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q-Multivector fields and q-forms on Weil bundles

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Abstract

Let M be a paracompact smooth manifold of dimension n, A a Weil algebra and M^A the associated Weil bundle. In this paper, we define the Schouten-Nijenhuis bracket on the $C^\infty(M^A, A)$-module $\mathfrak{X}^*(M^A)$ of multivector fields on M^A considered as multi-derivations from $C^\infty(M)$ into $C^\infty(M^A, A)$ and we show that the exterior algebra $\mathfrak{X}^*(M^A)$ of multivector fields on M^A is a Lie graded algebra over A. To finish, we estabish an isomorphism between $\mathfrak{X}^q(M^A)$ and the $C^\infty(M^A, A)$-module $\mathcal{L}^q_{alt}(\Omega(M^A, A), C^\infty(M^A, A))$ of skew-symmetric multilinear forms of degree q onto the $C^\infty(M^A, A)$-module $\Omega(M^A, A)$ of differential A-forms on M^A.

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1 Introduction

A local algebra in the sense of André Weil or simply a Weil algebra is a real commutative algebra A with unit of finite dimension admitting an unique maximal ideal $\mathfrak{m}$ of codimension 1 over $\mathbb{R}$ [7].
If M is a smooth manifold, $C^\infty(M)$ the algebra of smooth functions on M and A a Weil algebra, then we call an infinitely near point to $x \in M$ of kind A a homomorphism of algebras

$$\xi : C^\infty(M) \longrightarrow A$$

such that $[\xi(f) - f(x)] \in \mathfrak{m}$ for any $f \in C^\infty(M)$.
We denote M^A_x the set of all infinitely near points to x of kind A and

$$M^A = \bigcup_{x\in M} M^A_x,$$

the set M^A is a smooth manifold of dimension $\dim M \times \dim A$ called manifold of infinitely near points [6].

Hence (M^A, π_M, M) defines a bundle of infinitely near points or simply a Weil bundle.
If (U, φ) is a local chart of M with coordinate functions $(x_1, x_2, ..., x_n)$, the application

$$U^A \longrightarrow A^n, \xi \longmapsto (\xi(x_1), \xi(x_2), ..., \xi(x_n)),$$

is a bijection from U^A into an open of A^n. The manifold M^A is a smooth manifold modeled over A^n, that is to say an A-manifold of dimension n.
The set, $C^\infty(M^A, A)$ of smooth functions on M^A with value in A is a commutative , unitary A-algebra.When one identitifies $\mathbb{R}^A$ with A, for $f \in C^\infty(M)$, the application

$$f^A : M^A \longrightarrow A, \xi \longmapsto \xi(f)$$

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is smooth. Moreover the application

$$C^{\infty}(M) \longrightarrow C^{\infty}(M^A, A), f \longmapsto f^A,$$

is an injective homomorphism of algebras and we have:

$$(f+g)^A = f^A + g^A; (\lambda \cdot f)^A = \lambda \cdot f^A; (f \cdot g)^A = f^A \cdot g^A$$

with $\lambda \in \mathbb{R}$, f and g belongs to $C^{\infty}(M)$.
When $(a_\alpha)_{\alpha=1,2,\dots,\dim(A)}$ is a basis of A and when $(a^*_\alpha)_{\alpha=1,2,\dots,\dim(A)}$ is a dual basis of the basis $(a_\alpha)_{\alpha=1,2,\dots,\dim(A)}$, the application

$$\sigma : C^{\infty}(M^A, A) \longrightarrow A \otimes C^{\infty}(M^A), \varphi \longmapsto \sum_{\alpha=1}^{\dim A} a_\alpha \otimes a^*_\alpha \circ \varphi \tag{1}$$

is an isomorphism of A-algebras. That isomorphism does not depend of a choisen basis and the application

$$\gamma : C^{\infty}(M) \longrightarrow A \otimes C^{\infty}(M^A), f \longmapsto \sigma(f^A), \tag{2}$$

is a homomorphism of algebras.
A vector field X on M^A can be considered as a derivation from $C^{\infty}(M)$ into $C^{\infty}(M^A, A)$ i.e a linear application which verifies

$$X(fg) = X(f) \cdot g^A + f^A \cdot X(g)$$

for any $f, g \in C^{\infty}(M)$.

For any vector field X on M^A, considered as derivation from $C^{\infty}(M)$ into $C^{\infty}(M^A, A)$, there exists one and only one derivation

$$\widetilde{X} : C^{\infty}(M^A, A) \longrightarrow C^{\infty}(M^A, A)$$

such that:

1. $\widetilde{X}$ is A-linear;
2. $\widetilde{X}\left[C^{\infty}(M^A)\right] \subset C^{\infty}(M^A)$;
3. $\widetilde{X}(f^A) = X(f)$ for any $f \in C^{\infty}(M)$.

