\( q \)-Multivector fields and \( q \)-forms on Weil bundles

Olivier MABIALA MIKANOU\(^1\), Borhen Vann NKOU\(^2\), Basile Guy Richard BOSSOTO\(^3\)\(^4\)

**Abstract**

Let \( M \) be a paracompact smooth manifold of dimension \( n \), \( A \) a Weil algebra and \( M^A \) the associated Weil bundle. In this paper, we define the Schouten-Nijenhuis bracket on the \( \mathcal{C}^\infty(M^A, A) \)-module \( \mathfrak{X}^*(M^A) \) of multivector fields on \( M^A \) considered as multi-derivations from \( \mathcal{C}^\infty(M) \) into \( \mathcal{C}^\infty(M^A, A) \) and we show that the exterior algebra \( \mathfrak{X}^q(M^A) \) of multivector fields on \( M^A \) is a Lie graded algebra over \( A \). To finish, we establish an isomorphism between \( \mathfrak{X}^q(M^A) \) and the \( \mathcal{C}^\infty(M^A, A) \)-module \( \mathcal{L}_q^{\text{alt}}(\Omega(M^A, A), \mathcal{C}^\infty(M^A, A)) \) of skew-symmetric multilinear forms of degree \( q \) onto the \( \mathcal{C}^\infty(M^A, A) \)-module \( \Omega(M^A, A) \) of differential \( A \)-forms on \( M^A \).

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1. **Introduction**

A local algebra in the sense of André Weil or simply a Weil algebra is a real commutative algebra \( A \) with unit of finite dimension admitting an unique maximal ideal \( m \) of codimension 1 over \( \mathbb{R} \) [7]. If \( M \) is a smooth manifold, \( \mathcal{C}^\infty(M) \) the algebra of smooth functions on \( M \) and \( A \) a Weil algebra, then we call an infinitely near point to \( x \in M \) of kind \( A \) a homomorphism of algebras

\[
\xi : \mathcal{C}^\infty(M) \rightarrow A
\]

such that \([\xi(f) - f(x)] \in m\) for any \( f \in \mathcal{C}^\infty(M)\).

We denote \( M^A_x \) the set of all infinitely near points to \( x \) of kind \( A \) and

\[
M^A = \bigcup_{x \in M} M^A_x,
\]

the set \( M^A \) is a smooth manifold of dimension \( \dim M \times \dim A \) called manifold of infinitely near points [6].

Hence \((M^A, \pi_M, M)\) defines a bundle of infinitely near points or simply a Weil bundle.

If \((U, \varphi)\) is a local chart of \( M \) with coordinate functions \((x_1, x_2, ..., x_n)\), the application

\[
U^A \rightarrow A^n, \xi \mapsto (\xi(x_1), \xi(x_2), ..., \xi(x_n)),
\]

is a bijection from \( U^A \) into an open of \( A^n \). The manifold \( M^A \) is a smooth manifold modeled over \( A^n \), that is to say an \( A \)-manifold of dimension \( n \).

The set, \( \mathcal{C}^\infty(M^A, A) \) of smooth functions on \( M^A \) with value in \( A \) is a commutative, unitary \( A \)-algebra. When one identifies \( \mathbb{R}^A \) with \( A \), for \( f \in \mathcal{C}^\infty(M) \), the application

\[
f^A : M^A \rightarrow A, \xi \mapsto \xi(f)
\]

\(^1\)Université Marien NGOUABI , BP: 69, Brazzaville, Congo, E-mail: stive.elg@gmail.com
\(^2\)Université Marien NGOUABI , BP: 69, Brazzaville, Congo, Email: vannborhen@yahoo.fr
\(^3\)Université Marien NGOUABI , BP: 69, Brazzaville, Congo E-mail: bossotob@yahoo.fr
\(^4\)Institut de Recherche en Sciences Exactes et Naturelles (IRSEN), E-mail: bossotob@yahoo.fr
is smooth. Moreover the application
\[ C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto f^A, \]
is an injective homomorphism of algebras and we have:
\[ (f + g)^A = f^A + g^A; (\lambda \cdot f)^A = \lambda \cdot f^A; (f \cdot g)^A = f^A \cdot g^A \]
with \( \lambda \in \mathbb{R} \), and \( f \) and \( g \) belongs to \( C^\infty(M) \).
When \((a_\alpha)_{\alpha=1,2,\ldots,\dim(A)}\) is a basis of \( A \) and when \((a^*_\alpha)_{\alpha=1,2,\ldots,\dim(A)}\) is a dual basis of the basis \((a_\alpha)_{\alpha=1,2,\ldots,\dim(A)}\)
the application
\[ \sigma : C^\infty(M^A, A) \longrightarrow A \otimes C^\infty(M^A), \varphi \longmapsto \sum_{\alpha=1}^{\dim A} a_\alpha \otimes a^*_\alpha \circ \varphi \]
is an isomorphism of \( A \)-algebras. That isomorphism does not depend of a choisen basis and the application
\[ \gamma : C^\infty(M) \longrightarrow A \otimes C^\infty(M^A), f \longmapsto \sigma(f^A), \]
is a homomorphism of algebras.
A vector field \( X \) on \( M^A \) can be considered as a derivation from \( C^\infty(M) \) into \( C^\infty(M^A, A) \) i.e a linear application which verifies
\[ X(fg) = X(f) \cdot g^A + f^A \cdot X(g) \]
for any \( f, g \in C^\infty(M) \).
For any vector field \( X \) on \( M^A \), considered as derivation from \( C^\infty(M) \) into \( C^\infty(M^A, A) \), there exists one and only one derivation
\[ \widetilde{X} : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A) \]
such that:
1. \( \widetilde{X} \) is \( A \)-linear;
2. \( \widetilde{X}[C^\infty(M^A)] \subset C^\infty(M^A) \);
3. \( \widetilde{X}(f^A) = X(f) \) for any \( f \in C^\infty(M) \).

