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Numerical Comparison Between RBF Schemes With Respect to Other Approaches to Solve Fractional Partial Differential Equations and Their Advantages When Choosing Non-Uniform Nodes

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Abstract

In this work, some cases of Fractional Partial Differential Equations (FPDE) are considered and resolved numerically using a meshless method via Radial Basis Functions (RBF). Taking different types of fractional derivatives and non-uniform collocation nodes. The results are compared with those obtained by previous works. This type of approach sounds like a good choice to deal with problems in higher dimensions or non-uniform data.

Keywords: Fractional Partial Differential Equation (FPDE), Meshless Methods, Radial Basis Functions (RBF), fractional diffusion-wave equation (DWFE), Numerical solution

1. Introduction

The growing interest in Fractional Calculus has been motivated by applications of fractional equations in different fields of research. For example, models in Physics and Chemistry [1], [2], [3], [4], in problems of flow or heat transfer, but also in other fields such as finances [5], [6].

Recent important applications of these anomalous diffusion-convection models are those related to oil extraction and hydrological models for aquifers, food production and water distribution in large cities. C. Fuentes et al. ([7], [8]) proposed a model that represents anomalous diffusion of petroleum in three types of medium with different porosities: fractures, vugs and matrix. Determining the behavior of the fluid within the reservoir and the loss of permeability of the medium helps with research of oil migration mechanisms. In this context, the geometric and physical interpretation of the fractional derivatives in these differential equations is the **subdivusive** or **super-diffusive** diffusion.

But, fractional equations present serious numerical and mathematical difficulties in the context of diffusion equations in larger dimensions. There is no established theory as in the case of Partial Differential Equations (PDE), so it

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is important to count on efficient and precise numerical schemes. The proposal is to consider Radial Basis Functions (RBF) [9], taking into account that the geometry of the domain does not determine the efficiency of the algorithm and sounds like an immediate alternative to generalize to larger dimensions.

The objective of this paper is to show that the RBF methodology is an alternative to solve fractional partial differential equations, without necessarily considering uniformly distributed data, for different types of fractional derivatives, such as Riemann-Liouville, Riesz and Caputo, for time or for space. In addition, the results are compared with those obtained in the considered original works.

This type of technique approximates the solution by means of a linear combination of radial basis functions, which are easy to implement, globally defined and exponential convergence, but they produce dense and badly conditioned interpolation matrixes. On the other hand, some RBFs contain a shape parameter, a number that influences the precision of the numerical results ([10] and [11]). To attenuate the bad condition due to the shape parameter. With RBF-QR, very high precision and convergence are obtained without the need to increase a polynomial term to the interpolator [12]. Although the computational cost increases, so does the range of problems to which such a technique can be applied.

2. Fractional Calculus

The notation introduced by Leibniz in the seventeenth century gave rise to the idea of generalizing $\dfrac{d^n f(x)}{dx^n}$ for non integer n (see figure 1). He himself, in his letters to L'Hôpital (1695) and Wallis (1697) opens that possibility. But it is not until Euler (1738) that the first step was taken in observing that the derivative $\dfrac{d^n x^b}{dx^n}$ for non-integer n makes sense. Laplace (1812) proposed an idea to calculate non-integer derivatives, from an integral. Lacroix (1820) takes up the idea of Euler and using the gamma function Γ gives the exact formula to evaluate $\dfrac{d^{1/2} x^b}{dx^{1/2}}$.

Grünwald [13](1867) and Letnikov [14] (1868) developed a fractional derivative based on schemes in differences, for derivatives of positive integer order:

$$\begin{aligned}
f^{(1)}(x) &= \lim_{h\to 0} \frac{f(x) - f(x-h)}{h} \\
f^{(2)}(x) &= \lim_{h\to 0} \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2} \\
f^{(3)}(x) &= \lim_{h\to 0} \frac{f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h)}{h^3} \\
&\ \vdots \\
f^{(n)}(x) &= \lim_{h\to 0} \frac{1}{h^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x-ih)
\end{aligned}$$

and so, the right and left derivatives Grünwald-Letnikov with order $\alpha > 0$ of a given function f are defined as

$$
\begin{aligned}
f(x) &= \sin x \\
\frac{\partial f}{\partial x} &= \sin\left(x + 1 \cdot \tfrac{\pi}{2}\right) \\
\frac{\partial^2 f}{\partial x^2} &= \sin\left(x + 2 \cdot \tfrac{\pi}{2}\right) \\
\frac{\partial^3 f}{\partial x^3} &= \sin\left(x + 3 \cdot \tfrac{\pi}{2}\right) \\
&\vdots \\
\frac{\partial^n f}{\partial x^n} &= \sin\left(x + n \cdot \tfrac{\pi}{2}\right)
\end{aligned}
$$

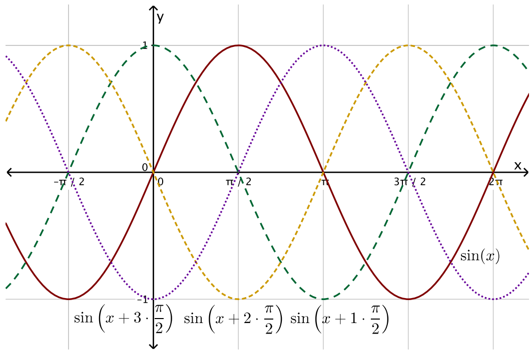


Figure 1: *If α is a non-natural number, does it make sense to put* $\dfrac{\partial^\alpha f}{\partial x^\alpha} = \sin\left(x + \alpha \cdot \dfrac{\pi}{2}\right)$*?*

$$
{}_{a}^{GL}D_x^\alpha f(x) = \lim_{\substack{h\to 0,\\ nh = x-a}} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(x - kh) \tag{1}
$$

and

$$
{}_{x}^{GL}D_b^\alpha f(x) = \lim_{\substack{h\to 0,\\ nh = b-x}} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(x + kh). \tag{2}
$$

where the binomial coefficient $\binom{\alpha}{k}$ is defined as $\dfrac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$.

Perhaps the first serious attempt to give a more general definition to the fractional derivative is due to Liouville (1832), setting the result $\dfrac{d^\nu}{dx^\nu} f(x) = a^\nu e^{ax}$ for ν positive.

Of equal importance to Liouville's works, Riemann in his article [15] written in 1847 but published in 1876, 10 years after his death, obtained an expression for fractional integration. Together with Liouville, they established one of the main definitions of the Differential-Integral Fractional Calculus.

From these works, great personalities of mathematics of the nineteenth and twentieth century were dedicated to formalize the concept of fractional derivative. To mention a few, there are Fourier (1822), Abel (1823), Liouville (1832), Holmgren (1865-1866), Grünwald (1867), Letnikov (1868), Nekrasov (1881-1891), Hadamard (1892), Weyl (1917), Riesz (1922-1923), M. Caputo (1967),etc.

In this part we define the fractional derivatives that we use in this work, first explaining some characteristics of the notation.