The set $\mathfrak{X}(M^A)$ of all vector fields on M^A is a $C^{\infty}(M^A, A)$-module and the application

$$[X, Y] = \widetilde{X} \circ Y - \widetilde{Y} \circ X : C^{\infty}(M) \longrightarrow C^{\infty}(M^A, A)$$

defines the structure of a Lie algebra over A [1], [**?**].
If $\theta : C^{\infty}(M) \longrightarrow C^{\infty}(M)$ is a vector field on M, then the application

$$\theta^A : C^{\infty}(M) \longrightarrow C^{\infty}(M^A, A), f \longmapsto [\theta(f)]^A,$$

is a vector field on M^A. We say that the vector field θ^A is the prolongation to M^A of the vector field θ. Then, for any $f \in C^{\infty}(M)$ we have:

$$\begin{aligned} (\theta_1 + \theta_2)^A &= \theta_1^A + \theta_2^A; \quad (f \cdot \theta)^A = f^A \cdot \theta^A; \\ \widetilde{(f \cdot \theta)^A} &= f^A \cdot \widetilde{\theta^A}; \quad [\theta_1^A, \theta_2^A] = [\theta_1, \theta_2]^A. \end{aligned}$$

The goal of this work, is to define the Schouten-Nijenhuis bracket on the $C^{\infty}(M^A, A)$-module $\mathfrak{X}^*(M^A)$ of multivector fields on M^A considered as multi-derivations [4] from $C^{\infty}(M)$ into $C^{\infty}(M^A, A)$ and to show that exterior algebra $\mathfrak{X}^*(M^A)$ of multivector fields on M^A is a Lie graded algebra over A. And to finish, we have to estabish an isomorphism between $\mathfrak{X}^q(M^A)$ and the $C^{\infty}(M^A, A)$-module $\mathfrak{L}^q_{alt}(\Omega(M^A, A), C^{\infty}(M^A, A))$ of skew-symmetric multilinear forms of degree q onto the $C^{\infty}(M^A, A)$-module $\Omega(M^A, A)$ of differential A-forms on M^A.

2 q-Multivector fields and q-forms on M^A

We denote $\mathfrak{X}^1(M^A) = \mathfrak{X}(M^A)$, the set of all vector fields on M^A i.e the set of smooth sections of tangent bundle (TM^A, π_{M^A}, M^A). More generally, one denotes, for $2 \leq q \leq m = \dim M^A$, $\mathfrak{X}^q(M^A)$ the set of multivector fields of degree q (or q-multivector fields [2],[3],[5]) on M^A, i.e the set of smooth sections of vector bundle $(\Lambda^q TM^A, \pi_{M^A}, M^A)$.
For any $\xi \in M^A$, $T_\xi M^A$ is an A-module [6]. In this section, we show that, a q-multivector field on M^A is a q-derivation [4].

2.1 The $C^\infty(M^A)$-module of q-multivector fields and of q-forms on M^A

Let $\pi_M : M^A \longrightarrow M$ be map the which assigns any infinitely near point ξ of M^A to its origin $x \in M$, and U be an open neighborhood of M with coordinate system $\{x_1, ..., x_n\}$. Hence $\{x_{i,\alpha}/i = 1, ..., n; \alpha = 1, ..., \dim A = r\}$ is coordinate system of $\pi_M^{-1}(U)$ where $x_{i\alpha} : \pi_M^{-1}(U) \longrightarrow \mathbb{R}, \xi \longmapsto x_{i,\alpha}(\xi)$ is such that

$$\xi(x_i) = \sum_\alpha x_{i\alpha}(\xi) a_\alpha$$

for any $x_i \in C^\infty(M)$, $\forall i = 1, ..., n$.

Lemma 1. *[6] Let $\{x_1, ..., x_n\}$ be a coordinate system on some neighborhood of M. Then we have*

$$a_\alpha \left(\frac{\partial}{\partial x_i}\right)^A = \frac{\partial}{\partial x_{i\alpha}}$$

for any $i = 1, ..., n$; $\alpha = 1, ..., r$.

Lemma 2. *Let $\{x_1, ..., x_n\}$ be a coordinate system on some neighborhood of M. Then we have*

$$a_\alpha^* \circ (dx_i)^A = dx_{i\alpha}.$$

for any $i = 1, ..., n$; $\alpha = 1, ..., r$.

