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A Validation for Dirichlet Problem

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Abstract

Euler method is often referred method because of ease of use and usefulness for ordinary differential equation. Usefulness is arises from simplicity and error analyses comfortableness. It is convenient for a second order differential equation because of reducibility. It is also used for Dirichlet problem. Analytical solution is compared with numerical approach. Analytical solution, numerical approach and error analyses are together reconfigured. Analytical solution is done Evolutionary Approach to Electromagnetics' frame. Despite the fact that the approach is to be required solution of Klein-Gordon equation, it is just validated with 0.02 step Dirichlet problems' numerical approach and analytical solution.

Keywords: Dirichlet boundary-value problem, analytical solution, numerical solution, Maxwell equations, TM time-domain mode

Introduction

Before starting with Euler method, substructure that will be solution space of analytical solution must be prepared. This preparation is studied in [1-5] with details so that we will mention shortly. $\varphi_m(\mathbf{r})$ potential in Maxwell equations is presented as κ_m eigen functions that are set of eigenvalue series but is not with additive to electromagnetic field.

3-component position vector $\mathbf{R} = \mathbf{x}x + \mathbf{y}y + \mathbf{z}z$ and the operator $\nabla = \mathbf{x}\partial_x + \mathbf{y}\partial_y + \mathbf{z}\partial_z$ onto their projections on the waveguide cross-section S and $\mathbf{z}$-axis,

$$\mathbf{R} = \mathbf{r} + \mathbf{z}z, \qquad \nabla = \nabla_\perp + \mathbf{z}\partial_z \tag{1}$$

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where *z is Oz* axis unit vector, **r** is the 2-component position vector in the domain S and $\nabla_\perp$ is latitudinal part of ∇. The differential operator $\nabla_\perp$ acts on the latitudinal waveguide coordinates (**r**) only. 3-component electromagnetic field vectors $\mathbf{E}$ and $\mathbf{H}$ are presented as the sum of 2-component and 1-component vectors.

$$\mathbf{E}_m(\mathbf{R},t) = \mathbf{E}(\mathbf{r},z,t) + \mathbf{z}E_z(\mathbf{r},z,t)$$
$$\mathbf{H}_m(\mathbf{R},t) = \mathbf{H}(\mathbf{r},z,t) + \mathbf{z}H_z(\mathbf{r},z,t) \tag{2}$$

The boundary conditions,

$$(\mathbf{n}\cdot\mathbf{H})_{|L} = 0, \qquad (\mathbf{l}\cdot\mathbf{E})_{|L} = 0, \qquad (\mathbf{z}\cdot\mathbf{E})_{|L} = 0 \tag{3}$$

where **n** is normal unit vector, **L** is tangent unit vector and **z** is axial coordinate. In this study Dirichlet problem for *TM* time-domain modes is only argued.

Model and Analysis

TM Time-Domain Mode

Dirichlet problem for *TM* time-domain modes are:

$$\left(\nabla_\perp^2 + \kappa_m^2\right)\varphi_m(\mathbf{r}) = 0$$
$$\varphi_m(\mathbf{r})_{|L} = 0 \tag{4}$$
$$\frac{\kappa_m^2}{S}\int_S \left|\varphi_m(\mathbf{r})\right|^2 ds = 1$$

where $\kappa_m^2 > 0, m = 1,2,....$ eigenvalues, *m* index is numerical values that are sequenced with increasing array in real axis, $\varphi_m(\mathbf{r})$ are eigenvectors corresponding to these eigenvalues. $\varphi_0(\mathbf{r})$ solution is zero that is corresponding to $\kappa_0^2 = 0$ eigenvalue. We accepted as rectangle S cross-section of waveguide. Homogeny Helmholtz equation is written as; $\left(\nabla_\perp^2 + \kappa_m^2\right)\varphi_m(\mathbf{r}) = 0$,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \kappa_m^2\right)\varphi_m(\mathbf{r}) = 0 \tag{5}$$

and if $\varphi_m(\mathbf{r})$ is separated its variables like $\varphi_m(\mathbf{r}) = X(x)Y(y)$ we can rewrite (5) equation;

$$Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} + \kappa_m^2 X(x)Y(y) = 0 \tag{6}$$

Then, if two part of the equation (6) is multiplied with $1/X(x)Y(y)$ we can rewrite (6) equation;

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} + \kappa_m^2 = 0 \tag{7}$$