For certain fractional derivatives, we start from the fact that derivation and integration processes are part of a single process, see figure 2. And so the fractional derivative is an interpolation of this sequence of operators (Podlubny

[16]). The notation used by Davis [17]

$$ {}_aD_x^{\alpha} f(x) \tag{3}$$

refers to the derivative of order α of a function $f(x)$, where the subscripts a and x (called *terminals* by Ross [18]) establish the differentiation domain. *Fractional integrals* correspond to negative α values.

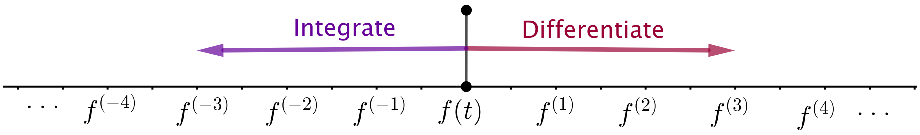


Figure 2: *Unifying the derivation and integration processes*

To define the fractional integral, let's start with an example of Basic Integral Calculus. Let f be a continuous function in an interval $[a, b]$ and we construct the operator $\mathcal{J}$ as

$$\mathcal{J}\left(f(t)\right) := \int_a^t f(\tau)d\tau \qquad \text{for } t \text{ between } a \text{ and } b, \tag{4}$$

and, using Fubini

$$\mathcal{J}^2\left(f(t)\right) = \int_a^t \left(\int_a^{\tau_1} f(\tau)d\tau\right) d\tau_1 = \int_a^t (t-\tau)f(\tau)d\tau. \tag{5}$$

Continuing with the inductive process we obtain the **Cauchy formula**, of repeated integration

$$\mathcal{J}^n\left(f(t)\right) := \int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau)d\tau \cdots d\tau_2 d\tau_1 = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau. \tag{6}$$

for $n \in \mathbb{N}$, $a, t \in \mathbb{R}$ and $t > a$. If we replace the natural number n with the positive real value α and the factorial with the Gamma function $\Gamma(\alpha)$, we obtain

Definition 1. *The* **left-sided Riemann-Liouville fractional integral** *of order α of function $f(x)$ is defined as*

$${}_a^{RL}D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)d\tau, \quad x > a. \tag{7}$$

Definition 2. *The* **right-sided Riemann-Liouville fractional integral** *of order α of function $f(x)$ is defined as*

$${}_x^{RL}D_b^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau)d\tau, \quad x < b. \tag{8}$$

Thus, according to Figure (2) we can obtain a fractional derivative (positive α value)"moving" to the right. And it is the idea to define the fractional operators that we will use in this work.

Definition 3. *The* **left-sided Riemann-Liouville fractional derivative** *of order α of function $f(x)$ is defined as*

$$ {}^{RL}_{a}D^{\alpha}_{x}f(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x (x-\tau)^{m-\alpha-1}f(\tau)d\tau, \quad x > a, \tag{9} $$

where $m = \lceil \alpha \rceil$.

Definition 4. *The* **right-sided Riemann-Liouville fractional derivative** *of order α of function $f(x)$ is defined as*

$$ {}^{RL}_{x}D^{\alpha}_{b}f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_x^b (\tau-x)^{m-\alpha-1}f(\tau)d\tau, \quad x < b, \tag{10} $$

where $m = \lceil \alpha \rceil$.

Another fractional derivative to consider is the fractional derivative of Riesz and it is used for equations that model superdiffusion, anomalous transport and kinetics of chaos dynamics; for further reference see [19, 20]. The definition can be stated as

Definition 5. *The* **Riesz fractional operator** *for α on a finite interval $0 \leq x \leq L$ is defined as*

$$ \frac{\partial^\alpha}{\partial |x|^\alpha} f(x,t) = -c_\alpha ({}^{RL}_{a}D^{\alpha}_{x} + {}^{RL}_{x}D^{\alpha}_{b}) f(x,t), \tag{11} $$

where

$$ c_\alpha = \frac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)}, \quad \alpha \neq 1, \tag{12} $$

$$ {}^{RL}_{a}D^{\alpha}_{x}f(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x (x-\tau)^{m-\alpha-1}f(\tau)d\tau, \tag{13} $$

$$ {}^{RL}_{x}D^{\alpha}_{b}f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_x^b (\tau-x)^{m-\alpha-1}f(\tau)d\tau. \tag{14} $$

Although the Fractional Calculus has been motivated by applications in different fields of research, approaches such as the fractional derivative of Riemann-Liouville have disadvantages when dealing with initial conditions. To counteract this drawback, we consider the fractional operator proposed by M. Caputo in his 1967 article [21] and two years later in his book [22]. Such an operator can be established as:

Definition 6. *The* **left-sided Caputo fractional derivative** *of order α of function $f(x)$ is defined as*

$$ {}^{C}_{a}D^{\alpha}_{x}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau, \quad x > a, \tag{15} $$

and $m = \lceil \alpha \rceil$.

The fact that derivative of integer order appears within the integral in definition (6), makes Caputo derivative the most suitable for dealing with initial conditions of the FPDEs.

A result that will be useful for us is that of the formula of the Caputo fractional derivative of the exponential $e^{\lambda x}$, with λ complex constant. The proof of the following theorem can be reviewed in the work of Ishteva [23].

Theorem 1. *Let $\alpha \in \mathbb{R}$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$. So the Caputo derivative of the exponential function has the form*

$$ {}_{x}^{C}D_{b}^{\alpha} e^{\lambda x} = \sum_{i=0}^{\infty} \frac{\lambda^{i+m} x^{i+m-\alpha}}{\Gamma(i+1+m-\alpha)} = \lambda^{m} x^{m-\alpha} E_{1,m-\alpha+1}(\lambda x), \tag{16}$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0 \quad \beta > 0, \tag{17}$$

is the **Mittag-Leffler function***.*

Example 1. *The fractional derivative of Caputo of the sine function $f(x) = \sin x$ can be calculated taking into account the representation*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}; \tag{18}$$

and using the linearity property of the Caputo derivative and the formula for the exponential (see theorem 1) we have

$$\begin{aligned} {}^{C}{}_{a}D_{x}^{\alpha} \sin \lambda x =& {}^{C}{}_{a}D_{x}^{\alpha} \frac{e^{i\lambda x} - e^{-i\lambda x}}{2i} \\ =& \frac{1}{2i} \left({}^{C}{}_{a}D_{x}^{\alpha} e^{i\lambda x} - {}^{C}{}_{a}D_{x}^{\alpha} e^{-i\lambda x} \right) \\ =& \tfrac{1}{2i}\left((i\lambda)^m x^{m-\alpha} E_{1,m-\alpha+1}(i\lambda x) - (-i\lambda)^m x^{m-\alpha} E_{1,m-\alpha+1}(-i\lambda x)\right) \\ =& -\tfrac{1}{2} i (i\lambda)^m x^{m-\alpha} (E_{1,m-\alpha+1}(i\lambda x) - (-1)^m E_{1,m-\alpha+1}(-i\lambda x)) \end{aligned}$$

where m is the natural number such that $m-1 < \alpha < m$, i is the complex unit and $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function.