Proof. We have, on one hand:

$$\begin{aligned} a_\alpha^* \left[(dx_i)^A \left[\left(\frac{\partial}{\partial x_j} \right)^A \right] \right] &= a_\alpha^* \left(dx_i \left(\frac{\partial}{\partial x_j} \right) \right)^A \\ &= a_\alpha^* \left((\delta_{ij})^A \right) = a_\alpha^*(\delta_{ij}). \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} dx_{i\alpha}[(\frac{\partial}{\partial x_j})^A] &= \left(\frac{\partial}{\partial x_i} \right)^A (x_{i\alpha}) \\ &= \left(\frac{\partial}{\partial x_i} \right)^A (a_\alpha^* \circ x_i^A) \\ &= a_\alpha^* \left[\left(\frac{\partial}{\partial x_j} \right)^A \right] (x_i^A) \\ &= a_\alpha^* \left[\left(\frac{\partial x_i}{\partial x_j} \right)^A \right] = a_\alpha^*(\delta_{ij}). \end{aligned}$$

Hence we get

$$a_\alpha^* \circ (dx_i)^A = dx_{i\alpha},$$

what ends the proof. ∎

In the coordinate system $\{x_{i\alpha}/i = 1, ..., n; \alpha = 1, ..., r\}$ a q-multivector field $Q \in \mathfrak{X}^q(M^A)$ is written

$$Q = \sum_{1\leq i_1<...<i_q\leq n} \sum_{1\leq\alpha_1<...<\alpha_q\leq r} Q_{i_1...i_q\,\alpha_1...\alpha_q} \frac{\partial}{\partial x_{i_1\alpha_1}} \wedge ... \wedge \frac{\partial}{\partial x_{i_q\alpha_q}},$$

where $Q_{i_1...i_q\,\alpha_1...\alpha_q} \in C^\infty(M^A)$. It follows that $\mathfrak{X}^q(M^A)$ is a $C^\infty(M^A)$-module of q-vector fields on M^A. We denote

$$\mathfrak{X}^*(M^A) = C^\infty(M^A) \oplus \mathfrak{X}^1(M^A) \oplus ... \oplus \mathfrak{X}^m(M^A)$$

the exterior algebra of $C^\infty(M^A)$-module of multivector fields.
In the other respects, for $1 \leq q \leq m$, $\Omega^q(M^A)$ denotes the space of differential q-forms on M^A. In the coordinate system $\{x_{i\alpha}/i = 1, ..., n; \alpha = 1, ..., r\}$ a differential q-form $\varpi \in \Omega^q(M^A)$ is written

$$\varpi = \sum_{1\leq i_1<...<i_q\leq n} \sum_{1\leq\alpha_1<...<\alpha_q\leq r} \varpi_{i_1...i_q\alpha_1...\alpha_q} dx_{i_1\alpha_1} \wedge ... \wedge dx_{i_q\alpha_q},$$

where $\varpi_{i_1...i_q,\alpha_1...\alpha_q} \in C^\infty(M^A)$. Thus $\Omega^q(M^A)$ is a $C^\infty(M^A)$-module of differentiable q-forms on M^A and

$$\Omega^*(M^A) = C^\infty(M^A) \oplus \Omega^1(M^A) \oplus \Omega^2(M^A) \oplus ... \oplus \Omega^m(M^A)$$

denotes exterior algebra of $C^\infty(M^A)$-module multilinear forms.

Proposition 3. *If Q is a q-multivector field on M^A then Q is a q-derivation from $C^\infty(M^A)$ into $C^\infty(M^A)$.*

Theorem 4. *Let $\mathfrak{X}^q(M^A)$ be the $C^\infty(M^A)$-module of q-multivector fields on M^A and $D^q(C^\infty(M^A))$ be the $C^\infty(M^A)$-module of q-derivations from $C^\infty(M^A)$ into $C^\infty(M^A)$. Then the map*

$$\Phi : \mathfrak{X}^q(M^A) \longrightarrow \mathcal{D}^q(C^\infty(M^A)), Q \longmapsto \Phi_Q$$

where

$$\Phi_Q : \overbrace{C^\infty(M^A) \times \cdots \times C^\infty(M^A)}^{q} \longrightarrow C^\infty(M^A), [\Phi_Q(f_1, f_2, ..., f_q)](\xi) = Q(\xi)(f_1, f_2, ..., f_q)$$

is defined by

$$[\Phi_Q(f_1, f_2, ..., f_q)](\xi) = Q(\xi)(f_1, f_2, ..., f_q)$$

for any $f_1, f_2, ..., f_q \in C^\infty(M^A)$ and $\xi \in M^A$, is an isomorphism of $C^\infty(M^A)$-modules.

2.2 The $C^\infty(M^A, A)$-module of q-multivector fields and of q-forms on M^A

Let U be an open neighborhood of M with coordinate system $\{x_1, ..., x_n\}$.
Then, $\left\{ \left(\frac{\partial}{\partial x_1}\right)^A_{|\xi}, ..., \left(\frac{\partial}{\partial x_n}\right)^A_{|\xi} \right\}$ is a free A-basis of $T_\xi M^A$ see [6]. Then, a q-multivector field Q on M^A is written:

$$Q = \sum_{1\leq i_1<...<i_q\leq n} Q_{i_1...i_q} \left(\frac{\partial}{\partial x_{i_1}}\right)^A \wedge ... \wedge \left(\frac{\partial}{\partial x_{i_q}}\right)^A,$$

where $Q_{i_1...i_q} \in C^\infty(U^A, A)$.
In this case $\mathfrak{X}^q(M^A)$ is a $C^\infty(M^A, A)$-module of q-multivector fields. Hence