If (7) equation is separated its variables like $\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -k_x^2$ and $\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -k_y^2$ we can obtain

$$\kappa_m^2 = k_x^2 + k_y^2 \tag{8}$$

eigenvalue series. Ultimately we can obtain κ_m eigenvalues set from (8) equation easily. In here somebody can ask why k_x^2 and k_y^2 constants are chosen negative. If k_x^2 and k_y^2 arbitrary constant were positive solution of equation (6) would be $\exp(\pm ik_x x)$ and $\exp(\pm ik_y y)$. So we would obtain $\varphi_m(\mathbf{r})$ as $\varphi_m(\mathbf{r}) = \exp(\pm ik_x x)\exp(\pm ik_y y)$. But these exponential functions involve divergences. Though our problem is solved under conditions of Evolutionary Approach to Electromagnetics Theory (EAE) so our solution must be convergent [6].

The solution of (7) equation is obtain as equation (9), $\varphi_m(\mathbf{r})_{|L} = 0$ under boundary conditions (*L* surface is rectangle)

$$\varphi_m(\mathbf{r}) = A_m^e \sin(p\pi x/a)\sin(q\pi y/b) \tag{9}$$

where *p* and *q* parameters are integers, $p + q \neq 0$ and $p, q = 0, 1, 2, ...$ A_m^e is normalization coefficient. From second line of Eq. (4) we obtain κ_m eigenvalues set as

$$\kappa_m^2 = \pi^2\left(p^2/a^2 + q^2/b^2\right) \tag{10}$$

Third line of Eq. (4) should be solved for A_m^e. In $\frac{\kappa_m^2}{S}\int_S |\varphi_m(\mathbf{r})|^2 ds = 1$ normalization integral if we accept $S = ab$ and

$$\frac{\kappa_m^2 \left(A_m^e\right)^2}{ab} \int_{x=0}^{x=a} \sin^2\left(p\pi x / a\right) dx \int_{y=0}^{y=b} \sin^2\left(q\pi y / b\right) dy = 1 \tag{11}$$

after necessary cancellation in Eq. (11), A_m^e normalization coefficient is obtain as

$$A_m^e = 2 / \kappa_m \tag{12}$$

$\varphi_m\left(\mathbf{r}\right)$ potential is a part of *TM* time-domain modes.

$$\begin{aligned} &H_{zm}^e = 0 \\ &\kappa_m^{-1} H_m^e = \left\langle -\partial_{(\kappa_m ct)} e_m\left(z,t\right) \right\rangle \left[z \times \sqrt[-2]{\mu_0} \nabla_\perp \varphi_m\left(\mathbf{r}\right) \right] \\ &\kappa_m^{-1} E_m^e = \left\langle \partial_{(\kappa_m z)} e_m\left(z,t\right) \right\rangle \left[\sqrt[-2]{\varepsilon_0} \nabla_\perp \varphi_m\left(\mathbf{r}\right) \right] \\ &\kappa_m^{-1} E_{zm}^e = \left\langle e_m\left(z,t\right) \right\rangle \left[\kappa_m \sqrt[-2]{\varepsilon_0} \varphi_m\left(\mathbf{r}\right) \right] \end{aligned} \tag{13}$$

where $\partial_{(\kappa_m ct)} = \dfrac{1}{c\kappa_m}\dfrac{\partial}{\partial t}, \partial_{(\kappa_m z)} = \dfrac{1}{\kappa_m}\dfrac{\partial}{\partial t}$ and $c = \sqrt[-2]{\varepsilon_0 \mu_0}$ is velocity of life in space.

Weyl Teoremi says that *TM* time-domain modes are complete in L_2 Hilbert space [7]. Amplitudes composing fields are specifies by a $e_m\left(z,t\right)$ that are obtained from Klein-Gordon equation.

$$\left(\partial_{\upsilon_m ct}^2 - \partial_{\upsilon_m z}^2 + 1\right) e_m\left(z,t\right) = 0 \tag{14}$$

In this study we will not mention $e_m\left(z,t\right)$ amplitudes that are obtained from the solution of Klein-Gordon equation.

***Notice 1*:**

For every *m* index that has constant value, (4) time-domain mode provides boundary conditions.

