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The Benders Decomposition for the Dual of the b- Complementary Multi-Semigroup Problem.

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Abstract

Multi-valued Additive Systems defined by Aráoz and Johnson in 1982, these finite algebraic structures are a generalization of finite groups and semigroups. A particular case of these systems are the b-complemetary multisemigroups. In 1980 Johnson studied the dual primal problem over a semigroup, and in 1985 Aráoz and Johnson presented a study that classifies the polyhedron associated with an additive system, a study that features vertices and faces of this polyhedron. Madriz in 2016 presents the duality results for the primal problem over a b-complementary multisemigroup. In this work, we show that systems of two different bases of the cone associated with an integer linear programming problem under a b-complementary Multisemigroup are equivalent. We also present the decomposition of Benders for the dual problem of the a b-Complementary Multisemigroup.

Keywords:Additive System; Multisemigruop; b-Complementary; Duality; Benders Decompostion

1 Introduction

The Group Problem (GP) was defined by Gomory in [8]. Gomory's work is grounded on the idea that the solution in the linear system of equations associated with an integer linear programming problem can be transformed into a system of an equation involving elements of a finite abelian group. In the same year, Gomory presents, using the subadditive cone the characterization of the vertices and faces associated to GP. Aráoz in [1] define the Semigroup Problem (SP), characterizes the polyhedra and shows the relation between minimal system of linear inequality of the polyhedra and extreme points and rays. Ellis Johnson, in [9], considers the dual of the master semigroup problem. Aráoz and Johnson, in [3], present the polyhedra of multivalued additive system problem. Aráoz and Johnson in [6] use bases of the subadditive cone to characterize or define polyhedron system associated with Multivalued Additive System (MAS).

A particular case of multivalued additive systems is the b-complementary Multisemigroups (b-CMS). In general, a b-CMS is an associative, an abelian, a b-consistent and a b-complementary MAS. Madriz, in [10], constructs the dual problem associated with a b-CMS problem, extending the duality result of semigroup by Johnson in [3], in this work are presented the conditionsto demonstrate the duality theorem for this type of combinatorial optimization problems. Madriz's work [10] is based on the theorem presented by Aráoz and Johnson in [5], where they determine that, given a base of the subadditive cone, it is possible to establish a system of equations and inequalities that define the polyhedron associated with a multivalued associative additive system. However, this result does not show what happens to the system for different bases. So, it this outcome lead us to the following question: If we have two bases of the cone, are the systems that they generate equivalent?. In general the dual problem is defined from a base of the subadditive cone of the b-CMS problem, for this dual problem in this work we present the decomposition of Benders.

In this work, we show that the systems associated with two different bases of the subadditive cone for the b-CMS Linear Programming Problem (b-CMSLIP) are equivalent. In addition, we present the dual problem associated with b-CMLIP and the Benders decomposition of this dual problem.

The paper was divided as follows. In Section 2 we present the definitions of additive system and the b-complementary multisemigroups. In the Section 3 we present the definition of a dual problem and the bases of the cone that we will use in addition to the basic definition of the subadditive cone and the optimization problem over an b-complementary multisemigroup. In Section 4, we prove that for two bases of the subadditivity cone the systems they define are equivalent. Finally, in Section 5 we present the decomposition of Benders for the dual problem associated with the problem of b-complementary multisemigroup dual problem.

2 Multisemigoup b-complementary

2.1 Additive System

An **additive system** is defined to be a non-empty finite set A together with addition $\widehat{+} : 2^A x 2^A \to 2^A$ $(2^A = \{H : H \subset A\})$ such that:

(i) $\{g\}\widehat{+}\{h\} \subseteq A$, for all g and h in A;

(ii) $S\widehat{+}T = \cup_{s\in S, t\in T}(\{s\}\widehat{+}\{t\})$, for all $S, T \subseteq A$.