The set \( \mathfrak{X}(M^A) \) of all vector fields on \( M^A \) is a \( C^\infty(M^A, A) \)-module and the application
\[ [X, Y] = \widetilde{X} \circ Y - \widetilde{Y} \circ X : C^\infty(M) \longrightarrow C^\infty(M^A, A) \]
defines the structure of a Lie algebra over \( A \) \cite{1}, \cite{2}.
If \( \theta : C^\infty(M) \longrightarrow C^\infty(M) \) is a vector field on \( M \), then the application
\[ \theta^A : C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto [\theta(f)]^A, \]
is a vector field on \( M^A \). We say that the vector field \( \theta^A \) is the prolongation to \( M^A \) of the vector field \( \theta \).
Then, for any \( f \in C^\infty(M) \) we have:
\[ (\theta_1 + \theta_2)^A = \theta_1^A + \theta_2^A; \quad (f \cdot \theta)^A = f^A \cdot \theta^A; \quad (f \cdot \theta)^A = f^A \cdot \theta^A; \quad [\theta_1^A, \theta_2^A] = [\theta_1, \theta_2]^A. \]

The goal of this work, is to define the Schouten-Nijenhuis bracket on the \( C^\infty(M^A, A) \)-module \( \mathfrak{X}^*(M^A) \) of multivector fields on \( M^A \) considered as multi-derivations \cite{4} from \( C^\infty(M) \) into \( C^\infty(M^A, A) \) and to show that exterior algebra \( \mathfrak{X}^*(M^A) \) of multivector fields on \( M^A \) is a Lie graded algebra over \( A \) .

And to finish, we have to establish an isomorphism between \( \mathfrak{X}^*(M^A) \) and the \( C^\infty(M^A, A) \)-module \( \Lambda_q^\Omega(M^A, A), C^\infty(M^A, A) \) of skew-symmetric multilinear forms of degree \( q \) onto the \( C^\infty(M^A, A) \)-module \( \Omega(M^A, A) \) of differential \( A \)-forms on \( M^A \).
2 \textit{q-Multivector fields and q-forms on }M^A\textit{ }

We denote $\mathfrak{X}^1(M^A) = \mathfrak{X}(M^A)$, the set of all vector fields on $M^A$ i.e the set of smooth sections of tangent bundle $(TM^A, \pi_{M^A}, M^A)$. More generally, one denotes, for $2 \leq q \leq m = \dim M^A$, $\mathfrak{X}^q(M^A)$ the set of multivector fields of degree $q$ (or $q$-multivector fields [2],[3],[5]) on $M^A$, i.e the set of smooth sections of vector bundle $(\Lambda^q TM^A, \pi_{M^A}, M^A)$.

For any $\xi \in M^A$, $T_\xi M^A$ is an $A$-module [6]. In this section, we show that, a $q$-multivector field on $M^A$ is a $q$-derivation [4].

2.1 \textit{The }C^\infty(M^A)\textit{-module of }q\text{-multivector fields and of }q\text{-forms on }M^A\textit{ }

Let $\pi_M : M^A \to M$ be map the which assigns any infinitely near point $\xi$ of $M^A$ to its origin $x \in M$, and $U$ be an open neighborhood of $M$ with coordinate system $\{x_1, ..., x_n\}$. Hence $\{x_{i,\alpha}/ i = 1, ..., n; \alpha = 1, ..., \dim A = r\}$ is coordinate system of $\pi^{-1}_M(U)$ where $x_{i,\alpha} : \pi^{-1}_M(U) \to \mathbb{R}, \xi \mapsto x_{i,\alpha}(\xi)$ is such that

$$\xi(x_i) = \sum_\alpha x_{i,\alpha}(\xi)a_\alpha$$

for any $x_i \in C^\infty(M)$, $\forall i = 1, ..., n$.