***Notice 2*:**

Eq. (4) is a specific solution of Maxwell equations systems for every time-domain $\left(\mathbf{E}_m, \mathbf{H}_m\right)$ modal field ∂_t that ensures time derivative:

$$\nabla \times \mathbf{E}_m = -\mu_0 \partial_t \mathbf{H}_m \text{ and } \nabla \times \mathbf{H}_m = \varepsilon_0 \partial_t \mathbf{E}_m \tag{15}$$

∂_t must be calculated that is suitable for time-domain modal fields special relativity theory as a solution of Maxwell equation that is protect time derivative. Electric and magnetic fields in the Maxwell equations are not studied in this study.

Numerical Approach

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \tag{16}$$

i. $f(x,y)$ function is defined and continuous $x_0 \le x \le b$ and $-\infty \le y \le +\infty$ intervals, where x_0 and b finite values,

ii. $f(x,y)$ function provides Lipschitz condition for any x values in the $[x_0,b]$ closed interval. Let's take Eq.**(16)** differential equation. $h = \Delta t$ is a step in the $[x_0,b]$ closed interval, y_i is numerical solution at the t_i step, Y_i is an analytical solution, $E_{i+1}$is a truncation error from t_i step to $t_{i+1}$and $e_i$is total error in the *i*. step: we can rewrite Euler equation as [8,9];

$$y_{i+1} = y_i + hf(y_i,t_i) \tag{17}$$

$$y^*_{i+1} = y_i + hf(Y_i,t_i) \tag{18}$$

$$E_{i+1} = -\frac{h^2}{2} f'(Y_i,u_i) \tag{19}$$

$$|e_i| \le \frac{f'(Y_i,u_i)h}{2\frac{\partial f}{\partial y}(z_i,t_i)}\left[\left(1 + h\frac{\partial f}{\partial y}(z_i,t_i)\right)^i - 1\right] \tag{20}$$

Where z_i is any value among, y_i, Y_i and $u_i \in (t_i,t_{i+1})$. In the Eq.**(7)** if y is constant we can obtain a new equation;

$$\frac{1}{X(t)}\frac{\partial^2 X(t)}{\partial t^2} = -k_t^2 \qquad \text{or} \qquad \frac{\partial^2 X(t)}{\partial t^2} + k_t^2 X(t) = 0 \tag{21}$$

$X(t) = \sin\frac{p\pi t}{a}$ is an analytical solution of Eq.(21).

In this study, privately; we take $p,a \in \mathbb{Z}$ values as $p = 1$, $a = 1$. We can obtain that $k_t = \pi$. Under these conditions we rewrite Eq.**(21)** as,

$$X'' + X\pi^2 = 0 \tag{22}$$

Eq. (22) is converted to a second order linear differential equation with constant coefficients. We solved Eq. (22) at the $t = 0$, $X(0) = 0$, $\frac{dX}{dt} = u(t)$, $u(0) = \pi$ under beginning conditions with $h = 0.02$ step length for $t = 1$we solved our problem numerical with Euler method.

Before solution under conditions that are given Eq. (21) and Eq. (22)are like that,

$$\begin{aligned} &t = t_0 = 0 \\ &u(0) = u_0 = \pi \\ &x_{i+1} = x_i + u_i.h \\ &u_{i+1} = u_i + h.(-x_i.\pi^2) \\ &x^*_{i+1} = x_i + u_i.h \end{aligned} \tag{23}$$

If we solve Eq. (22) with Eq. (23) couple by numerical methods, we can obtain equations below;

for $i = 0$ $\quad x_1 = x_0 + u_0.h \quad u_1 = u_0 + h.(-x_0.\pi^2)$

for $i = 1$ $\quad x_2 = x_1 + u_1.h \quad u_2 = u_1 + h.(-x_1.\pi^2)$

for $i = 2$ $\quad x_3 = x_2 + u_2.h \quad u_3 = u_2 + h.(-x_2.\pi^2)$

$\vdots$

for $i = 49$ $\quad x_{50} = x_{49} + u_{49}.h \quad u_{50} = u_{49} + h.(-x_{49}.\pi^2)$

If all numeric calculations are made with MATLAB2019a, we obtain values of x_i, $x^*_i$and u_i like below;

Comparison of Analytical Solution and Numerical Approach

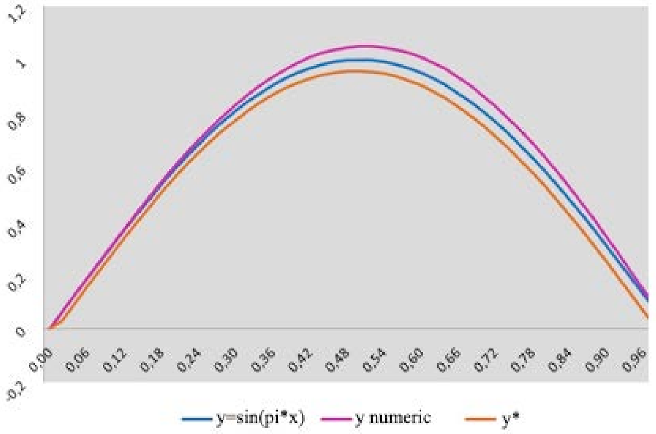

Figure 1 (Obtained with Table 1 values.)