In this paper we denoted $\{s\}\widehat{+}\{t\}$ by $s\widehat{+}t$ and an additive system by the pair $(A, \widehat{+})$. And, the additive system $(A, \widehat{+})$ is **associative** if satisfies $(S\widehat{+}T)\widehat{+}U = S\widehat{+}(T\widehat{+}U)$, for all $S, T, U \subseteq A$, and it's **abelian** if $S\widehat{+}T = T\widehat{+}S$, for all $S, T \subseteq A$. We assume, without loss of generality, that there exists an element $\widehat{0} \in A$ such that $\widehat{0}\widehat{+}g = g\widehat{+}\widehat{0} = g$, for all $g \in A$, if there isn't element in A that fulfills this property, we could adjoin one element to A without changing $(A, \widehat{+})$, the $\widehat{0}$ is clearly unique. Besides that, we assume that the additive system has at most one infeasible element denoted by $\widehat{\infty}$, and we denote by A_+ the set of proper elements in $A - \{\widehat{0}, \widehat{\infty}\}$.

One **expression** of an additive system $(A, \widehat{+})$ is defined recursively by

- The null string ξ is an **expression** called the **null expression**
- The string (g) is called a **primitive expression**;
- The string $E = (E_1\widehat{+}E_2)$ is an **expression**, where E_1 and E_2 are called a **non-null subexpressions** of E.

The **evaluation** is a function $\gamma : \widehat{E} \to 2^A$ defined by

- $\gamma(\xi) = \{\widehat{0}\}$;
- $\gamma((g)) = \{g\}$, for all $g \in A$;
- $\gamma((E_1\widehat{+}E_2)) = \gamma(E_1)\widehat{+}\gamma(E_2)$.

Let E be an expression of the additive system $(A, \widehat{+})$. A vector $t \in \mathbb{Z}_+^A$ is called the **incidence vector** of E if and only if $t(g) \neq 0$ is the number of times with (g) appears as primitive expression in E. $t \in \mathbb{Z}_+^A$ **represent** $g \in A$ if there are an expression E such that t is the incidence vector of E and $g \in \gamma(E)$. An expression E is called the **solution expression** for $b \in A$, if $b \in \gamma(E)$, and the vector $t \in {\mathbb{Z}_+}^A$ is called a **solution vector** for b if there are an expression E such that: E is a solution expression for b and t is an incidence vector of E.

2.2 Multisemigroup.

Let $(A, \widehat{+})$ an abelian, associative additive system. For $g \in A$ and $k \in \mathbb{Z}_+^*$ we denote by E_g^k the set of all expression E where (g) is the only primitive expression to appear k times in E. Now, let γ be an evaluation and $g \in A$, γ_g^k the set $\{\gamma(E) : E \in E_g^k\}$. Since $(A, \widehat{+})$ is an abelian and associative additive system, γ_g^k is a single set, we denoted by $k.g$ this element.

2.3 Loops

We assume, without loss of generality, that $0.g = \widehat{0}$ and $1.g = g$. Now, since there are only a finite number of subsets of A in the sequence

$$0.g, 1.g, 2.g, \ldots, k.g, \ldots$$

there are sets which appear infinitely many times, such sets are called **loop sets of** g.

The **loop of** g is the union of all the loop sets of g. We define g **goes to** ϕ and write $g \to \phi$ when the loop of g is empty, otherwise we write $g \longleftarrow \phi$.

Let $(A, \widehat{+})$ be an abelian associative additive system and $b \in A_+$. $(A, \widehat{+})$ is **b-consistent**, if and only if $b \in b \widehat{+} kg$, for all $g \in A$ such that $g \longleftarrow \phi$ for all $k \in \mathbb{Z}_+$.

A abelian, associative additive sytem $(A, \widehat{+})$ is a **multisemigroup** if it's g-consistent for all $g \in A$.

2.4 b-complementary

Let $(A, \widehat{+})$ be a multisemigroup and $b \in A$. We define

$$b \sim g = \{x \in A : b \in x \hat{+} g\}.$$

These sets induce a partial order in A, we say $g \tilde{<} h$ when $b \sim g \subseteq b \sim h$. When the set $b \sim g$ has a minimum element, this minimum element is called b-**complement** of g and is denoted by $\widehat{g}$. A multisemigroup A is called **b-complementary** when every element has a b-complement.

An element $g \in A$ is **infeasible** whenever there is not solution of the equation $b \in g \hat{+} x$, that is , $b \sim g = \emptyset$.

3 Dual, Bases and The Optimization Problem.

3.1 The Dual Problem

In this work we will use the following formulation of duality.