\textbf{Lemma 1.} [6] Let $\{x_1, ..., x_n\}$ be a coordinate system on some neighborhood of $M$. Then we have

$$a_\alpha(\frac{\partial}{\partial x_i})^A = \frac{\partial}{\partial x_{i,\alpha}}$$

for any $i = 1, ..., n; \alpha = 1, ..., r$.

\textbf{Lemma 2.} Let $\{x_1, ..., x_n\}$ be a coordinate system on some neighborhood of $M$. Then we have

$$a_\alpha^* \circ (dx_i)^A = dx_{i,\alpha}.$$

for any $i = 1, ..., n; \alpha = 1, ..., r$.

\textbf{Proof.} We have, on one hand:

$$a_\alpha^* \left[ (dx_i)^A \left[ \left( \frac{\partial}{\partial x_j} \right)^A \right] \right] = a_\alpha^* \left( dx_i \left( \frac{\partial}{\partial x_j} \right)^A \right) = a_\alpha^* \left( (\delta_{ij})^A \right) = a_\alpha^* (\delta_{ij}).$$

On the other hand, we have:

$$dx_{i,\alpha} \left[ \left( \frac{\partial}{\partial x_j} \right)^A \right] = \left( \frac{\partial}{\partial x_i} \right)^A (x_{i,\alpha}) = \left( \frac{\partial}{\partial x_i} \right)^A (a_\alpha^* \circ x_i^A) = a_\alpha^* \left[ \left( \frac{\partial}{\partial x_j} \right)^A \right] (x_i^A) = a_\alpha^* \left[ \left( \frac{\partial_{x_i}}{\partial x_j} \right)^A \right] = a_\alpha^* (\delta_{ij}).$$

Hence we get

$$a_\alpha^* \circ (dx_i)^A = dx_{i,\alpha},$$

what ends the proof. \qed
In the coordinate system \( \{x_{i\alpha} / i = 1, \ldots, n; \alpha = 1, \ldots, r \} \) a \( q \)-multivector field \( Q \in \mathcal{X}(M^A) \) is written

\[
Q = \sum_{1 \leq i_1 < \ldots < i_q \leq n \atop 1 \leq \alpha_1 < \ldots < \alpha_q \leq r} Q_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q} \frac{\partial}{\partial x_{i_1 \alpha_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_q \alpha_q}},
\]

where \( Q_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q} \in C^\infty(M^A) \). It follows that \( \mathcal{X}^q(M^A) \) is a \( C^\infty(M^A) \)-module of \( q \)-vector fields on \( M^A \). We denote

\[
\mathcal{X}^*(M^A) = C^\infty(M^A) \oplus \mathcal{X}^1(M^A) \oplus \ldots \oplus \mathcal{X}^n(M^A)
\]

the exterior algebra of \( C^\infty(M^A) \)-module of multivector fields.

In the other respects, for \( 1 \leq q \leq m \), \( \Omega^q(M^A) \) denotes the space of differential \( q \)-forms on \( M^A \). In the coordinate system \( \{x_{i\alpha} / i = 1, \ldots, n; \alpha = 1, \ldots, r \} \) a differential \( q \)-form \( \varpi \in \Omega^q(M^A) \) is written

\[
\varpi = \sum_{1 \leq i_1 < \ldots < i_q \leq n \atop 1 \leq \alpha_1 < \ldots < \alpha_q \leq r} \varpi_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q} dx_{i_1 \alpha_1} \wedge \ldots \wedge dx_{i_q \alpha_q},
\]

where \( \varpi_{i_1 \ldots i_q \alpha_1 \ldots \alpha_q} \in C^\infty(M^A) \). Thus \( \Omega^q(M^A) \) is a \( C^\infty(M^A) \)-module of differentiable \( q \)-forms on \( M^A \) and

\[
\Omega^*(M^A) = C^\infty(M^A) \oplus \Omega^1(M^A) \oplus \Omega^2(M^A) \oplus \ldots \oplus \Omega^m(M^A)
\]

denotes exterior algebra of \( C^\infty(M^A) \)-module multilinear forms.

**Proposition 3.** If \( Q \) is a \( q \)-multivector field on \( M^A \) then \( Q \) is a \( q \)-derivation from \( C^\infty(M^A) \) into \( C^\infty(M^A) \).

**Theorem 4.** Let \( \mathcal{X}^q(M^A) \) be the \( C^\infty(M^A) \)-module of \( q \)-multivector fields on \( M^A \) and \( D^q(C^\infty(M^A)) \) be the \( C^\infty(M^A) \)-module of \( q \)-derivations from \( C^\infty(M^A) \) into \( C^\infty(M^A) \). Then the map

\[
\Phi : \mathcal{X}^q(M^A) \rightarrow D^q(C^\infty(M^A)), Q \mapsto \Phi_Q
\]

where

\[
\Phi_Q : C^\infty(M^A) \times \cdots \times C^\infty(M^A) \rightarrow C^\infty(M^A), [\Phi_Q(f_1, f_2, \ldots, f_q)](\xi) = Q(\xi)(f_1, f_2, \ldots, f_q)
\]

is defined by

\[
[\Phi_Q(f_1, f_2, \ldots, f_q)](\xi) = Q(\xi)(f_1, f_2, \ldots, f_q)
\]

for any \( f_1, f_2, \ldots, f_q \in C^\infty(M^A) \) and \( \xi \in M^A \), is an isomorphism of \( C^\infty(M^A) \)-modules.