$$\mathfrak{X}^*(M^A) = C^\infty(M^A, A) \oplus \mathfrak{X}^1(M^A) \oplus ... \oplus \mathfrak{X}^n(M^A)$$

denotes the exterior algebra of $C^\infty(M^A, A)$-module of multivector field on M^A.
In the other respects, for $1 \leq q \leq n$, $\Omega^q(M^A, A)$ denotes the space of differential A-forms of degree q on M^A see[1]. Any differential A-form [1] of degree q on M^A is written

$$\varpi = \sum_{1\leq i_1<\ldots<i_q\leq n} \varpi_{i_1\ldots i_q}(dx_{i_1})^A \wedge \ldots \wedge (dx_{i_q})^A,$$

for any $\varpi_{i_1\ldots i_q} \in C^\infty(U^A, A)$. Then, $\Omega^q(M^A, A)$ is a $C^\infty(M^A, A)$-module of differentiable A-forms of degree q on M^A.
We denote

$$\Omega^*(M^A, A) = C^\infty(M^A, A) \oplus \Omega^1(M^A, A) \oplus \Omega^2(M^A, A) \oplus \ldots \oplus \Omega^n(M^A, A)$$

the exterior algebra of $C^\infty(M^A, A)$-module of differentiable multilinear A-forms on M^A.

Proposition 5. *If Q is a q-multivector field on M^A then Q is a q-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$.*

Proposition 6. *If Q is a q-vector field on M then the map*

$$Q^A : \overbrace{C^\infty(M) \times \cdots \times C^\infty(M)}^{q} \longrightarrow C^\infty(M^A, A), (f_1, \quad \ldots, \quad f_q) \longrightarrow [Q(f_1, \quad \ldots, \quad f_q)]^A$$

is a q-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$.

Theorem 7. *Let $\mathfrak{X}^q(M^A)$ be the $C^\infty(M^A, A)$-module of q-vector fields on M^A and $D^q(C^\infty(M), C^\infty(M^A, A))$ be the $C^\infty(M^A, A)$-module of q-derivations from $C^\infty(M)$ into $C^\infty(M^A, A)$. The map*

$$\Theta : \mathfrak{X}^q(M^A) \longrightarrow \mathcal{D}^q(C^\infty(M), C^\infty(M^A, A)) \ , Q \longmapsto \Theta_Q$$

where

$$\Theta_Q : \overbrace{C^\infty(M) \times \cdots \times C^\infty(M)}^{q} \longrightarrow C^\infty(M^A, A)$$

is defined by

$$[\Theta_Q(f_1, f_2, \ldots, f_q)](\xi) = Q(\xi)(f_1, f_2, \ldots, f_q)$$

for any $f_1, f_2, \ldots, f_q \in C^\infty(M)$ and $\xi \in M^A$; is an isomorphism of $C^\infty(M^A, A)$-modules.

We now define two q-multilinear applications σ_q and γ_q as follows:

$$\sigma_q : \overbrace{C^\infty(M^A, A) \times \ldots \times C^\infty(M^A, A)}^{q} \longrightarrow A \otimes C^\infty(M^A), (\varphi_1, \ldots, \varphi_q) \longmapsto \sum_\alpha a_\alpha \otimes a^*_\alpha(\varphi_1 \times \ldots \times \varphi_q)$$

and

$$\gamma_q : \overbrace{C^\infty(M) \times \ldots \times C^\infty(M)}^{q} \longrightarrow A \otimes C^\infty(M^A), (f_1, \ldots, f_q) \longmapsto \sigma_q(f_1^A, \ldots, f_q^A).$$

For $q = 1$, we have $\sigma_1 = \sigma$ and $\gamma_1 = \gamma$.

Theorem 8. *Let Q be a multivector field on M^A considered as q-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$ then there exists an unique q derivation:*

$$\widetilde{Q} : \overbrace{C^\infty(M^A, A) \times \ldots \times C^\infty(M^A, A)}^{q} \longrightarrow C^\infty(M^A, A) \tag{3}$$

such that

1. *Q be A-multilinear;*
2. *$\widetilde{Q}\left[C^\infty(M^A) \times \ldots \times C^\infty(M^A)\right] \subset C^\infty(M^A)$;*

3. $\widetilde{Q}(f_1^A, f_2^A, ..., f_q^A) = Q(f_1, f_2, ..., f_q)$, *for any* $f_1, f_2, ..., f_q) \in C^\infty(M)$.