Figure 3 shows how the graph of the sine function changes over the interval $[0, 2\pi]$, when the order of the Caputo derivative goes from 0 to 2.

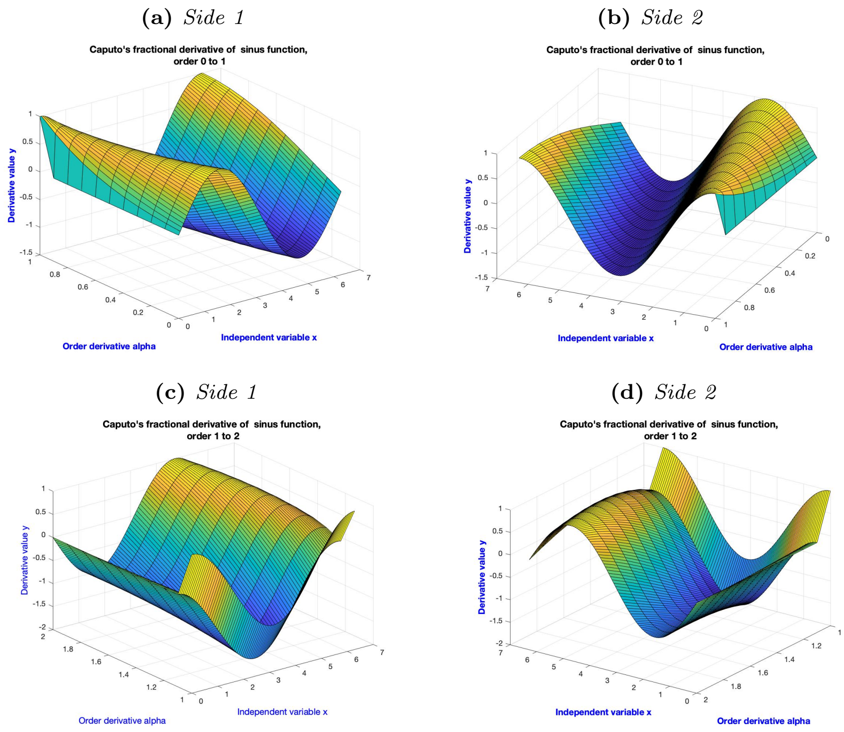
3. Radial Basis Function Methodology

Radial Basis Function's method is a kind of a meshless method, i.e. the approximation doesn't need an structured mesh, whose interpolators are defined in terms of a distance. It has several advantages like easy implementation, being adaptable to spaces in higher dimensions and spectrally accurate.

A standard interpolator in terms of radial basis functions (see [12]), given the data u_k, of some real function u, in the corresponding **collocation nodes** $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, has the form

$$s_{\varepsilon}(\mathbf{x}) = \sum_{k=1}^{N} \lambda_k \phi_k(\mathbf{x}) \tag{19}$$

Figure 3: *Caputo's fractional derivatives for the sine function, $y = \sin x$, $x \in [0, 2\pi]$.*



(a) *Side 1* **(b)** *Side 2* **(c)** *Side 1* **(d)** *Side 2*

where $\phi_k(\mathbf{x}) = \phi(\varepsilon||\mathbf{x} - \mathbf{x_k}||)$, ϕ is one real variable function and the constant value ε is called **shape parameter**. Generally $||\cdot||$ denotes the Euclidean norm.

The unknown coefficients λ_k can be determined by interpolation conditions

$$s_\varepsilon(\mathbf{x}_i) = u_i, \qquad i = 1, \ldots, N.$$

producing a linear equation system which can be described in matrix form

$$A\lambda = u_X \tag{20}$$

where $A = \begin{bmatrix} \Phi(\mathbf{x}_1)^T \\ \vdots \\ \Phi(\mathbf{x}_N)^T \end{bmatrix}$, $\Phi(\mathbf{x}) := \begin{bmatrix} \phi_1(\mathbf{x}) \\ \vdots \\ \phi_N(\mathbf{x}) \end{bmatrix}$, $\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix}$ and $u_X = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}$. A is called **Gram matrix**.

The main disadvantage of this method is that Gram matrix is an ill-conditioned matrix. Besides this, there is no general method to choose shape parameter and much less in fractional problems. In some cases, can be estimated as noted in [24] and [9].

The next step is estimate the action of a differential operator $\mathcal{L}$ to the function $u(\mathbf{x})$, in every **evaluation node** from the set $Z = \{\mathbf{z}_1, \ldots, \mathbf{z}_{N_e}\}$, given the values of the function at collocation nodes $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$.

Assuming that $\mathcal{L}$ is linear, we apply it to the RBF interpolator, for each of the evaluation nodes, as an approximation to the values of $\mathcal{L}$ in u:

$$\mathcal{L}s_\varepsilon(\mathbf{z}_i) = \sum_{k=1}^{N} \lambda_k \mathcal{L}\phi_k(\mathbf{z}_i), \qquad i = 1, 2, \ldots, N_e; \tag{21}$$

which leads to the matrix form

$$\mathcal{L}s_\varepsilon = B\lambda \tag{22}$$

where

$$\mathcal{L}s_\varepsilon = \begin{bmatrix} \mathcal{L}s_\varepsilon(\mathbf{z}_1) \\ \vdots \\ \mathcal{L}s_\varepsilon(\mathbf{z}_{N_e}) \end{bmatrix}, B = \begin{bmatrix} \mathcal{L}\phi_1(\mathbf{z}_1) & \cdots & \mathcal{L}\phi_N(\mathbf{z}_1) \\ \vdots & \vdots & \vdots \\ \mathcal{L}\phi_1(\mathbf{z}_{N_e}) & \cdots & \mathcal{L}\phi_N(\mathbf{z}_{N_e}) \end{bmatrix}.$$

And the coefficient matrix λ is calculated by solving for λ in equation (20). In this way, from equation (22) we obtain

$$\mathcal{L}s_\varepsilon = (BA^{-1})u_X. \tag{23}$$

Thus, the matrix BA^{-1} approximates the values of the differential operator applied to the function.

Table (1) shows some examples, about the more used real functions $\phi(r)$. But in this work the chosen function is the Gaussian function.

Table 1: *Common Elections for $\phi(r)$*

RBF name	RBF $\phi(r)$
Piecewiese smooth, global	
Polyharmonic spline	r^m, $m = 1, 3, 5, \ldots$
	$r^m \ln(r)$, $m = 2, 4, 6, \ldots$
Compact support ('Wendland')	$(1 - \varepsilon r)_+^m p(\varepsilon r)$, p polynomial.
Smooth, global	
Gaussian (GA)	$e^{-(\varepsilon r)^2}$
Multiquadric (MQ)	$\sqrt{1 + (\varepsilon r)^2}$
Inverse Quadric (IQ)	$1/(1 + (\varepsilon r)^2)$
Inverse multiquadric (IMQ)	$1/\sqrt{1 + (\varepsilon r)^2}$
Bessel (BE) ($d = 1, 2, \ldots$)	$J_{d/2-1}(\varepsilon r)/(\varepsilon r)^{d/2-1}$

3.1. QR decomposition and RBF for Interpolation

In order to reduce the bad conditioning of the Gram matrix due in part to the form parameter ε, a change of basis is made [25], in terms of Chebyshev polynomials. With RBF-QR, very high precision and convergence are obtained without the need to increase a polynomial term to the interpolator (Larsson et al [12]). The QR decomposition interpolation routines that were used for the numerical part were developed by Larsson et. al. ([25, 12]) and can be reviewed in www.it.uu.se/research/scientific_computing/software/rbf_qr.