### 2.2 The \( C^\infty(M^A, A) \)-module of \( q \)-multivector fields and of \( q \)-forms on \( M^A \)

Let \( U \) be an open neighborhood of \( M \) with coordinate system \( \{x_1, \ldots, x_n\} \).

Then, \( \left\{ \left( \frac{\partial}{\partial x_1} \right)^A, \ldots, \left( \frac{\partial}{\partial x_n} \right)^A \right\} \) is a free \( A \)-basis of \( T_\xi M^A \) see [6]. Then, a \( q \)-multivector field \( Q \) on \( M^A \) is written:

\[
Q = \sum_{1 \leq i_1 < \ldots < i_q \leq n} Q_{i_1 \ldots i_q} \left( \frac{\partial}{\partial x_{i_1}} \right)^A \wedge \ldots \wedge \left( \frac{\partial}{\partial x_{i_q}} \right)^A,
\]

where \( Q_{i_1 \ldots i_q} \in C^\infty(U^A, A) \).

In this case \( \mathcal{X}^q(M^A) \) is a \( C^\infty(M^A, A) \)-module of \( q \)-multivector fields. Hence

\[
\mathcal{X}^*(M^A) = C^\infty(M^A, A) \oplus \mathcal{X}^1(M^A) \oplus \ldots \oplus \mathcal{X}^n(M^A)
\]
denotes the exterior algebra of $C^\infty(M^A, A)$-module of multivector field on $M^A$.

In the other respects, for $1 \leq q \leq n$, $\Omega^q(M^A, A)$ denotes the space of differential $A$-forms of degree $q$ on $M^A$ see[1]. Any differential $A$-form [1] of degree $q$ on $M^A$ is written
\[
\varpi = \sum_{1 \leq i_1 < \ldots < i_q \leq n} \varpi_{i_1 \ldots i_q} (dx_{i_1})^A \wedge \ldots \wedge (dx_{i_q})^A,
\]
for any $\varpi_{i_1 \ldots i_q} \in C^\infty(U^A, A)$. Then, $\Omega^q(M^A, A)$ is a $C^\infty(M^A, A)$-module of differentiable $A$-forms of degree $q$ on $M^A$.

We denote
\[
\Omega^*(M^A, A) = C^\infty(M^A, A) \oplus \Omega^1(M^A, A) \oplus \Omega^2(M^A, A) \oplus \ldots \oplus \Omega^n(M^A, A)
\]
the exterior algebra of $C^\infty(M^A, A)$-module of differentiable multilinear $A$-forms on $M^A$.

**Proposition 5.** If $Q$ is a $q$-multivector field on $M^A$ then $Q$ is a $q$-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$.

**Proposition 6.** If $Q$ is a $q$-vector field on $M$ then the map
\[
Q^A : C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M^A, A), (f_1, \ldots, f_q) \mapsto [Q(f_1, \ldots, f_q)]^A
\]
is a $q$-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$.

**Theorem 7.** Let $\mathfrak{X}^q(M^A)$ be the $C^\infty(M^A, A)$-module of $q$-vector fields on $M^A$ and $\mathcal{D}^q(C^\infty(M), C^\infty(M^A, A))$ be the $C^\infty(M^A, A)$-module of $q$-derivations from $C^\infty(M)$ into $C^\infty(M^A, A)$. The map
\[
\Theta : \mathfrak{X}^q(M^A) \to \mathcal{D}^q(C^\infty(M), C^\infty(M^A, A)) \; , \; Q \mapsto \Theta_Q
\]
where
\[
\Theta_Q : C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M^A, A)
\]
is defined by
\[
[\Theta_Q(f_1, f_2, \ldots, f_q)(\xi) = Q(\xi)(f_1, f_2, \ldots, f_q)
\]
for any $f_1, f_2, \ldots, f_q \in C^\infty(M)$ and $\xi \in M^A$; is an isomorphism of $C^\infty(M^A, A)$-modules.

We now define two $q$-multilinear applications $\sigma_q$ and $\gamma_q$ as follows:
\[
\sigma_q : C^\infty(M^A, A) \times \ldots \times C^\infty(M^A, A) \to A \otimes C^\infty(M^A), (\varphi_1, \ldots, \varphi_q) \mapsto \sum_a a_\alpha \otimes a_\alpha^* (\varphi_1 \times \ldots \times \varphi_q)
\]
and
\[
\gamma_q : C^\infty(M) \times \ldots \times C^\infty(M) \to A \otimes C^\infty(M^A), (f_1, \ldots, f_q) \mapsto \sigma_q(f_1^A, \ldots, f_q^A).
\]
For $q = 1$, we have $\sigma_1 = \sigma$ and $\gamma_1 = \gamma$.