Proof. If Q is a multivector field on M^A considered as q-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$ and if

$$\overline{Q} : C^\infty(M^A) \longrightarrow C^\infty(M^A)$$

is an unique derivation such that

$$\sigma^{-1} \circ (id_A \otimes \overline{Q}) \circ \gamma_q = Q,$$

then the map

$$\widetilde{Q} = \sigma^{-1} \circ (id_A \otimes \overline{Q}) \circ \sigma_q : \overbrace{C^\infty(M^A, A) \times ... \times C^\infty(M^A, A)}^{q} \longrightarrow C^\infty(M^A, A)$$

answers to the question. □

Proposition 9. *If* $\mu : A \longrightarrow A$ *is an endomorphism and* Q *is a multivector field on* M^A *considered as* q*-derivation from* $C^\infty(M)$ *into* $C^\infty(M^A, A)$*, then*

$$\widetilde{Q}(\mu \circ f_1^A, \mu \circ f_2^A, ..., \mu \circ f_q^A) = \mu \circ Q(f_1, f_2, ..., f_q) \tag{4}$$

for any $f_1, f_2, ..., f_q) \in C^\infty(M)$.

Proof. From $\widetilde{Q}(f_1^A, f_2^A, ..., f_q^A) = Q(f_1, f_2, ..., f_q)$, we have

$$\widetilde{Q}\left[\sum_{\alpha=1}^{r}(a_\alpha^* \circ f_1^A, a_\alpha^* \circ f_2^A, ..., a_\alpha^* \circ f_q^A) \cdot a_\alpha\right] = \sum_{\alpha=1}^{r} a_\alpha^* \circ Q(f_1, f_2, ..., f_q) \cdot a_\alpha$$

that implies

$$\sum_{\alpha=1}^{r} \widetilde{Q}(a_\alpha^* \circ f_1^A, a_\alpha^* \circ f_2^A, ..., a_\alpha^* \circ f_q^A) \cdot a_\alpha = \sum_{\alpha=1}^{r} a_\alpha^* \circ Q(f_1, f_2, ..., f_q) \cdot a_\alpha.$$

Hence $\widetilde{Q}(a_\alpha^* \circ f_1^A, a_\alpha^* \circ f_2^A, ..., a_\alpha^* \circ f_q^A) = a_\alpha^* \circ Q(f_1, f_2, ..., f_q)$ for all $(a_\alpha^*)_{\alpha=1,...,r}$. Since

$$(\mu \circ f_1^A, \mu \circ f_2^A, ..., \mu \circ f_q^A) = \sum_{\alpha=1}^{r}(a_\alpha^* \circ f_1^A, a_\alpha^* \circ f_2^A, ..., a_\alpha^* \circ f_q^A) \cdot \mu(a_\alpha)$$

it follows that

$$\begin{aligned} \widetilde{Q}(\mu \circ f_1^A, \mu \circ f_2^A, ..., \mu \circ f_q^A) &= \sum_{\alpha=1}^{r} \widetilde{Q}(a_\alpha^* \circ f_1^A, a_\alpha^* \circ f_2^A, ..., a_\alpha^* \circ f_q^A) \cdot \mu(a_\alpha) \\ &= \sum_{\alpha=1}^{r} a_\alpha^* \circ Q(f_1, f_2, ..., f_q) \cdot \mu(a_\alpha) \\ &= \mu \circ Q(f_1, f_2, ..., f_q) \end{aligned}$$

therefore we have the result. □

2.3 Schouten-Nijenhuis bracket on $C^\infty(M^A, A)$-module $\mathfrak{X}^q(M^A)$

Let $\mathfrak{X}^q(M^A)$ be the set of q-multivector fields on M^A considered as q-derivations from $C^\infty(M)$ into $C^\infty(M^A, A)$, we define the Schouten-Nijenhuis bracket (see [2],[5]) as follows:

Theorem 10. *If P and Q are both multivector fields on M^A of degree p and q, respectively, considered as p-derivation and q-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$, respectively. Then, the bracket of P and Q defined by:*

$$
\begin{aligned}
[P,Q]_S(f_1, f_2..., f_{p+q-1}) &= \sum_{\sigma\in\mathfrak{S}_{q,p-1}} \varepsilon(\sigma)\widetilde{P}\left(Q(f_{\sigma(1)},...,f_{\sigma(q)}), f^A_{\sigma(q+1)},...,f^A_{\sigma(p+q-1)}\right)\\
&\quad -(-1)^{(p-1)(q-1)} \sum_{\sigma\in\mathfrak{S}_{p,q-1}} \varepsilon(\sigma)\widetilde{Q}\left(P(f_{\sigma(1)},...,f_{\sigma(p)}), f^A_{\sigma(p+1)},...,f^A_{\sigma(p+q-1)}\right)
\end{aligned}
$$

for any $f_1, f_2..., f_{p+q-1} \in C^\infty(M)$, is a multivector field on M^A of degree $p+q-1$.

Proposition 11. *If P and Q both are p and q-multivector fields on M^A considered as p and q-derivations, respectively from $C^\infty(M)$ into $C^\infty(M^A, A)$, and if $\varphi \in C^\infty(M^A, A)$, then we have:*

1. $[\widetilde{P},\widetilde{Q}]_S = \widetilde{[P,Q]_S}$.
2. $\widetilde{\varphi\cdot P} = \varphi\cdot\widetilde{P}$.
3. $\widetilde{P\wedge Q} = \widetilde{P}\wedge\widetilde{Q}$.