Let the linear programming problem

$$\min \widetilde{c}x \tag{1}$$
$$\text{s.t: } \widetilde{A}x = \widetilde{b} \tag{2}$$
$$\widetilde{E}x \geq \widetilde{h} \tag{3}$$
$$x \geq 0 \tag{4}$$

where x and $\widetilde{c}$ are n vector, $\widetilde{b}$ is an m vector and $\widetilde{h}$ is a p vector, $\widetilde{A}$ is an $m \times n$ matrix and $\widetilde{E}$ is a $p \times n$ matrix $(n, m, p \in \mathbb{N}^*)$. Corresponding to this problem, called *primal problem*, consider the following linear problem

$$\max \widetilde{\pi}\widetilde{b} + \widetilde{\mu}\widetilde{h} \tag{5}$$
$$\text{s.t: } \widetilde{\pi}\widetilde{A} + \widetilde{\mu}\widetilde{E} \leq \widetilde{c} \tag{6}$$
$$\widetilde{\pi} \text{ unrestricted, and } \widetilde{\mu} \geq 0 \tag{7}$$

where $\widetilde{\pi}$ and $\widetilde{\mu}$ are row vetor of size m and p, respectively. The problem defined by (5) - (7) is called the **dual problem** of the primal problem. (see 2.5 in [12]).

3.1.1 Bases of the Convex Cone

Let C be a closed convex cone in $\mathbb{R}^n$, we denote L_C the linearity of C,

$$L_C = \{x \in C : -x \in C\}.$$

We extend the definition of **extreme point** to mean that $x \in C$ is an extreme point if $x = x^1 + x^2$, both x^1 and x^2 belong to C imply

$$x^i = \alpha_i x + l^i,$$

$\alpha_i \leq 0$, $l^i \in L_C$ for either, (and hence both) $i = 1$ or $i = 2$.

When $L_C = \{\overrightarrow{0}\}$ an extreme point is any vector on an extreme ray of C. But, a non-zero linearity is present ($L_C \neq \{\overrightarrow{0}\}$), any vector in the linearity is extreme, and adding a vector in L_C to an extreme point gives another equivalent extreme point.

In general, intersecting the cone with the orthogonal complement of the linearity gives a pointed cone generated by non-negative combinations of its extreme rays. The original cone is generated by linear combinations of a basis of the linearity plus non-negative combinations of the extreme rays. In terms of the original cone, we do not have extreme rays, but instead, we might say, extreme half-subspaces of dimension two or higher. These extreme half-subspaces can be formed as an extreme ray plus the linearity. Any vector in such an extreme half-subspace is an extreme vector, and C is equal to the non-negative combinations of its extreme vectors.

When C has a linearity L_C, this linearity form a vector subspace of R^n, hence has a finite basis. The extreme vectors can be taken module de linearity L_C, i.e., two extreme x, y are equivalents if one is a positive multiple of the other plus a vector in the linearity, in this case we write $x \sim y$ when x is equivalent to y. Being polyhedral for C means that, in this sense, there are a finite number of non-equivalent extreme vectors only.

A basis (E, B) of C are two disjoint sets contained in C such that B is a basis of L_C an E is a set of pair wise non-equivalent extreme points such that for any extreme point not in L_C there is a point equivalent to it in E. In this case we have $C = cone(E) + lin(B)$, where $cone(E)$ is the cone generated by E and $lin(B)$ is the subspace generated by B(recall that $cone(\emptyset) = lin(\emptyset) = \{\overrightarrow{0}\}$) and (E, B) is a minimal representation of C.

When $L_C = \{\overrightarrow{0}\}$, E correspond to a unique set of rays, in general the elements of E are one to one equivalents to the unique basis of the intersection of C with the orthogonal complement of the linearity of C ([12]).

3.2 Subadditivity Cone.

Let $(A, \widehat{+})$ be a additive system, the function $\pi : A \to R$ is **subadditivity** if satisfies:

(i) $\pi(\emptyset) = -\infty$;

(ii) $\pi(G) = \max\{\pi(g) : g \in G\}$ for all $G \subseteq A$;

(iii) $\pi(\left\{\widehat{0}\right\}) = 0$;

(iv) $\pi(G) + \pi(H) \geq \pi(G\widehat{+}H)$ for all $G, H \subseteq A$.

The **Subadditivity Cone** is the set

$$C(A) = \{(\pi(g); g \in A_+) : \pi \text{ is a subadditivity function } \}$$

We denote the linearity of $C(A)$ by $L(A)$, and $\pi(\{g\})$ by $\pi(g)$.