**Theorem 8.** Let $Q$ be a multivector field on $M^A$ considered as $q$-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$ then there exists an unique $q$ derivation:
\[
\overline{Q} : C^\infty(M^A, A) \times \ldots \times C^\infty(M^A, A) \to C^\infty(M^A, A)
\]
such that

1. $Q$ be $A$-multilinear;
2. $\overline{Q} [C^\infty(M^A) \times \ldots \times C^\infty(M^A)] \subset C^\infty(M^A)$;
3. \( \bar{Q}(f_1^A, f_2^A, ..., f_q^A) = Q(f_1, f_2, ..., f_q) \), for any \( f_1, f_2, ..., f_q \) ∈ \( C^\infty(M) \).

Proof. If \( Q \) is a multivector field on \( M^A \) considered as \( q \)-derivation from \( C^\infty(M) \) into \( C^\infty(M^A, A) \) and if

\[
\overline{Q} : C^\infty(M^A) \longrightarrow C^\infty(M^A)
\]

is an unique derivation such that

\[
\sigma^{-1} \circ (id_A \otimes \overline{Q}) \circ \gamma_q = Q,
\]

then the map

\[
\tilde{Q} = \sigma^{-1} \circ (id_A \otimes \overline{Q}) \circ \sigma_q : C^\infty(M^A, A) \times ... \times C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)
\]

answers to the question. \( \square \)

**Proposition 9.** If \( \mu : A \longrightarrow A \) is an endomorphism and \( Q \) is a multivector field on \( M^A \) considered as \( q \)-derivation from \( C^\infty(M) \) into \( C^\infty(M^A, A) \), then

\[
\bar{Q}(\mu \circ f_1^A , \mu \circ f_2^A , ..., \mu \circ f_q^A) = \mu \circ Q(f_1, f_2, ..., f_q)
\]

(4)

for any \( f_1, f_2, ..., f_q \) ∈ \( C^\infty(M) \).

Proof. From \( \bar{Q}(f_1^A, f_2^A, ..., f_q^A) = Q(f_1, f_2, ..., f_q) \), we have

\[
\bar{Q} \left[ \sum_{\alpha=1}^{r} (a_\alpha^* \circ f_1^A, a_\alpha^* \circ f_2^A, ..., a_\alpha^* \circ f_q^A) \cdot a_\alpha \right] = \sum_{\alpha=1}^{r} a_\alpha^* \circ Q(f_1, f_2, ..., f_q) \cdot a_\alpha
\]

that implies

\[
\sum_{\alpha=1}^{r} \bar{Q}(a_\alpha^* \circ f_1^A, a_\alpha^* \circ f_2^A, ..., a_\alpha^* \circ f_q^A) \cdot a_\alpha = \sum_{\alpha=1}^{r} a_\alpha^* \circ Q(f_1, f_2, ..., f_q) \cdot a_\alpha.
\]

Hence \( \bar{Q}(a_\alpha^* \circ f_1^A, a_\alpha^* \circ f_2^A, ..., a_\alpha^* \circ f_q^A) = a_\alpha^* \circ Q(f_1, f_2, ..., f_q) \) for all \( (a_\alpha^*)_{\alpha=1}^{r} \). Since

\[
(\mu \circ f_1^A, \mu \circ f_2^A, ..., \mu \circ f_q^A) = \sum_{\alpha=1}^{r} (a_\alpha^* \circ f_1^A, a_\alpha^* \circ f_2^A, ..., a_\alpha^* \circ f_q^A) \cdot \mu(a_\alpha)
\]

it follows that

\[
\bar{Q}(\mu \circ f_1^A, \mu \circ f_2^A, ..., \mu \circ f_q^A) = \sum_{\alpha=1}^{r} \bar{Q}(a_\alpha^* \circ f_1^A, a_\alpha^* \circ f_2^A, ..., a_\alpha^* \circ f_q^A) \cdot \mu(a_\alpha)
\]

\[
= \sum_{\alpha=1}^{r} a_\alpha^* \circ Q(f_1, f_2, ..., f_q) \cdot \mu(a_\alpha)
\]

\[
= \mu \circ Q(f_1, f_2, ..., f_q)
\]

therefore we have the result. \( \square \)