Proof. 1. For any $f_1, f_2..., f_{p+q-1} \in C^\infty(M)$, we have

$$
\begin{aligned}
& [\widetilde{P},\widetilde{Q}]_S(f^A_1, f^A_2..., f^A_{p+q-1})\\
= & \sum_{\sigma\in\mathfrak{S}_{q,p-1}} \varepsilon(\sigma)\widetilde{P}\left(\widetilde{Q}(f^A_{\sigma(1)},...,f^A_{\sigma(q)}), f^A_{\sigma(q+1)},...,f^A_{\sigma(p+q-1)}\right)\\
& -(-1)^{(p-1)(q-1)} \sum_{\sigma\in\mathfrak{S}_{p,q-1}} \varepsilon(\sigma)\widetilde{Q}\left(\widetilde{P}(f^A_{\sigma(1)},...,f^A_{\sigma(p)}), f^A_{\sigma(p+1)},...,f^A_{\sigma(p+q-1)}\right)\\
= & \sum_{\sigma\in\mathfrak{S}_{q,p-1}} \varepsilon(\sigma)\widetilde{P}\left(Q(f_{\sigma(1)},...,f_{\sigma(q)}), f^A_{\sigma(q+1)},...,f^A_{\sigma(p+q-1)}\right)\\
& -(-1)^{(p-1)(q-1)} \sum_{\sigma\in\mathfrak{S}_{p,q-1}} \varepsilon(\sigma)\widetilde{Q}\left(P(f_{\sigma(1)},...,f_{\sigma(p)}), f^A_{\sigma(p+1)},...,f^A_{\sigma(p+q-1)}\right)\\
= & [P,Q]_S(f_1, f_2..., f_{p+q-1}).
\end{aligned}
$$

Since $\widetilde{[P,Q]_S}$ is the unique $p+q-1$-derivation from $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$ such that

$$\widetilde{[P,Q]_S}(f^A_1, f^A_2..., f^A_{p+q-1}) = [P,Q]_S(f_1, f_2..., f_{p+q-1})$$

for any $f_1, f_2..., f_{p+q-1} \in C^\infty(M)$, hence we have

$$[\widetilde{P},\widetilde{Q}]_S = \widetilde{[P,Q]_S};$$

2. We have

$$\left(\varphi\cdot\widetilde{P}\right)(f^A_1, f^A_2..., f^A_p) = \varphi\cdot\widetilde{P}(f^A_1, f^A_2..., f^A_p) = \varphi\cdot P(f_1, f_2..., f_p) = (\varphi\cdot P)(f_1, f_2..., f_p).$$

Since $\widetilde{\varphi\cdot P}$ is the unique p-derivation from $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$ such that

$$\widetilde{\varphi\cdot P}(f^A_1, f^A_2..., f^A_p) = (\varphi\cdot P)(f_1, f_2..., f_p)$$

for any $f_1, f_2..., f_{p+q-1} \in C^\infty(M)$, hence we have

$$\widetilde{\varphi\cdot P} = \varphi\cdot\widetilde{P}.$$

3. For any $f_1, f_2..., f_{p+q-1} \in C^\infty(M)$, we have

$$\begin{aligned}\widetilde{P}\wedge\widetilde{Q}(f_1^A, \overline{f}_2^A, ..., f_{p+q}^A) &= \sum_{\sigma\in\mathfrak{S}_{p,q}} \varepsilon(\sigma)\widetilde{P}(f^A_{\sigma(1)}, ..., f^A_{\sigma(p)})\cdot\widetilde{Q}(f^A_{\sigma(q+1)}, ..., f^A_{\sigma(p+q)}) \\ &= \sum_{\sigma\in\mathfrak{S}_{p,q}} \varepsilon(\sigma)P(f^A_{\sigma(1)}, ..., f_{\sigma(q)})\cdot Q(f_{\sigma(q+1)}, \cdots, f_{\sigma(p+q)}) = P\wedge Q(f_1, f_2, ..., f_{p+q}).\end{aligned}$$

In the other hand, since $\widetilde{P\wedge Q}$ is the unique $p+q$-derivation from $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$, such that

$$\widetilde{P\wedge Q}(f_1^A, f_2^A, ..., f_{p+q}^A) = P\wedge Q(f_1, f_2, ..., f_{p+q})$$

hence we have

$$\widetilde{P\wedge Q} = \widetilde{P}\wedge\widetilde{Q}.$$

□

Let us give now the intrinsic properties of Schouten-Nijenhuis bracket.