Since the Chebyshev polynomials have domain $[-1, 1]$ and starting from an interval of the form $[a, b]$, $a < b$, a change of variable is applied:

$$y = \frac{x - \left(\frac{b+a}{2}\right)}{\frac{b-a}{2}}, \quad x \in [a, b]. \tag{24}$$

so, for the variable $y \in [-1,1]$, the RBF basis is rewritten as

$$
\begin{aligned}
\phi_k(x) &= e^{-\varepsilon^2(x-x_k)^2} \qquad k = 1,2,\ldots,N \\
&= e^{-\varepsilon^2\left(\frac{b-a}{2}\right)^2(y-y_k)^2} \qquad \left[y_k := \frac{x_k - \left(\frac{b+a}{2}\right)}{\frac{b-a}{2}}\right] \\
&= e^{-\varepsilon_y^2(y-y_k)^2} \qquad \left[\varepsilon_y := \left(\frac{b-a}{2}\right)\varepsilon\right] \\
&= \sum_{j=0}^{\infty} d_j c_j(y_k) \underbrace{e^{-\varepsilon_y^2 y^2} T_j(y)}_{V_j(y)}, \qquad (25)
\end{aligned}
$$

where $d_j = \dfrac{2\varepsilon_y^{2j}}{j!}$, $c_j(y_k) = t_j e^{-\varepsilon_y^2 y_k^2} y_k^j {}_0F_1([], j+1, \varepsilon_y^4 y_k^2)$, with $t_0 = \frac{1}{2}$ and $t_j = 1$ para $j > 0$, ${}_mF_n$ hypergeometric function, and T_j is the Chebyshev polynomial of degree j.

The series 25 is truncated in $j = p \geq N$, based on the size of its terms (see [25]). So

$$
\begin{aligned}
\Phi(x)^T &= [\phi_1(x) \cdots \phi_N(x)] \\
&\approx V(y)^T \begin{bmatrix} c_0(y_1) & c_0(y_2) & \cdots & c_0(y_N) \\ d_1c_1(y_1) & d_1c_1(y_2) & \cdots & d_1c_1(y_N) \\ \vdots & \vdots & & \vdots \\ d_pc_p(y_1) & d_pc_p(y_2) & \cdots & d_pc_p(y_N) \end{bmatrix} \\
&= V(y)^T \underbrace{\begin{bmatrix} d_0 & & \\ & \ddots & \\ & & d_p \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} c_0(y_1) & c_0(y_2) & \cdots & c_0(y_N) \\ c_1(y_1) & c_1(y_2) & \cdots & c_1(y_N) \\ \vdots & \vdots & & \vdots \\ c_p(y_1) & c_p(y_2) & \cdots & c_p(y_N) \end{bmatrix}}_{C^T}
\end{aligned}
$$

where $V(y)^T = \left[e^{-\varepsilon_y^2 y^2} T_0(y) \quad \cdots \quad e^{-\varepsilon_y^2 y^2} T_p(y)\right]$, the elements of the matrix C are $O(1)$ and each element d_k of the matrix D is proportional to $\varepsilon_y^{2m_k}$ con $m_{k+1} \geq m_k$.

Matrix C is factored into the QR form

$$
\begin{aligned}
\Phi(x)^T &\approx V(y)^T D C^T \\
&= V(y)^T D R^T Q^T.
\end{aligned}
$$

The Q matrix is split into blocks

$$
\begin{aligned}
\Phi(x)^T &\approx V(y)^T \begin{bmatrix} D_1 & \mathbf{O} \\ \mathbf{O} & D_2 \end{bmatrix} \begin{bmatrix} R_1^T \\ R_2^T \end{bmatrix} Q^T \\
&= V(y)^T \begin{bmatrix} D_1 R_1^T \\ D_2 R_2^T \end{bmatrix} Q^T
\end{aligned}
$$

where R_1 and D_1 are square of order $N \times N$ and invertible.

Multiplying both terms by the matrix $Q\left(R_1^T\right)^{-1}D_1^{-1}$

$$\Phi(x)^T Q\left(R_1^T\right)^{-1}D_1^{-1} \qquad \approx V(y)^T \begin{bmatrix} I_N \\ D_2R_2^T\left(R_1^T\right)^{-1}D_1^{-1}\end{bmatrix}$$

Thus, the new basis is

$$\Psi(y)^T = \begin{bmatrix}\psi_1(y) & \cdots & \psi_N(y)\end{bmatrix} = V(y)^T \begin{bmatrix} I_N \\ \tilde{R}^T\end{bmatrix} \tag{26}$$

where $\tilde{R}^T = D_2R_2^T\left(R_1^T\right)^{-1}D_1^{-1}$.

The Gram matrix is taken as

$$A = \begin{bmatrix}\Phi(x_1)^T \\ \vdots \\ \Phi(x_N)^T\end{bmatrix} \approx \begin{bmatrix}V(y_1)^T \\ \vdots \\ V(y_N)^T\end{bmatrix}\begin{bmatrix} I_N \\ \tilde{R}^T\end{bmatrix} D_1R_1^TQ^T \tag{27}$$

The problem of interpolation (20) is rewritten in the form

$$A_\Psi\lambda_\Psi = u_X \tag{28}$$

where $A_\Psi = \begin{bmatrix}V(y_1)^T \\ \vdots \\ V(y_N)^T\end{bmatrix}\begin{bmatrix} I_N \\ \tilde{R}^T\end{bmatrix}$ y $\lambda_\Psi = D_1R_1^TQ^T\lambda$.

In some way, the numerical singularity that induces the shape parameter has been removed, and this process will be used to discretize fractional operators using RBF with small shape parameters, see [9].

3.2. Descomposición QR para discretización de operadores diferenciales

Now we want to use the new basis (26) to approximate the values of a linear differential operator $\mathcal{L}$ applied to a function u, on a set of evaluation nodes $Z = \{z_1, \ldots, z_{N_e}\}$. Taking into account that each Ψ depends linearly on the functions $V(y)$,

$$\mathcal{L}\Psi(y)^T = \mathcal{L}V(y)^T \begin{bmatrix} I_N \\ \tilde{R}^T\end{bmatrix} \tag{29}$$

with $\mathcal{L}V(y)^T = \begin{bmatrix}\mathcal{L}V_0(y) & \cdots & \mathcal{L}V_p(y)\end{bmatrix}$.