### 2.3 Schouten-Nijenhuis bracket on \( C^\infty(M^A, A) \)-module \( \mathfrak{X}^q(M^A) \)

Let \( \mathfrak{X}^q(M^A) \) be the set of \( q \)-multivector fields on \( M^A \) considered as \( q \)-derivations from \( C^\infty(M) \) into \( C^\infty(M^A, A) \), we define the Schouten-Nijenhuis bracket (see [2],[5]) as follows:
Theorem 10. If $P$ and $Q$ are both multivector fields on $M^A$ of degree $p$ and $q$, respectively, considered as $p$-derivation and $q$-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$, respectively. Then, the bracket of $P$ and $Q$ defined by:

$$[P, Q]_S(f_1, f_2, ..., f_{p+q-1}) = \sum_{\sigma \in \Theta_{p+q-1}} \varepsilon(\sigma)\tilde{P}\left(Q(f_{\sigma(1)}, ..., f_{\sigma(q)}), f^A_{\sigma(q+1)}, ..., f^A_{\sigma(p+q-1)}\right)$$

$$-(-1)^{(p-1)(q-1)} \sum_{\sigma \in \Theta_{p+q-1}} \varepsilon(\sigma)\tilde{Q}\left(P(f_{\sigma(1)}, ..., f_{\sigma(p)}), f^A_{\sigma(p+1)}, ..., f^A_{\sigma(p+q-1)}\right)$$

for any $f_1, f_2, ..., f_{p+q-1} \in C^\infty(M)$, is a multivector field on $M^A$ of degree $p + q - 1$.

Proposition 11. If $P$ and $Q$ both are $p$ and $q$-multivector fields on $M^A$ considered as $p$ and $q$-derivations, respectively from $C^\infty(M)$ into $C^\infty(M^A, A)$, and if $\varphi \in C^\infty(M^A, A)$, then we have:

1. $[\tilde{P}, \tilde{Q}]_S = [\tilde{P}, \tilde{Q}]_S$.
2. $\varphi \cdot \tilde{P} = \varphi \cdot \tilde{P}$.
3. $\tilde{P} \wedge Q = \tilde{P} \wedge Q$.

Proof. 1. For any $f_1, f_2, ..., f_{p+q-1} \in C^\infty(M)$, we have

$$[\tilde{P}, \tilde{Q}]_S(f^A_1, f^A_2, ..., f^A_{p+q-1})$$

$$= \sum_{\sigma \in \Theta_{p+q-1}} \varepsilon(\sigma)\tilde{P}\left(Q(f^A_{\sigma(1)}, ..., f^A_{\sigma(q)}), f^A_{\sigma(q+1)}, ..., f^A_{\sigma(p+q-1)}\right)$$

$$-(-1)^{(p-1)(q-1)} \sum_{\sigma \in \Theta_{p+q-1}} \varepsilon(\sigma)\tilde{Q}\left(P(f^A_{\sigma(1)}, ..., f^A_{\sigma(p)}), f^A_{\sigma(p+1)}, ..., f^A_{\sigma(p+q-1)}\right)$$

$$= \sum_{\sigma \in \Theta_{p+q-1}} \varepsilon(\sigma)\tilde{P}\left(Q(f^A_{\sigma(1)}, ..., f^A_{\sigma(q)}), f^A_{\sigma(q+1)}, ..., f^A_{\sigma(p+q-1)}\right)$$

$$-(-1)^{(p-1)(q-1)} \sum_{\sigma \in \Theta_{p+q-1}} \varepsilon(\sigma)\tilde{Q}\left(P(f^A_{\sigma(1)}, ..., f^A_{\sigma(p)}), f^A_{\sigma(p+1)}, ..., f^A_{\sigma(p+q-1)}\right)$$

$$= [P, Q]_S(f_1, f_2, ..., f_{p+q-1}).$$

Since $[\tilde{P}, \tilde{Q}]_S$ is the unique $p + q - 1$-derivation from $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$ such that

$$[\tilde{P}, \tilde{Q}]_S(f^A_1, f^A_2, ..., f^A_{p+q-1}) = [P, Q]_S(f_1, f_2, ..., f_{p+q-1})$$

for any $f_1, f_2, ..., f_{p+q-1} \in C^\infty(M)$, hence we have

$$[\tilde{P}, \tilde{Q}]_S = [\tilde{P}, \tilde{Q}]_S.$$
3. For any $f_1, f_2, ..., f_{p+q-1} \in C^\infty(M)$, we have
\[
\tilde{P} \wedge \tilde{Q}(f_1^A, f_2^A, ..., f_{p+q}^A) = \sum_{\sigma \in \mathcal{S}_{p,q}} \varepsilon(\sigma) \tilde{P}(f_{\sigma(1)}, ..., f_{\sigma(p)}) \cdot \tilde{Q}(f_{\sigma(q+1)}, ..., f_{\sigma(p+q)})
\]
\[
= \sum_{\sigma \in \mathcal{S}_{p,q}} \varepsilon(\sigma)P(f_{\sigma(1)}, ..., f_{\sigma(q)}) \cdot Q(f_{\sigma(q+1)}, ..., f_{\sigma(p+q)}) = P \wedge Q(f_1, f_2, ..., f_{p+q}).
\]

In the other hand, since $\tilde{P} \wedge \tilde{Q}$ is the unique $p+q$-derivation from $C^\infty(M, A)$ into $C^\infty(M, A)$, such that
\[
\tilde{P} \wedge \tilde{Q}(f_1^A, f_2^A, ..., f_{p+q}^A) = P \wedge Q(f_1, f_2, ..., f_{p+q})
\]

hence we have
\[
\tilde{P} \wedge \tilde{Q} = \tilde{P} \wedge Q.
\]

Let us give now the intrinsic properties of Schouten-Nijenhuis bracket.