Theorem 12. *Let P, Q, R be p-multivector, q-multivector and r-multivector fields on M^A, respectively, considered as p,q and r-derivations from $C^\infty(M)$ into $C^\infty(M^A, A)$, respectively. Let φ, ψ both be smooth functions on M^A with values in A and X a vector field on M^A considered as derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$. The Schouten-Nijenhuis bracket verifies the following equalities:*

1. $[\varphi, \psi]_S = 0$;
2. $[X, P]_S = L_X P$ *the Lie derivative with respect to the vector field X;*
3. $[P, Q]_S = -(-1)^{(p-1)(q-1)}[Q, P]_S$;
4. $[P, Q\wedge R]_S = [P, Q]_S\wedge R + (-1)^{(p-1)q}Q\wedge[P, R]_S$.

let us give some indications of the proof of theorem12.

Proof. (of the theorem12)
We put $P = X_1\wedge X_2\wedge\cdots\wedge X_p$, $Q = Y_1\wedge Y_2\wedge\cdots\wedge Y_q$ and $R = Z_1\wedge Z_2\wedge\cdots\wedge Z_r$. Hence

$$[\widetilde{X_1}\wedge\cdots\wedge\widetilde{X_p}, \widetilde{Y_1}\wedge\cdots\wedge\widetilde{Y_q}]_S = \sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{i+j}[\widetilde{X_i}, \widetilde{Y_j}]_S\wedge\widetilde{X_1}\wedge\cdots\wedge\widehat{\widetilde{X_i}}\cdots\wedge\widetilde{X_p}\wedge\widetilde{Y_1}\wedge\cdots\wedge\widehat{\widetilde{Y_i}}\cdots\wedge\widetilde{Y_p}$$

in this case the proof of the theorem is in the same way like in the classical case see [3]. For achieving the proof, we consider the fact that

$$[\widetilde{P}, \widetilde{Q}]_S(f_1^A, f_2^A, ..., f_{p+q-1}^A) = [P, Q]_S(f_1, f_2, ..., f_{p+q-1}),$$

for any $f_1, f_2..., f_{p+q-1} \in C^\infty(M)$. □

Theorem 13. *Endowed with Schouten-Nijenhuis bracket $\mathfrak{X}^*(M^A)$ is a Lie graded algebra over A and its graded Jacobi identity is given by:*

$$(-1)^{(p-1)(r-1)}[P, [Q, R]_S]_S + (-1)^{(q-1)(p-1)}[Q, [R, P]_S]_S + (-1)^{(r-1)(q-1)}[R, [P, Q]_S]_S = 0$$

for any $P\in\mathfrak{X}^p(M^A)$, $Q\in\mathfrak{X}^q(M^A)$, $R\in\mathfrak{X}^r(M^A)$ considered as p, q and r-derivations from $C^\infty(M)$ into $C^\infty(M^A, A)$, respectively.

Proof. The proof of graded Jacobi identity is made in the same way like in the proof of the theorem12. □

Proposition 14. *If $X_1, \cdots, X_p$ are vector fields on M and $P\in\mathfrak{X}^p(M))$, $Q\in\mathfrak{X}^q(M)$, then we have:*

1. $X_1^A \wedge \cdots \wedge X_p^A = (X_1 \wedge \cdots \wedge X_p)^A$.

2. $P^A + Q^A = (P+Q)^A$, *if* $p = q$.

3. $P^A \wedge Q^A = (P \wedge Q)^A$.

4. $[P^A, Q^A]_S = [P, Q]_S^A$.

Proof. 1. For any $f_1, \cdots, f_q \in C^\infty(M)$, we have:

$$\begin{aligned}(X_1 \wedge \cdots \wedge X_p)(f_1, \cdots, f_p) &= \sum_{\sigma \in S_p} \varepsilon(\sigma) X_1^A(f_{\sigma(1)}) \cdots X_q^A(f_{\sigma(q)}) = \sum_{\sigma \in S_p} \varepsilon(\sigma)(X_1(f_{\sigma(1)}))^A \cdots (X_p(f_{\sigma(p)}))^A \\ &= \left(\sum_{\sigma \in S_p} \varepsilon(\sigma) X_1(f_{\sigma(1)}) \cdots X_p(f_{\sigma(p)}) \right)^A \\ &= ((X_1 \wedge \cdots \wedge X_p)(f_1, \cdots, f_p))^A \\ &= (X_1 \wedge \cdots \wedge X_p)^A (f_1, \cdots, f_p)\end{aligned}$$

hence

$$X_1^A \wedge \cdots \wedge X_p^A = (X_1 \wedge \cdots \wedge X_p)^A.$$

2. For any $f_1, \cdots, f_q \in C^\infty(M)$, we have:

$$\begin{aligned}(P^A + Q)(f_1, \cdots, f_p) &= P^A(f_1, \cdots, f_p) + Q^A(f_1, \cdots, f_p) \\ &= (P(f_1, \cdots, f_p))^A + (Q(f_1, \cdots, f_p))^A \\ &= = (P(f_1, \cdots, f_p) + Q(f_1, \cdots, f_p))^A \\ &= [(P+Q)(f_1, \cdots, f_p)]^A = (P+Q)^A(f_1, \cdots, f_p),\end{aligned}$$

so that

$$P^A + Q^A = (P+Q)^A.$$

3. Let us take $P = X_1 \wedge \cdots \wedge X_p$ and $Q = Y_1 \wedge \cdots \wedge Y_q$. Then, we have:

$$\begin{aligned}P^A \wedge Q^A &= (X_1 \wedge \cdots \wedge X_p)^A \wedge (Y_1 \wedge \cdots \wedge Y_p)^A \\ &= X_1^A \wedge \cdots \wedge X_p^A \wedge Y_1^A \wedge \cdots \wedge Y_p^A \\ &= (X_1 \wedge \cdots \wedge X_p \wedge Y_1 \wedge \cdots \wedge Y_q)^A \\ &= (P \wedge Q)^A;\end{aligned}$$

