateral boundary $\partial D \times (0,1)$, and the top and bottom boundary will be $(x_3 = 0\,,1)$, whichever the so-called rigid or free boundary conditions are given:

$$T = 0, u = 0 \ (rigid\ boundary), \tag{12}$$

$$T = 0, \quad u_3 = 0, \ \frac{\partial(u_1, u_2)}{\partial x_3} = 0 \ (free\ boundary). \tag{13}$$

Usually different combinations are used at the top and bottom boundary conditions in different physical setting such as the system of rigid-rigid, rigid-free, free-rigid, as well as free-free. And also for the lateral boundary $\partial D \times [0,1]$, is usually used one of the following boundary conditions as in (Tian Ma and Shouhong Wang, 2004):

1. The Periodic condition:

$$(u,T)(x_1 + k_1L_1, x_2 + k_2L_2, x_3) = (u,T)(x_1, x_2, x_3) \tag{14}$$

For any $k_1, k_2 \in Z$.

2. The Dirichlet boundary condition:

$$u = 0, \ T = 0 \ \left(Or \ \frac{\partial T}{\partial n} = 0\right); \tag{15}$$

3. The Free boundary condition:

$$T = 0, \ u_n = 0 \ \left(Or \ \frac{\partial u_r}{\partial n} = 0\right), \tag{16}$$

In which n and τ are the unit normal and tangent vectors on $\partial D \times [0,1]$ correspondingly, and $u_n = u \cdot n, u_r = u \cdot \tau$.

For minimalism, here the study will proceed with the following set of boundary conditions, and the whole results hold true even for other combinations of boundary conditions.

$$\begin{cases} T = 0, \ u = 0 \qquad at\ x_3 = (0,1) \\ (u,T)(x_1 + k_1L_1, x_2 + k_2L_2, x_3\ t) = (u,T)(u,t) \end{cases} \tag{17}$$

For any $k_1, k_2 \in Z$.

Dynamic Bifurcation and Stability in the Rayleigh-Benard Convection

The main contribution here is to attempt to establish a nonlinear theory for the Rayleigh-Benard convection by a new notion of bifurcation, called attractor bifurcation, and the corresponding theories in the literature (Tian Ma and Shouhong Wang 2002; 2004) all these followed the three features of a complete theory for the problem just mentioned along with the main idea with the methods to be used.

The Dynamic bifurcation for the nonlinear progression equations:

In this part, we have to recall some results from the literature for the dynamic bifurcation of abstract nonlinear evolution equations which established many researchers especially the one published by the authors in (T. Ma and S. Wang, 2004), which is essential in the study of the Benard problem in this thesis. Indeed, in this section this study intends to provide a formula or method for proving dynamic bifurcations for complications that deal with symmetric linear operators.

Attractor bifurcation

Consider H and $H1$ to be two Hilbert spaces, and $H1 \rightarrow H$ to be a compressed and compact insertion. The study considered the resulting nonlinear evolution equations

$$\frac{\delta y}{\delta x} = L_{\beta Y} + G(Y, \beta) \tag{18}$$

$$Y\,(0) = Y_0 \tag{19}$$

Where Y: (0,∞) ⟶ H and referred as the unknown function, β ϵ as the system parameter, however, $L_\beta : H1 \rightarrow H$ are parameterized linear completely continuous fields continuously depending on $\lambda \in R1$, which satisfy the following equations:

$$\begin{aligned} &L_\beta : -A + B_\beta \ldots\ldots\ldots \text{is a sectorial operator,} \\ &A: H1 \rightarrow H \ldots\ldots\ldots \text{is a linear homeomorphism,} \\ &B_\beta : H1 \rightarrow H \ldots \text{is the parameterized linear compact operators} \end{aligned} \tag{20}$$

It is easy to identify as in some literatures (D. Henry, 1981; A. Pazzy, 1983) that $L\beta$ generates an analytic semi-group as $\{e^{tL\beta}\}t \geq 0$. Hence, we can define fractional power operators $L_\beta^\propto$ for any $0 \leq \propto \leq 1$ with its domain $H_\propto = D(L_\beta^\propto)$ such that $H_{\propto 1} \subset H_{\propto 2}\ if\ \alpha 1 > \alpha\ 2$, and $H_0 = H$.

Moreover, this study will adopt that the nonlinear terms $G(y, \beta): H_\propto \rightarrow H$ for some $1 > \alpha \geq 0$ are the family of parameterized C^r bounded operator ($r \geq 1$) constantly depending on the parameter $\lambda \in R_1$, in which

$$G(y, \beta) = 0(\|y\|H_\propto) \tag{21}$$

In the applications, this study have interested in the sectorial operator $L_\beta = -A + B_\beta$ in which a real eigenvalue sequence there exist $\{\rho k\} \subset R_1$ and an eigenvector sequence of $\{ek\} \subset H_1\ of\ A$:

$$\begin{aligned} &A_{ek} = \rho k^e k \\ &0 < \rho_1 < \rho_2 < \cdots \ldots. \\ &\rho k \rightarrow \infty\ (k \rightarrow \infty) \end{aligned} \tag{22}$$

In which $\{ek\}$ is an orthogonal basis of H.

Therefore, for the compact operator $B_\beta : H_1 \rightarrow H$, the study will also presume that there will be a constant $0 < \theta < 1$ such that

$$B_\beta : H_\theta \rightarrow H \text{ Bounded}, \forall \lambda \in R_1 \quad (23)$$

Let consider $\{S_\beta(t)\}t \geq 0$ to be an operator semi-group created by the equation (1) which delight in the properties.

For any $t \geq 0$, $S_\beta(t): H \rightarrow H$ is a linear continuous operator, $S_\beta(0) = I : H \rightarrow H$ is the identity on H, and

Hence, for any t, $s \geq 0$, $S_\beta(t + s) = S_\beta(t) \cdot S_\beta(s)$

Therefore, the solution of equation (1) and equation (2) can be articulated as

$$y(t) = S_\beta(t) y_0, \quad t \geq 0.$$

Conclusion

In this article we tried as we mentioned before to clarify the structure of the eigenvectors of the linearized problem in which this study plays an important role and studying the onset of the Rayleigh-B´enard convection. The dimension $m + 1$ of the eingenspace E_0 regulates the dimension of the bifurcated attractor AR as well. However, the thesis will further explain by examine the detail of the first eingenspace for different geometry of the spatial domain and for different geometry and boundary condions.

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