**Theorem 12.** Let $P, Q, R$ be $p$-multivector, $q$-multivector and $r$-multivector fields on $M^A$, respectively, considered as $p,q$ and $r$-derivations from $C^\infty(M)$ into $C^\infty(M^A, A)$, respectively. Let $\varphi, \psi$ both be smooth functions on $M^A$ with values in $A$ and $X$ a vector field on $M^A$ considered as derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$. The Schouten-Nijenhuis bracket verifies the following equalities:

1. $[\varphi, \psi]_S = 0$;
2. $[X, P]_S = L_X P$ the Lie derivative with respect to the vector field $X$;
3. $[P, Q]_S = -(-1)^{(p-1)(q-1)}[Q, P]_S$;
4. $[P, Q \wedge R]_S = [P, Q]_S \wedge R + (-1)^{(p-1)q}Q \wedge [P, R]_S$.

Let us give some indications of the proof of theorem 12.

**Proof.** (of the theorem 12)

We put $P = X_1 \wedge X_2 \wedge \cdots \wedge X_p$, $Q = Y_1 \wedge Y_2 \wedge \cdots \wedge Y_q$ and $R = Z_1 \wedge Z_2 \wedge \cdots \wedge Z_r$. Hence
\[
[\tilde{X}_1 \wedge \cdots \wedge \tilde{X}_p, \tilde{Y}_1 \wedge \cdots \wedge \tilde{Y}_q]_S = \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [\tilde{X}_i, \tilde{Y}_j]_S \wedge \tilde{X}_1 \wedge \cdots \wedge \tilde{X}_{i-1} \wedge \tilde{X}_{i+1} \wedge \cdots \wedge \tilde{X}_p \wedge \tilde{Y}_1 \wedge \cdots \wedge \tilde{Y}_{j-1} \wedge \tilde{Y}_{j+1} \wedge \cdots \wedge \tilde{Y}_q
\]
in this case the proof of the theorem is in the same way like in the classical case see [3]. For achieving the proof, we consider the fact that
\[
[\tilde{P}, \tilde{Q}]_S(f_1^A, f_2^A, ..., f_{p+q-1}^A) = [P, Q]_S(f_1, f_2, ..., f_{p+q-1})
\]
for any $f_1, f_2, ..., f_{p+q-1} \in C^\infty(M)$.

**Theorem 13.** Endowed with Schouten-Nijenhuis bracket $\mathfrak{X}^r(M^A)$ is a Lie graded algebra over $A$ and its graded Jacobi identity is given by:
\[
(-1)^{(p-1)(r+q)}[P, [Q, R]_S]_S + (-1)^{(q-1)(p-1)}[Q, [R, P]_S]_S + (-1)^{(r-1)(q-1)}[R, [P, Q]_S]_S = 0
\]
for any $P \in \mathfrak{X}^p(M^A)$, $Q \in \mathfrak{X}^q(M^A)$, $R \in \mathfrak{X}^r(M^A)$ considered as $p,q$ and $r$-derivations from $C^\infty(M)$ into $C^\infty(M^A, A)$, respectively.

**Proof.** The proof of graded Jacobi identity is made in the same way like in the proof of the theorem 12.

**Proposition 14.** If $X_1, \cdots, X_p$ are vector fields on $M$ and $P \in \mathfrak{X}^p(M)$, $Q \in \mathfrak{X}^q(M)$, then we have:
1. \(X_1^A \wedge \cdots \wedge X_p^A = (X_1 \wedge \cdots \wedge X_p)^A\).
2. \(P^A + Q^A = (P + Q)^A\), if \(p = q\).
3. \(P^A \wedge Q^A = (P \wedge Q)^A\).
4. \([P^A, Q^A]_S = [P, Q]_S^A\).

**Proof.**

1. For any \(f_1, \ldots, f_q \in C^\infty(M)\), we have:

\[
(X_1 \wedge \cdots \wedge X_p)(f_1, \ldots, f_p) = \sum_{\sigma \in S_p} \varepsilon(\sigma)X_1^A(f_{\sigma(1)}) \cdots X_p^A(f_{\sigma(q)}) = \sum_{\sigma \in S_p} \varepsilon(\sigma)(X_1(f_{\sigma(1)})) \cdots (X_p(f_{\sigma(p)}))^A
\]

hence

\(X_1^A \wedge \cdots \wedge X_p^A = (X_1 \wedge \cdots \wedge X_p)^A\).