Taking into account that the matrix D discretizes the operator $\mathcal{L}u(x)$ on the set of evaluation nodes, see (23)

$$\begin{aligned} D &= BA^{-1} \\ &= \begin{bmatrix}\mathcal{L}\phi_1(z_1) & \cdots & \mathcal{L}\phi_N(z_1) \\ \vdots & \vdots & \vdots \\ \mathcal{L}\phi_1(z_{N_e}) & \cdots & \mathcal{L}\phi_N(z_{N_e})\end{bmatrix} A^{-1} = \begin{bmatrix}\mathcal{L}\Phi^T(z_1) \\ \vdots \\ \mathcal{L}\Phi^T(z_{N_e})\end{bmatrix} A^{-1}, \end{aligned}$$

scaling the interval $[-1, 1]$, we have

$$\eta_i = \frac{z_i - \left(\frac{b+a}{2}\right)}{\frac{b-a}{2}}, \quad z_i \in [a, b], \quad i = 1, \ldots, N_e. \tag{30}$$

and using (27)

$$D \approx \begin{bmatrix} \mathcal{L}V(\eta_1)^T \\ \vdots \\ \mathcal{L}V(\eta_{N_e})^T \end{bmatrix} \begin{bmatrix} I_N \\ \tilde{R}^T \end{bmatrix} D_1 R_1^T Q^T \left(\begin{bmatrix} V(y_1)^T \\ \vdots \\ V(y_N)^T \end{bmatrix} \begin{bmatrix} I_N \\ \tilde{R}^T \end{bmatrix} D_1 R_1^T Q^T \right)^{-1}$$

we obtain

$$D \approx B_{\mathcal{L}\Psi} A_{\Psi}^{-1} \tag{31}$$

where $B_{\mathcal{L}\Psi} = \begin{bmatrix} \mathcal{L}V(\eta_1)^T \\ \vdots \\ \mathcal{L}V(\eta_{N_e})^T \end{bmatrix} \begin{bmatrix} I_N \\ \tilde{R}^T \end{bmatrix}$ and $A_{\Psi} = \begin{bmatrix} V(y_1)^T \\ \vdots \\ V(y_N)^T \end{bmatrix} \begin{bmatrix} I_N \\ \tilde{R}^T \end{bmatrix}$

The interpolation matrix A (20) for the Gaussian RBF is non-singular for collocation nodes different from each other and positive shape parameters. The corresponding matrix A_{Ψ} is no singular if the change of basis is "well defined", as discussed in [12].

As it is of our interest to apply fractional operators on a set of nodes not necessarily equispaced, we will use formulas in their representation in series. In addition, we must specifically calculate the fractional derivatives of $\phi_k(x) = e^{-\varepsilon^2(x-x_k)^2}$ for the simple case of RBF's scheme and $V_j(y) = e^{-\varepsilon_y^2 y^2} \cdot T_j(y)$ for the RBF-QR case.

The following formulas that were applied for the fractional part can be reviewed in the work of Mohammadi, Maryam and Schaback, Robert [26].

For all $x, y \in \mathbb{R}$ and $x > a$ we have

$${}_a^{RL}D_x^{\alpha}\phi(|x-y|) = \xi^{-\alpha}\left({}_{\xi(a-y)}^{RL}D_x^{\alpha}\phi\right)(|x-y|), \tag{32}$$

and

$${}_a^{C}D_x^{\alpha}\phi(|x-y|) = \xi^{-\alpha}\left({}_{\xi(a-y)}^{C}D_x^{\alpha}\phi\right)(|x-y|), \tag{33}$$

where $\xi = \operatorname{sign}(x-y)$.

Theorem 2. *For $a \neq 0, n \in \mathbb{N}$ and $x > a$ we have*

$${}_a^{RL}I_x^{\alpha}x^n = n!(x-a)^{\alpha}\sum_{k=0}^{n}\frac{a^{n-k}(x-a)^k}{(n-k)!\Gamma(\alpha+k+1)}, \tag{34}$$

$${}_a^{RL}D_x^{\alpha}x^n = n!(x-a)^{-\alpha}\sum_{k=0}^{n}\frac{a^{n-k}(x-a)^k}{(n-k)!\Gamma(k-\alpha+1)}, \tag{35}$$

$${}_a^{C}D_x^{\alpha}x^n = n!a^{-m}(x-a)^{m-\alpha}\sum_{k=0}^{n-m}\frac{a^{n-k}(x-a)^k}{(n-m-k)!\Gamma(m-\alpha+k+1)} \tag{36}$$

Thus, for example, for the calculation of the fractional derivative of the function $\phi_k(x) = e^{-\varepsilon^2(x-x_k)^2}$ we first develop its expansion in MacLaurin series and then the operator is applied to each term

$$\begin{aligned} D^{\alpha}\phi_k(x) &= D^{\alpha}e^{-\varepsilon^2|x-x_k|^2} \\ &= \sum_{j=0}^{\infty}\frac{(-1)^j\varepsilon^{2j}}{j!}\left(\sum_{i=0}^{2j}\frac{(2j)!(-1)^i x_k^{2j-i}}{i!(2j-i)!}D^{\alpha}x^i\right) \end{aligned} \tag{37}$$

where D^{α} is a fractional derivative.

4. Numerical Examples

4.1. Fractional Partial Differential Equation with Riesz space fractional derivatives

We consider the fractional partial differential equation proposed by Yang et. al. [27] and again considered by Mohammadi and Schaback [26]:

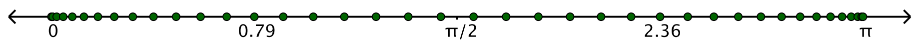
$$\frac{\partial u(x,t)}{\partial t} = -K_\alpha \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha}, \quad x \in [0,\pi], \quad t \in (0,T], \tag{38}$$
$$u(x,0) = u_0(x),$$
$$u(0,t) = u(\pi,t) = 0,$$

where u could be for example the concentration of a dissolute substance, and K_α the dispersion coefficient. Here we use the Riesz fractional derivative and $1 < \alpha < 2$.

Since RBF applies for non-equispaced nodes, we choose a Chebyshev distribution of the collocation nodes over the interval $[0,\pi]$ (see figure 4):

$$x_i = \frac{\pi}{2}\cos(\theta_i) + \frac{\pi}{2}, \qquad \theta_i = \pi - i\frac{\pi}{N-1}, \quad i = 0,1,\ldots,N-1$$

Figure 4: *Chebyshev nodes distribution over* $[0,\pi]$ *interval.*



Then, we solve the FPDE by the method of lines based on the spatial trial spaces spanned by the Lagrange basis $L_1(x),\ldots,L_N(x)$ associated to RBFs $\phi_j(x) = \phi(|x - x_j|)$, $j = 1,2,\ldots,N$.

$$L(x)^T = \Phi(x)^T A^{-1} \tag{39}$$

where

$$L(x)^T = \begin{bmatrix} L_1(x) & \cdots & L_N(x) \end{bmatrix},$$
$$\Phi(x)^T = \begin{bmatrix} \phi_1(x) & \cdots & \phi_N(x) \end{bmatrix}$$

and the Gram matrix

$$A = \begin{bmatrix} \phi_1(x_1) & \cdots & \phi_N(x_1) \\ \vdots & \vdots & \vdots \\ \phi_1(x_N) & \cdots & \phi_N(x_N) \end{bmatrix}$$

