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Minimal Completion of the Lie Algebra arising from Fractional Calculus

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Abstract

We showed the group $G_{\mathbb{R}}$ and Lie algebra $\mathfrak{g}_{\mathbb{R}}$ generated by $\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$ and $\{x^a|a \in \mathbb{R}\}$ and their generating operators $\log(\frac{d}{dx})$ and $\log x$ have unique maximal normal subgroup $\mathsf{A}_{\mathbb{R}}$ and maximal ideal $\mathfrak{a}$. It was shown taking suitable completion $\bar{\mathfrak{a}}$, ϑ; $\vartheta g = g^{-1}\frac{dg}{dx}$ gives a monomorphism from $\mathsf{A}_{\mathbb{R}}$ into $\bar{\mathfrak{a}}$([6].). But to search "good" completion is remained as a problem..

In this paper, we propose the good completion (in some sense, minimal) is $\mathfrak{a}_{\exp}$;

$$\mathfrak{a}_{\exp} = \{\sum_n c_n \Psi^{(n)}(1+s) | \sum_n c_n x^n \in \mathrm{Exp}(\mathbb{C})\}.$$

Here $\mathrm{Exp}(\mathbb{C})$ is the space of finite exponential type functions. Corresponding extension of $\mathsf{A}_{\mathbb{R}}$ is also proposed.

1. Introduction

Fractional calculus is now studied and applied not only in mathematics, but also in several area of Sciences (cf.[9]). As a tool to the study of fractional calculus, we have introduced an integral transform $\mathcal{R}$;

$$\mathcal{R}[f(s)](x) = \int_{-\infty}^{\infty} \frac{x^s}{\Gamma(1+s)} f(s) ds,$$

and show for suitable class of function f; *e.g.* $\dfrac{f(s)}{\Gamma(1+s)}$ is rapidly decreasing at $s \to \pm\infty$,

$$\frac{d^a}{dx^a}\mathcal{R}[f(s)](x) = \mathcal{R}[\tau_a f(s)](x), \quad \tau_a f(s) = f(s+a).$$

([5]). Precisely saying, if $x < 0$, we need to fix x^s in the calculation of $\mathcal{R}$ This is done for example, to fix $\log x$ as $\log|x| - \pi i$ Another way to consider $\mathcal{R}$ for $x < 0$ is consider $\mathcal{R}[f]$ to be a many valued function on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. But in this paper, we skip to think this kinds of problems (cf.[5]).

Applying $\mathcal{R}$, we determined structure of the group $G_{\mathbb{R}}$ generated by 1-parameter groups $\{\frac{d^a}{dx^a}|a \in \mathbb{R}\}$ and $\{x^a|a \in \mathbb{R}\}$, and the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ generated their generating operators $\log(\frac{d}{dx})$ and $\log x$([6] As for definition and properties of $\log(\frac{d}{dx})$, we refer [4].[11]). Via conjugation by $\mathcal{R}$, $G_{\mathbb{R}}$ is an extension of the multiplicative free abelian group $\mathsf{A}_{\mathbb{R}}$ generated by $\{\frac{\Gamma(1+s)}{\Gamma(1+s+a)}|a \in \mathbb{R}^{\times}, \mathbb{R}^{\times} = \mathbb{R}\backslash\{0\}\}$ by $\mathbb{R} = \{\tau_a|a \in \mathbb{R}\}$; $\tau_a f(s) = f(s+a)$. Hence $\mathsf{A}_{\mathbb{R}}$ is isomorphic to the free abelian group $\mathsf{D}_{\mathbb{R}^{\times},\mathbb{Z}}$ generated by $\{\delta_a|a \in \mathbb{R}^{\times}\}$. The isomorphism from $\mathsf{D}_{\mathbb{R}^{\times};\mathbb{Z}}$ to $\mathsf{A}_{\mathbb{Z}}$ is given by $\mu_{-x:\Psi}$;

$$u_{-x;\Psi}T = \exp(T\int_{-x}^{s}\Psi(1+x+t)dt).$$

$\mathfrak{g}_{\mathbb{R}}$ has unique maximal ideal $\mathfrak{a}$ generated by $\Psi(1+s)$. Taking suitable completion $\bar{\mathfrak{a}}$ of $\mathfrak{a}$, it is shown there is a monomorphism $\vartheta : \mathsf{A}_{\mathbb{R}} \to \bar{\mathfrak{a}}$; $\vartheta f = f^{-1}\frac{df(s)}{ds}$. Precisely, $\vartheta(\mathsf{A}_{\mathbb{R}})$ is contained in the completion of $\mathfrak{a}_2$, the subideal of $\mathfrak{a}$ generated by $\Psi'(1+s)$.