3.3 The Optimization Problem

Let $(A, \widehat{+})$ be a b-complementary multisemigroup and $M \subseteq A_+$. The **multisemigoup b-complementary problem** is

$$min \sum_{g \in M} c(g)t(g)$$

$$\text{s.t: } b \in \widehat{\sum_{g \in M}} t(g)g.$$

$$t \in \mathbb{Z}_+^M$$

where $c \in \mathbb{R}^M$.

The problem is called the **Master Problem** if $M = A_+$, and the **Non-Master Problem** when $M \neq A_+$. In this paper we denoted by $P(A, b)$ *the hull convex* of the set

$$\{t \in \mathbb{Z}_+^M : b \in \widehat{\sum_{g \in M}} t(g)g\}.$$

4 Invariant System of $P(A, b)$

In ([6]) Araoz and Johnson show the following theorem:

Theorem 4.1. *[6, Theorem 3.8] Let (L, E) be a base of $C(A)$. The following system defined a $P(A, b)$*

(i) $\sum_{g\in A_+} \rho(g)t(g) = \rho(b)$, *for all* $\rho \in L$

(ii) $\sum_{g\in A_+} \pi(g)t(g) \geq \pi(b)$, *for all* $\pi \in E$

(ii) $t(g) \geq 0$, *for all* $g \in A_+$.

4.1 Equivalent Systems for $P(A, b)$

Let (L, E) be a base of $C(A)$, we denote by $S_{L,E}$ the system:

(1) $\sum_{g\in A_+} \rho(g)t(g) = \rho(b)$, for all $\rho \in L$

(2) $\sum_{g\in A_+} \pi(g)t(g) \geq \pi(b)$, for all $\pi \in E$

(3) $t(g) \geq 0$, for all $g \in A_+$;

Theorem 4.2. *Let (L_1, E_1) and (L_2, E_2)be a base of $C(A)$. Them, the systems S_{L_1,E_1} and S_{L_2,E_2} are equivalent.*

Proof. Let $t \in \mathbb{R}^{A_+}$ such that it verifies (1), (2) and (3), since (L_1, E_1) is a base of $C(A)$, for all $\rho^{'} \in L_2$ and $\pi^{'} \in E_2$, there are scalars $\alpha_\rho \geq 0, \rho \in L_1$ and $\beta_\pi \geq 0, \pi \in E_1$ such that

$$\rho^{'} = \sum_{\rho\in L_1} \alpha_\rho \rho$$

and

$$\pi^{'} = \sum_{\pi\in E_1} \beta_\pi \pi.$$

Then

$$\sum_{g\in A_+} \rho^{'}(g)t(g) = \sum_{g\in A_+} (\sum_{\rho\in L_1} \alpha_\rho \rho)(g)t(g) = \sum_{g\in A_+} (\sum_{\rho\in L_1} \alpha_\rho \rho(g)t(g))$$

$$= \sum_{\rho \in L_1} \alpha_\rho (\sum_{g \in A_+} \rho(g)t(g)) = \sum_{\rho \in L_1} \alpha_\rho \rho(b) = \rho^{'}(b)$$

therefore

$$\sum_{g \in A_+} \rho^{'}(g)t(g) = \rho^{'}(b)$$

On the other hand,

$$\sum_{g \in A_+} \pi^{'}(g)t(g) = \sum_{g \in A_+} (\sum_{\pi \in E_1} \beta_\pi \pi)(g)t(g) = \sum_{g \in A_+} (\sum_{\pi \in e_1} \beta_\pi \pi(g)t(g))$$

$$\sum_{\pi \in E_1} \beta_\pi (\sum_{g \in A_+} \pi(g)t(g)) \geq \sum_{\pi \in E_1} \beta_\pi \pi(b) = \pi^{'}(b)$$

then

$$\sum_{g \in A_+} \pi^{'}(g)t(g) \geq \pi^{'}(b)$$

In an analogous way, we prove that if $t \in \mathbb{R}^{A+}$ verify $S(L_2, E_2)$, then it verify $S(L_1, E_1)$ $\diamond$

5 Bender Descomposition of the b-complementary multisemigroup dual problem

5.1 The b-complementary multisemigroup dual problem

Let $(A, \widehat{+})$ be a multisemigroup b-complementary. We denote by $P_{A,b}$ the following linear programming problem

$$min \sum_{g \in A_+} c(g)t(g)$$

$$\text{s.t: } t \in P(A, b)$$

where $c(g) \in \mathbb{R}$ for all $g \in A_+$.