4. For any $f_1, \cdots, f_{p+q-1} \in C^\infty(M)$, we have:

$$\begin{aligned}& [P^A, Q^A]_S(f_1, \cdots, f_{p+q-1}) \\ =& \sum_{\sigma \in \mathfrak{S}_{q,p-1}} \varepsilon(\sigma) P^A \left(Q^A(f_{\sigma(1)}, ..., f_{\sigma(q)}), f_{\sigma(q+1)}, ..., f_{\sigma(p+q-1)} \right) \\ & -(-1)^{(p-1)(q-1)} \sum_{\sigma \in \mathfrak{S}_{p,q-1}} \varepsilon(\sigma) Q^A \left(P^A(f_{\sigma(1)}, ..., f_{\sigma(p)}), f^A_{\sigma(p+1)}, ..., f^A_{\sigma(p+q-1)} \right) \\ =& \sum_{\sigma \in \mathfrak{S}_{q,p-1}} \varepsilon(\sigma) \left(P \left(Q(f_{\sigma(1)}, ..., f_{\sigma(q)}), f_{\sigma(q+1)}, ..., f_{\sigma(p+q-1)} \right) \right)^A \\ & -(-1)^{(p-1)(q-1)} \sum_{\sigma \in \mathfrak{S}_{p,q-1}} \varepsilon(\sigma) \left(Q \left(P(f_{\sigma(1)}, ..., f_{\sigma(p)}), f_{\sigma(p+1)}, ..., f_{\sigma(p+q-1)} \right) \right)^A \\ =& \begin{bmatrix} \sum_{\sigma \in \mathfrak{S}_{q,p-1}} \varepsilon(\sigma) P \left(Q(f_{\sigma(1)}, ..., f_{\sigma(q)}), f_{\sigma(q+1)}, ..., f_{\sigma(p+q-1)} \right) \\ -(-1)^{(p-1)(q-1)} \sum_{\sigma \in \mathfrak{S}_{p,q-1}} \varepsilon(\sigma) Q \left(P(f_{\sigma(1)}, ..., f_{\sigma(p)}), f_{\sigma(p+1)}, ..., f_{\sigma(p+q-1)} \right) \end{bmatrix}^A \\ =& [[P, Q]_S(f_1, \cdots, f_{p+q-1})]^A \\ =& [P, Q]_S^A(f_1, \cdots, f_{p+q-1}),\end{aligned}$$

hence

$$[P^A, Q^A]_S = [P, Q]_S^A.$$

□

We presently are going to estabish the relation between a q-multivector field and a q-form on M^A through the following result.

Theorem 15. *Let $\mathfrak{X}^q(M^A)$ be the set of all multivector fields on M^A of degree q considered as q-derivation from $C^\infty(M)$ into $C^\infty(M^A, A)$ and $\mathfrak{L}^q_{alt}(\Omega(M^A, A)$, the $C^\infty(M^A, A)$-module of skew-symmetric multilinear forms of degree q on $\Omega(M^A, A)(=\Omega^1(M^A, A))$. Then the map*

$$\Theta : \mathfrak{X}^q(M^A) \longrightarrow \mathfrak{L}^q_{alt}(\Omega(M^A, A), C^\infty(M^A, A))$$

such that

$$([\Theta(Q)](\varpi_1, \cdots, \varpi_q))(\xi) = \widetilde{Q}(\xi)(\varpi_1(\xi), \cdots, \varpi_q(\xi)),$$

for any $\xi \in M^A$, $Q \in \mathfrak{X}^q(M^A)$ and $\varpi_1, ...\varpi_q \in \Omega(M^A, A)$ is an isomorphism of $C^\infty(M^A, A)$-modules.

Theorem 16. *Let Q be a multivector field of degree q on M. Then there exists an unique multivector field of degree q on M^A*

$$Q^A : \overbrace{\Omega(M^A, A) \times \cdots \times \Omega(M^A, A)}^{q} \longrightarrow C^\infty(M^A, A)$$

such that

$$Q^A(a_1\omega_1^A, \cdots, a_q\omega_q^A) = a_1 \times \cdots \times a_q[Q(\omega_1, \cdots, \omega_q)]^A \tag{5}$$

for any $a_1, ..., a_q \in A$ and $\omega_1, ..., \omega_q \in \Omega(M)$.

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