Note. Extension of these results to several variables case shows ϑ should be replaced to ρ; $\rho g = g^{-1}dg$. Since ρ is essential in the non-abelian de Rham theory ([1].[2]), this suggests there might exist relations between non-abelian de Rham theory and fractional calculus.

These results suggest there might be exist some lack in $G_{\mathbb{R}}$. We proposed this lack is supplied to add $\Gamma(1+s)$ to $\mathsf{A}_{\mathbb{R}}$ ([6]). We denote this extended group by $\mathsf{A}^{\natural}_{\mathbb{R}}$. It is isomorphic to the group $\mathsf{D}_{\mathbb{R};\mathbb{Z}}$, the free abelian group generated by $\{\delta_a|a \in \mathbb{R}\}$. The map ϑ is defined on $\mathsf{A}^{\natural}_{\mathbb{R}}$ and gives an isomorphism into $\bar{\mathfrak{a}}$. But in our former studies, we only demand $\sum_n \frac{a^n}{n!}\Psi^{(n)}(1+s) \in \bar{\mathfrak{a}}$ and can not fix the completion. In this paper, we propose

Proposal. We take $\mathfrak{a}_{\exp}$;

$$\mathfrak{a}_{\exp} = \{\sum_n c_n\Psi^{(n)}(1+s)|\sum_n c_n x^n \in \mathrm{Exp}(\mathbb{C})\},$$

where $\mathrm{Exp}(\mathbb{C})$ is the space of finite exponential type functions, as the appropriate completion of $\mathfrak{a}$.

By definition, $\mathfrak{a}_{\exp}$ is isomorphic to $\mathrm{Exp}(\mathbb{C})$ as a vector space. Convergence of a series $\{f_n\}$ of $\mathrm{Exp}(\mathbb{C})$ is defined $|f_n(x)| \leq Ce^{K|x|}$ for all n, where C, K are positive constants, and converges to a function f uniformly on $\mathbb{C}$ in wider sense. Then $\mathrm{Exp}(\mathbb{C})$ is complete by this definition of convergence. Topology of $\mathfrak{a}_{\exp}$ is imposed from this topology of $\mathrm{Exp}(\mathbb{C})$.

Note. Roughly speaking, $\mathfrak{a}_{\exp}$ is the minimal completion of $\mathfrak{a}\otimes\mathbb{C}$ which satisfies the demand $\sum_n \dfrac{a_n}{n!}\Psi^{(n)}(1+s) \in \bar{\mathfrak{a}}, a \in \mathbb{C}$. But to take $\mathfrak{a}_{\exp}$ as the good completion seems to restrict studies of fractional calculus only in the analytic category.

For a power series $f(x) = \sum_n c_n x^n$, we set $f(\delta) = \sum_n c_n \delta^{(n)}$. $f(\delta)$ is not a distribution in the sense of Schwartz, in general. But it may be a generalized function on suitable function space. We note this kind of sum of delta functions and interpretations them as generalized functions (may not be distributions in the sense of Schwartz) are appear in physical literatures (cf.[8],[13],[14],[16]).

Especially if $f \in \mathrm{Exp}(\mathbb{C})$, then $f(\delta)$ acts on $\mathrm{Ent}(\mathbb{C})$, the space of entire functions on $\mathbb{C}$ convergence of its series $\{f_n\}$ is defined by the uniform convergence in wider sense. In fact, denoting $\mathrm{Exp}(\mathbb{C})_\delta = \{f(\delta) | f \in \mathrm{Exp}(\mathbb{C})\}$, we have

$$\mathrm{Exp}(\mathbb{C})_\delta = \mathrm{Ent}(\mathbb{C})^\dagger.$$

By definition, $\mathsf{D}_{\mathbb{R};\mathbb{Z}} \subset \mathrm{Exp}(\mathbb{C})_\delta$. Since $\mu_{-x;\Psi}$ can be defined on $\mathrm{Exp}(\mathbb{C})_\delta$, we set $\mu_{-x;\Psi}\mathrm{Exp}(\mathbb{C})_\delta = \mathsf{A}^\natural_{\mathbb{C}:\exp}$. then we have