If $\mathcal{L}$ is a differential operator and the RBF ϕ is soft enough, then the application of such an operator to the Lagrange base is calculated via the relationship

$$(\mathcal{L}L)(x) = (\mathcal{L}\phi)A^{-1}.$$

Due to standard Lagrange conditions, the zero boundary conditions in $x_1 = 0$ and $x_N = \pi$ are satisfied if we use only an approximation generated by the functions $L_2,\ldots,L_{N-1}$. This approximation is then represented as

$$u(x,t) = \sum_{j=2}^{N-1} \beta_j(t) L_j(x),$$

with the unknown vector

$$\beta(t) = \begin{bmatrix} \beta_2(t) \\ \vdots \\ \beta_{N-1}(t) \end{bmatrix}.$$

By evaluating the interpolator in the FPDE for each node x_i, you get

$$\sum_{j=2}^{N-1} \beta_j'(t) L_j(x_i) = -K_\alpha \sum_{j=2}^{N-1} \beta_j(t) \frac{\partial^\alpha}{\partial |x|^\alpha} L_j(x_i)$$

and the initial conditions

$$\beta_j(0) = u_0(x_j), \quad 2 \leq j \leq N-1.$$

From these last two equations the following system of ordinary differential equations is obtained

$$\beta'(t) = -K_\alpha \left(\frac{\partial^\alpha}{\partial |\mathbf{x}|^\alpha} \mathbf{L} \right) \cdot \beta(t),$$

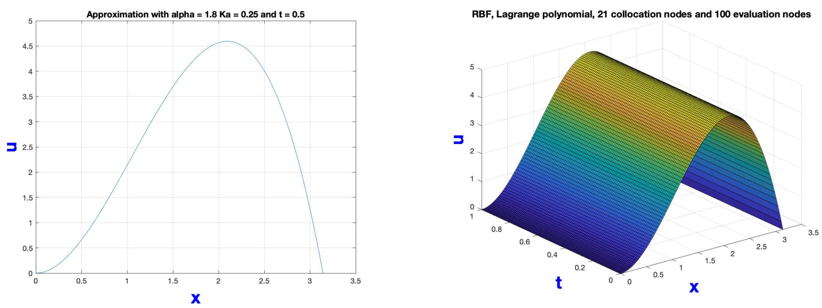
initial conditions

$$\beta(0) = U_0,$$

$$\frac{\partial^\alpha}{\partial |\mathbf{x}|^\alpha} \mathbf{L} = \begin{bmatrix} \frac{\partial^\alpha}{\partial |x|^\alpha} L_2(x_2) & \cdots & \frac{\partial^\alpha}{\partial |x|^\alpha} L_{N-1}(x_2) \\ \vdots & \cdots & \vdots \\ \frac{\partial^\alpha}{\partial |x|^\alpha} L_2(x_N) & \cdots & \frac{\partial^\alpha}{\partial |x|^\alpha} L_{N-1}(x_N) \end{bmatrix} \quad \text{and} \quad U_0 = \begin{bmatrix} u_0(x_2) \\ \vdots \\ u_0(x_{N-1}) \end{bmatrix}.$$

The results obtained via the RBF methodology and Lagrange polynomials for problem (38) are consistent with those obtained by Yang et al [27] and Cheng et al [28]. Taking into account the data $\alpha = 1.8$, $K_\alpha = 0.25$, $u_0(x) = x^2(\pi - x)$ and shape parameter $\varepsilon = 0.8$. The graphs with the results are shown in figure (5).

Figure 5: *Numerical approximation via RBF to the solution of equation (38), with* $\alpha = 1.8$, $\varepsilon = 0.8$ *and* $K_\alpha = 0.25$



4.2. Riemann-Liouville space-fractional diffusion equation

In this part, let us consider the partial fractional differential equation (FDP) introduced by Sousa [29] and taken up in Piret et. al. [9], the fractional diffusion equation, in one dimension

$$\frac{\partial f(x,t)}{\partial t} = d(x)_0 D_x^\alpha f(x,t) + q(x,t) \quad \text{para } x \in [0,1] \text{ y } t > 0 \tag{40}$$

where ${}_{0}^{RL}D_x^{\alpha}$ is the fractional derivative of Riemann-Liouville, $1 < \alpha \le 2$,

$$d(x) = \frac{\Gamma(5-\alpha)}{24}x^{\alpha} \quad \text{y} \quad q(x,t) = -2e^{-t}x^4.$$

with the initial conditions

$$f(x,0) = x^4, \quad x \in (a,b) \tag{41}$$

and Dirichlet boundary conditions

$$f(0,t) = 0, \quad f(1,t) = e^{-t}. \tag{42}$$

The exact solution (40) to this equation is

$$f(x,t) = e^{-t}x^4.$$

The numerical approximation to the solution via RBF methodology is obtained using Gaussian $\phi(r) = e^{-\varepsilon r^2}$ and multiquadric $\phi(r) = \sqrt{1+\varepsilon^2 r^2}$ RBFs, together with Lagrange polynomials. With $N = 21$ Chebyshev type collocation nodes, $Ne = 100$ evaluation nodes, several values for the order of the derivative α and shape parameters $\varepsilon = 0.6, 1.2$.

The results obtained by Sousa [29] were obtained by applying implicit Crank-Nicholson schemes for time. The discretization of the fractional derivative was done using splines, of second order precision. The results, both from Sousa and from RBF, are shown in the table (2), taking into account the absolute error

$$||u_{exact} - u_{approximation}||_{\infty} \tag{43}$$

where $||\cdot||_{\infty}$ is ℓ_{∞} norm.

Table 2: *Comparison of results to the numerical approximation of the solution to FPDE (40)*

(a) *Sousa results [29]*

Global ℓ_{∞} error (43) of time converged solution for three mesh resolutions at $t = 1$ for $\alpha = 1.2, 1.4, 1.5, 1.8$ and $\Delta t = \Delta x$.

Δx	$\alpha = 1.2$	Rate	$\alpha = 1.4$	Rate
1/15	0.1275×10^{-2}		0.9070×10^{-3}	
1/20	0.7571×10^{-3}	1.8	0.5327×10^{-3}	1.8
1/25	0.5030×10^{-3}	1.8	0.3486×10^{-3}	1.9
1/30	0.3566×10^{-3}	1.9	0.2461×10^{-3}	1.9
Δx	$\alpha = 1.5$	Rate	$\alpha = 1.8$	Rate
1/15	0.7660×10^{-3}		0.4380×10^{-3}	
1/20	0.4493×10^{-3}	1.9	0.2540×10^{-3}	1.9
1/25	0.2929×10^{-3}	1.9	0.1649×10^{-3}	1.9
1/30	0.2067×10^{-3}	1.9	0.1150×10^{-3}	2.0

(b) *RBF results.*

Global ℓ_{∞} error (43) of time converged solution at $t = 1$, for $\alpha = 1.2, 1.4, 1.5, 1.8$ and shape parameters $\varepsilon = 0.6, 1.2$.