Numerical Approach of Equation

x	y=sin(pi*x)	y numeric	y*
0	0	0	0
0,02	0,06279052	0,06283185	0,03141593
0,04	0,12533323	0,12566371	0,09420645
0,06	0,18738131	0,18824751	0,1564393
0,94	0,18738131	0,20973603	0,12615814
0,96	0,12533323	0,1420801	0,06362233
0,98	0,06279052	0,07359618	0,00064956
1	0	0,00455134	-0,06251165

Table 1 Calculated with MATLAB

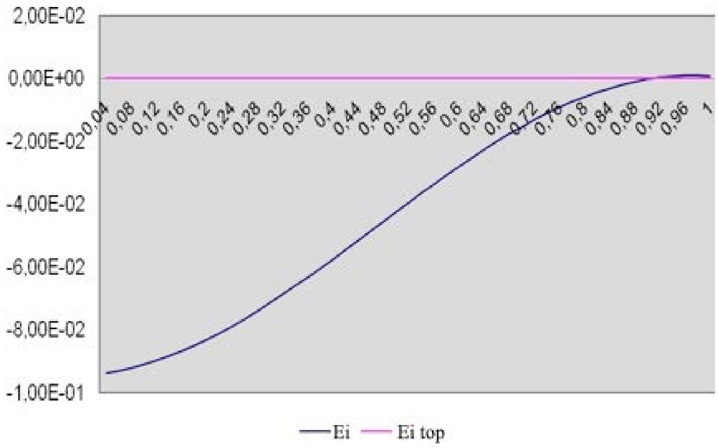

Figure 2 (Obtained with Table 2 values.)

Error Analysis

x	E_i	E_i top
0,04	-9,39E-02	1,57E-04
0,06	-9,32E-02	1,57E-04
0,08	-9,23E-02	1,56E-04
0,96	8,25E-04	1,48E-05
0,98	8,32E-04	9,86E-06
1	6,50E-04	4,93E-06

Table 2 Calculated with MATLAB

E_i shows the error numerator in error analysis. E_i top shows the top bound of error.

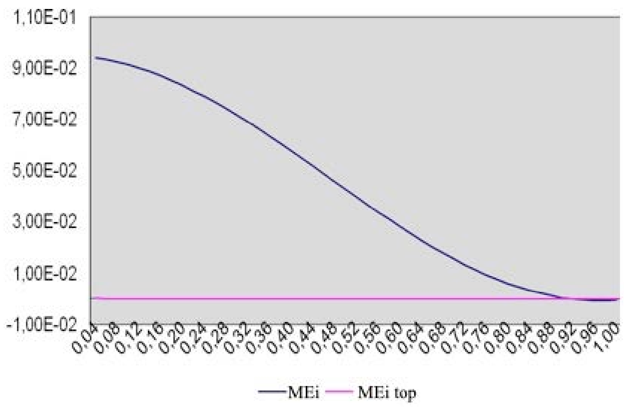

Figure 3 (Obtained with Table 3.)

Absolute Error Analysis

x	MEi	MEi top
0,04	9,39E-02	-1,57E-04
0,06	9,32E-02	-1,57E-04
0,08	9,23E-02	-1,56E-04
0,96	-8,25E-04	-1,48E-05
0,98	-8,32E-04	-9,86E-06
1	-6,50E-04	-4,93E-06

Table 3 Calculated with MATLAB

Conclusions

In this study a validation is observed. Along (x, y) axis firstly elliptic Helmholtz equation is solved analytical, secondly in this solution y is taken constant, studied for numerical values for x only and a comparison is made between analytical solution and numerical solution. This comparison gives a validation with error analysis to readers. We can comment visually thanks to figures.

Surely if we approach along (x, y) axis, we can apply lots of numeric method for two independent coefficients like x and y. These will be studied later. In addition to these discussions more serious validations are possible with Finite Difference in Time Domain (FDTD).

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