In [12] we shown the following theorems.

Theorem 5.1. *The $P_{A,b}$ problem is equivalent to the P_p problem*

$$\min \sum_{g\in A_+} c(g)t(g)$$

$$\sum_{g\in A_+} \rho(g)t(g) = \rho(b), \quad \rho \in L;$$

$$\sum_{g\in A_+} \pi(g)t(g) \geq \pi(b), \quad \pi \in E;$$

$$t(g) \geq 0, \quad g \in A_+,$$

where (L, E) is a base for $C(A)$ and $c \in R^{A_+}$

Theorem 5.2. *The dual problem of P_p is the problem P_d*

$$max \ \left(\sum_{\rho\in L} \rho(b)v(\rho) + \sum_{\pi\in E} \pi(b)w(\pi)\right)$$

$$\sum_{\rho\in L} \rho(g)v(\rho) + \sum_{\pi\in E} \pi(g)w(\pi) \leq \quad c(g), \quad g \in A_+$$

$$v(\rho) \text{ unrestricted, } \rho \in L$$

$$w(\pi) \geq 0, \pi \in E.$$

5.2 The Benders Decompostion of P_d

We present the Benders decomposition for problem P_d. From $v \in \mathbb{R}^{A_+}$ we denote with P_v the following problem

$$max \ \sum_{\pi\in E} \pi(b)w(\pi)$$

$$\sum_{\pi\in E} \pi(g)w(\pi) \leq \quad c(g) - \textstyle\sum_{\rho\in L} \rho(g)v(\rho), \quad g \in A_+$$

$$w(\pi) \geq 0, \pi \in E.$$

As

$$
\begin{aligned}
max \ \left(\sum_{\rho \in L} \rho(b) v(\rho) + \sum_{\pi \in E} \pi(b) w(\pi) \right) \\
\sum_{\rho \in L} \rho(g) v(\rho) + \sum_{\pi \in E} \pi(g) w(\pi) \leq \quad & c(g), \quad g \in A_+ \\
v(\rho) \text{ unrestricted}, \ \rho \in L \\
w(\pi) \geq 0, \pi \in E.
\end{aligned}
$$

is iqual to

$$
\begin{aligned}
max \ \left(\sum_{\rho \in L} \rho(b) v(\rho) + P_v \right) \\
v(\rho) \text{ unrestricted}, \ \rho \in L.
\end{aligned}
$$

And the dual of the P_v is the problem DP_v

$$
\begin{aligned}
min \ \sum_{g \in A_+} \left(c(g) - \sum_{\rho \in L} \rho(g) v(\rho) \right) t(g) \\
\sum_{g \in A_+} \pi(g) t(g) \geq \quad & \pi(b), \pi \in E \\
t(g) \geq 0, g \in A_+
\end{aligned}
$$

then, the Benders descomposition of theP_d is the problem

$$
\begin{aligned}
max \ \left(\sum_{\rho \in L} \rho(b) v(\rho) + DP_v \right) \\
v(\rho) \text{ unrestricted}, \ \rho \in L
\end{aligned}
$$

Thus for the construction of the master problem of Bendres we consider the set

$$
X = \{ t \in \mathbb{R}_+^{A_+} : \sum_{g \in A_+} \pi(g) t(g) \geq \pi(b), \quad \pi \in E \},
$$

and denote with $V(X)$ the set of vertices the polhyedra X. If the set X is empty, the dual problem DP_v it is infeasible, and from duality theory, the primal problem P_v has no feasible or is unbounded. Therefore, we can assume that the set X is nonempty.

As the convex polhyedron X is independent of ρ, thus we have the Bender master problem for the dual b-complemantary of the Multisemigrou Problem as the problem:

$$max\{\gamma : \gamma \leq \sum_{\rho \in L} \rho(b)v(\rho) + \sum_{g \in A_+} (c(g) - \sum_{g \in A_+} \rho(g)v(\rho))t(g), t \in V(X)\}$$

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