Order α	Gaussian		Multiquadric
	$\varepsilon = 0.6$	$\varepsilon = 1.2$	$\varepsilon = 0.6$
1.2	0.35574×10^{-4}		0.13190×10^{-3}
1.4	0.46508×10^{-4}		0.14486×10^{-3}
1.5	0.60498×10^{-4}		0.19793×10^{-3}
1.8	0.16994×10^{-4}	0.18125×10^{-4}	0.26661×10^{-3}

4.3. Caputo time fractional partial differential equations

In this part we consider three examples of partial differential equations (Uddin and Haq, see [30]) of the type

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \delta \frac{\partial u(x,t)}{\partial x} + \gamma \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \tag{44}$$

where $t > 0, x \in [a,b], \quad 0 < \alpha \leq 1$, δ and γ are real parameters, bounded initial condition $u(x,0) = u_0(x)$ and boundary conditions Dirichlet $u(a,t) = g_1(t)$ y $u(b,t) = g_2(t)$ for $t \geq 0$. The fractional derivative is Caputo derivative.

4.3.1. Example 1

Putting $\delta = 1, \gamma = -1$ and $f(x,t) = \dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2$ in equation (44), we obtain a fractional, linear and non-homogeneous Burger equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2. \tag{45}$$

with initial condition

$$u(x,0) = x^2, \tag{46}$$

and Dirichlet boundary conditions

$$u(0,t) = t^2, \quad u(1,t) = 1 + t^2. \tag{47}$$

The exact solution (see [31]) is

$$u(x,t) = x^2 + t^2, \tag{48}$$

The problem is solved using Lagrange polynomials, RBF-QR for the spatial part and an implicit scheme in finite differences for the temporal part, for $x \in [0,1]$ and $t \in [0,2]$, fractional derivative of order $\alpha = 0.5$. The program considers several shape parameters and chooses the one with the lowest maximum error.

Figure 6 compares the errors that result from choosing uniform collocation nodes, with a fixed step size, against nodes of Chebyshev. Chebyshev's choice of nodes is due to the fact that they attenuate bad behavior, which manifests itself in large oscillations in the numerical approach, called the Gibbs phenomenon (see [32], [33]). In Table (3) we compared the results we obtained with respect to those obtained by Uddin-Haq, where they also took RBF but multiquadric (MQ) RBF and potential functions .

4.3.2. Example 2

We take equation (44) when $\delta = 1$, $\gamma = 0$ and $f(x,t) = \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t\cos(x)$,

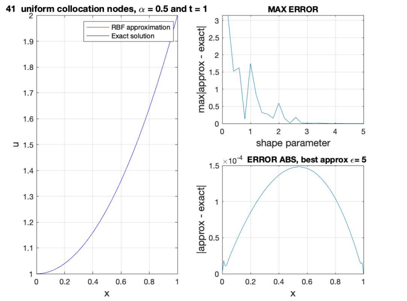
$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial x} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t\cos(x). \tag{49}$$

With the initial condition

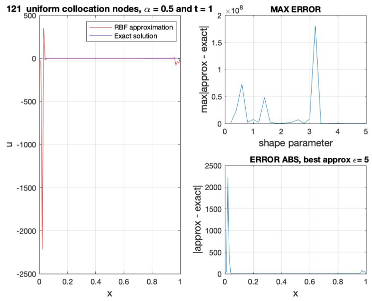
$$u(x,0) = 0, \tag{50}$$

Figure 6: *Comparison of results for the equation (45), $\alpha = 0.5$ and $t = 1$.*

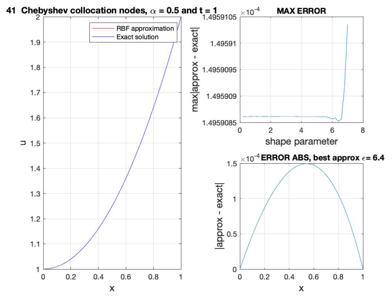
(a) *Max error for $\varepsilon = 5$ is 1.4802×10^{-4}*



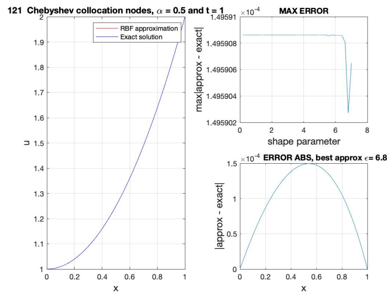
(b) *Max error for $\varepsilon = 5$ is 2.2165×10^{3}*



(c) *Max error for $\varepsilon = 6.4$ is 1.4959×10^{-4}*



(d) *Max error for $\varepsilon = 6.8$ is 1.4959×10^{-4}*



The exact solution to this problem (see [31]) is

$$u(x,t) = t \sin x. \tag{51}$$

The problem is solved by Lagrange and RBF-QR, for $\alpha = 0.6$, $N = 51$ and $N = 121$ collocation nodes for $x \in [-3, 3]$. Figure (7) shows that when the number of uniform collocation nodes grows, the instability increases as the solution approaches. And to mitigate this bad behavior Chebyshev nodes are chosen. Table (4) compares the results obtained for this problem with respect to Uddin-Haq results.

4.3.3. Example 3

Consider the equation (44) where $\delta = 0$ and $\gamma = -1$ and $f(x,t) = 0$,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{52}$$

We use the initial condition

$$u(x,0) = 4x(1-x), \tag{53}$$

and boundary conditions

$$u(0,t) = u(1,t) = 0. \tag{54}$$

Table 3: *Comparison of results to the numerical approximation of the solution to FPDE (45), $\alpha = 0.5$ in spatial interval* $[0, 1]$

(a) *Uddin-Haq results [30]*

Error norms corresponding for $\delta t = 0.01$, $N = 51$, MQ shape parameter $\varepsilon = 0.1$.

Time	0.1	0.5	1.0	1.5	2.0
MQ					
L_∞	6.086×10^{-2}	2.958×10^{-2}	2.114×10^{-2}	1.732×10^{-2}	1.503×10^{-2}
L_2	2.613×10^{-1}	1.277×10^{-1}	9.134×10^{-2}	7.485×10^{-2}	6.494×10^{-2}
r^7					
L_∞	6.086×10^{-2}	2.958×10^{-2}	2.112×10^{-2}	1.728×10^{-2}	1.495×10^{-2}
L_2	2.612×10^{-1}	1.277×10^{-1}	9.120×10^{-2}	7.454×10^{-2}	6.440×10^{-2}
r^5					
L_∞	6.087×10^{-2}	2.961×10^{-2}	2.122×10^{-2}	1.748×10^{-2}	1.529×10^{-2}
L_2	2.613×10^{-1}	1.279×10^{-1}	9.189×10^{-2}	7.597×10^{-2}	6.688×10^{-2}

(b) *RBF-QR Gaussian results.*

Error norms corresponding for $N = 121$ and several shape parameters.