2. For any \(f_1, \ldots, f_q \in C^\infty(M)\), we have:

\[
(P^A + Q)(f_1, \ldots, f_p) = P^A(f_1, \ldots, f_p) + Q^A(f_1, \ldots, f_p)
= (P(f_1, \ldots, f_p))^A + (Q(f_1, \ldots, f_p))^A
= (P(f_1, \ldots, f_p) + Q(f_1, \ldots, f_p))^A
= [(P + Q)(f_1, \ldots, f_p)]^A = (P + Q)^A(f_1, \ldots, f_p),
\]

so that

\(P^A + Q^A = (P + Q)^A\).

3. Let us take \(P = X_1 \wedge \cdots \wedge X_p\) and \(Q = Y_1 \wedge \cdots \wedge Y_q\). Then, we have:

\[
P^A \wedge Q^A = (X_1 \wedge \cdots \wedge X_p)^A \wedge (Y_1 \wedge \cdots \wedge Y_p)^A
= X_1^A \wedge \cdots \wedge X_p^A \wedge Y_1^A \wedge \cdots \wedge Y_p^A
= (X_1 \wedge \cdots \wedge X_p \wedge Y_1 \wedge \cdots \wedge Y_q)^A
= (P \wedge Q)^A;
\]

4. For any \(f_1, \ldots, f_{p+q-1} \in C^\infty(M)\), we have:

\[
[P^A, Q^A]_S(f_1, \ldots, f_{p+q-1}) = \sum_{\sigma \in \Theta_{p+q-1}} \varepsilon(\sigma)P^A(Q^A(f_{\sigma(1)}, \ldots, f_{\sigma(q)}), f_{\sigma(q+1)}, \ldots, f_{\sigma(p+q-1)})
- (-1)^{(p-1)(q-1)} \sum_{\sigma \in \Theta_{p+q-1}} \varepsilon(\sigma)Q^A(P^A(f_{\sigma(1)}, \ldots, f_{\sigma(p)}), f_{\sigma(p+1)}, \ldots, f_{\sigma(p+q-1)})

= \sum_{\sigma \in \Theta_{p+q-1}} \varepsilon(\sigma)P^A(Q(f_{\sigma(1)}, \ldots, f_{\sigma(q)}), f_{\sigma(q+1)}, \ldots, f_{\sigma(p+q-1)})
- (-1)^{(p-1)(q-1)} \sum_{\sigma \in \Theta_{p+q-1}} \varepsilon(\sigma)Q^A(P(f_{\sigma(1)}, \ldots, f_{\sigma(p)}), f_{\sigma(p+1)}, \ldots, f_{\sigma(p+q-1)})
\]

\[
=[P, Q]_S^A(f_1, \ldots, f_{p+q-1})
=[P, Q]_S^A(f_1, \ldots, f_{p+q-1}).
\]
hence
\[ [P^A, Q^A]_S = [P, Q]^A_S. \]

We presently are going to establish the relation between a \( q \)-multivector field and a \( q \)-form on \( M^A \) through the following result.

**Theorem 15.** Let \( \mathfrak{X}^q(M^A) \) be the set of all multivector fields on \( M^A \) of degree \( q \) considered as \( q \)-derivation from \( C^\infty(M) \) into \( C^\infty(M^A, A) \) and \( \Omega^q_{alt}(\Omega(M^A, A)) \), the \( C^\infty(M^A, A) \)-module of skew-symmetric multilinear forms of degree \( q \) on \( \Omega(M^A, A) \). Then the map
\[ \Theta : \mathfrak{X}^q(M^A) \to \Omega^q_{alt}(\Omega(M^A, A), C^\infty(M^A, A)) \]
such that
\[ (\Theta(Q)\varpi_1, \cdots, \varpi_q) = \widetilde{Q}(\varpi_1, \cdots, \varpi_q), \]
for any \( \xi \in M^A \), \( Q \in \mathfrak{X}^q(M^A) \) and \( \varpi_1, \cdots, \varpi_q \in \Omega(M^A, A) \) is an isomorphism of \( C^\infty(M^A, A) \)-modules.

**Theorem 16.** Let \( Q \) be a multivector field of degree \( q \) on \( M \). Then there exists an unique multivector field of degree \( q \) on \( M^A \)
\[ Q^A : \Omega(M^A, A) \times \cdots \times \Omega(M^A, A) \to C^\infty(M^A, A) \]
such that
\[ Q^A(a_1 \varpi_1^A, \cdots, a_q \varpi_q^A) = a_1 \times \cdots \times a_q [Q(\varpi_1, \cdots, \varpi_q)]^A \]
for any \( a_1, \cdots, a_q \in A \) and \( \varpi_1, \cdots, \varpi_q \in \Omega(M) \).

**References**