Time	ε	L_∞
0.1	7.6	1.2021×10^{-4}
0.5	6.8	1.4385×10^{-4}
1	6.8	1.4959×10^{-4}
1.5	6.8	1.5215×10^{-4}
2	6.8	1.5368×10^{-4}

The exact solution to this problem is not known, but it is shown as an example of subdiffusive and superdiffusive phenomena, as discussed in the introduction. Again, the problem is solved by Lagrange polynomials, RBF-QR scheme for the spatial part and an implicit scheme for the temporal part, for $x, t \in [0, 1]$. The results for $\alpha = 0.5$ and $\alpha = 0.7$ go according to those shown in Podlubny et al. 2009 [34] and Uddin et. al. 2011 [30]. The first two rows of figure 8 show the numerical solution of problem 52 for $\alpha = 0.5$ and $\alpha = 0.7$ and are an example of subdiffusive phenomena; while the third row of figure 8 is added to exemplify a superdiffusive phenomenon, with $\alpha = 1.2$.

5. Conclusions

As shown, RBF schemes are efficient and they are on par with schemes like Finite Differences, they allow non-uniform data like Chebyshev nodes and Halton nodes. The challenge is to adapt them to solve multidimensional fractional systems of equations that consider that the medium does not have a single characteristic.

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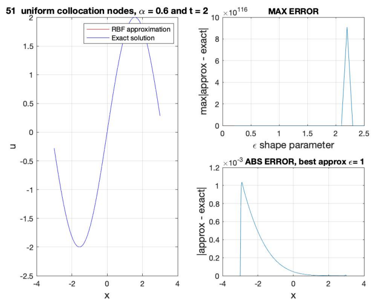
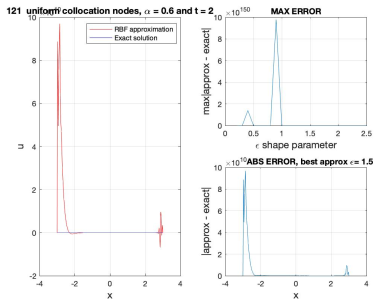
Figure 7: *Instability due to the choice of nodes, for the equation (49).*

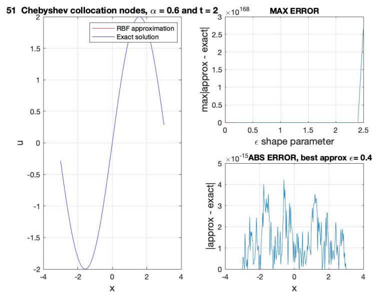
(a) *Max error for* $\varepsilon = 1$ *is* 0.0010



(b) *Max error for* $\varepsilon = 1.5$ *is* 9.6981×10^{10}



(c) *Max error for* $\varepsilon = 0.4$ *is* 4.2188×10^{-15}



(d) *Max error for* $\varepsilon = 0.2$ *is* 6.4393×10^{-15}

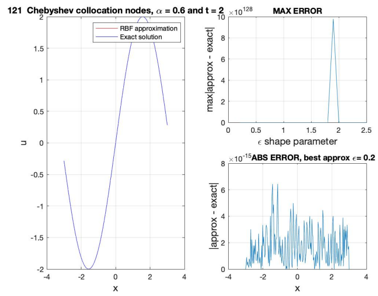


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Table 4: *Comparison of results to the numerical approximation of the solution to FPDE (49), $\alpha = 0.6$ in spatial interval* $[-3, 3]$

(a) *Uddin-Haq results [30]*

Error norms corresponding for $\delta t = 0.01$, $N = 121$, MQ shape parameter $\varepsilon = 0.48$.

Time	0.1	0.5	1.0	1.5	2.0
MQ					
L_∞	1.152×10^{-6}	8.673×10^{-6}	1.894×10^{-5}	2.984×10^{-5}	4.070×10^{-5}
L_2	9.299×10^{-7}	1.730×10^{-5}	5.717×10^{-5}	1.142×10^{-4}	1.854×10^{-4}
r^7					
L_∞	1.045×10^{-7}	7.000×10^{-7}	1.657×10^{-6}	2.718×10^{-6}	3.822×10^{-6}
L_2	8.765×10^{-8}	1.368×10^{-6}	4.769×10^{-6}	9.916×10^{-6}	1.655×10^{-5}
r^5					
L_∞	1.303×10^{-6}	8.035×10^{-6}	1.878×10^{-5}	3.113×10^{-5}	4.439×10^{-5}
L_2	1.104×10^{-6}	1.612×10^{-5}	5.431×10^{-5}	1.125×10^{-4}	1.890×10^{-4}

(b) *RBF-QR Gaussian results.*

Error norms corresponding for $N = 121$ and several shape parameters.

Time	ε	L_∞
0.1	0.2	2.3592×10^{-16}
0.5	0.2	1.3878×10^{-15}
1	0.6	3.1086×10^{-15}
1.5	0.2	5.3291×10^{-15}
2	0.2	6.4393×10^{-15}

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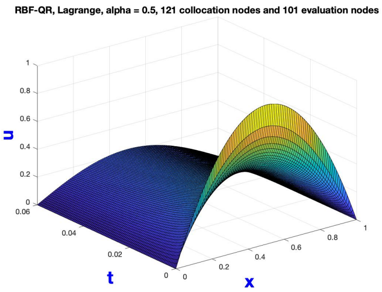
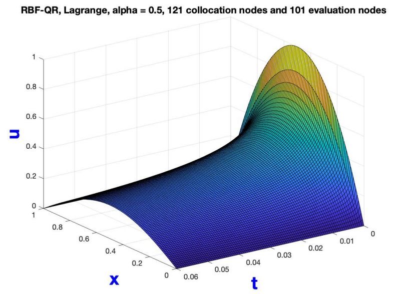
Figure 8: *Numerical solution to equation (52), orders* $\alpha = 0.5, 0.7$ *and* 1.2

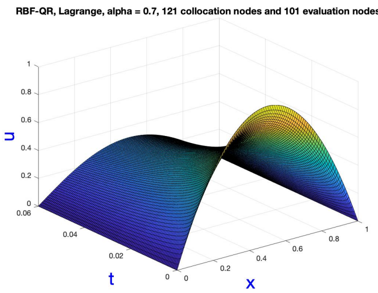
(a) *Side 1*



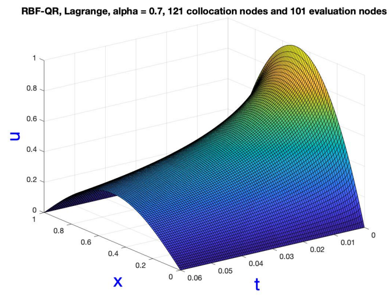
(b) *Side 2*



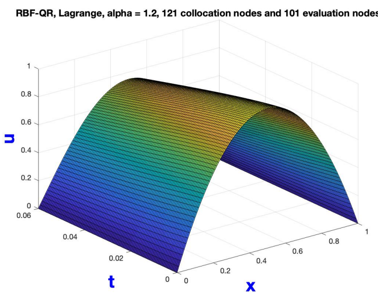
(c) *Side 1*



(d) *Side 2*



(e) *Side 1*



(f) *Side 2*