$$\vartheta : \mathsf{A}^\natural_{\mathbb{C};\exp} \cong \mathfrak{a}_{\exp}.$$

The map $\iota_\Psi = \vartheta \circ \mu_{-x;\Psi} : (\mathrm{Exp}(\mathbb{C})_\delta \cong \mathfrak{a}_{\exp}$ is given by

$$\iota_\Psi T = T_t \Psi(1+x+t).$$

When $f(x) = e^{ax}$ and $T = f(\delta)$, we have $\iota_\Psi T = \Psi(1+x+a)$, via analytic continuation. This suggest if $f(x) = \int_{-\infty}^{\infty} e^{axs} g(s)ds$, it may be

$$\iota_\Psi T = \int_{-\infty}^{\infty} g(s)\Psi(1+s+a)ds.$$

If $f \in \mathrm{Exp}(\mathbb{C})$, then $f = \mathcal{B}[g]$, where g is a holomorphic function at the origin. Therefore,we may ask if $T = f(\delta)$, does the formula

$$\iota_\Psi T = \frac{1}{2\pi i}\int_{|s|=\epsilon} g(s)\Psi(1+x+\frac{1}{s})ds,$$

is hold taking suitable ϵ?.

This paper is consisted by seven Sections. In §2, we study $\mathrm{Exp}(\mathbb{C})_\delta$ and show it is the dual space of $\mathrm{Ent}(\mathbb{C})$. $\mathrm{Ent}(\mathbb{C})_\delta = \{f(\delta)|f \in \mathrm{Ent}(\mathbb{C})\}$ is also discussed and show $\mathrm{Ent}(\mathbb{C})_\delta = \mathrm{Exp}(\mathbb{C})^\dagger$. At a glance, δ_a, $a \neq 0$ seems does not belong to $\mathrm{Exp}(\mathbb{C})_\delta$. But regarding $\mathrm{Exp})(\mathbb{C})_\delta = \mathrm{Ent}(\mathbb{C})^\dagger$, we can identify δ_a and $\sum_n \frac{(-a)^n}{n!}\delta^{(n)} \in \mathrm{Exp}(\mathbb{C})_\delta$. Based on this fact, we introduce $\mathfrak{a}_{\exp}$, *etc.* in §3. §4 treats the map $\iota_\Psi = \vartheta \circ \mu_{-x;\Psi}$. Originally, elements of $G_\mathbb{R}$ *etc.* are regarded as operators. But elements of $\mathfrak{a}_{\exp}$ and $\mathsf{A}^\natural_{\mathbb{C};\exp}$ are defined by completion, that is formal infinite sum (or product) of fractional differentiations. So it is a problem whether elements of $\mathfrak{a}_{\exp}$, *etc.* can be interpreted as operators acting on some function space. §5 treats this problem. Related to this problem, the problem whether the sum $\sum_n c_n\delta_{a_n}$ can be regarded as an operator acting on some space has been treated in [7]. In §6, we reconsider this problem and give alternative proof of results in [7].

We denote $\mathbf{a} = (a_1, \dots, a_n)$, *etc.*.Then the group $G_{\mathbb{R}^n}$ and Lie algebra $\mathfrak{g}_{\mathbb{R}^n}$ generated by $\{\frac{\partial^{a_1}}{\partial x_1^{a_1}}, \dots, \frac{\partial^{a_n}}{\partial x_n^{a_n}}|\mathbf{a} \in \mathbb{R}^n\}$ and $\{x_1^{a_1}, \dots, x_n^{a_n}|\mathbf{a} \in \mathbb{R}^n\}$, and their generating operators $\{\log(\frac{\partial}{\partial x_1}), \dots, \log(\frac{\partial}{\partial x_n})\}$ and $\{\log x_1, \dots, \log x_n\}$ are direct sum (product) of $G_\mathbb{R}$ and $\mathfrak{g}_\mathbb{R}$:

$$G_{\mathbb{R}^n} = \overbrace{G_\mathbb{R}|_{s=s_1} \times \cdots \times G_\mathbb{R}|_{s=s_n}}^{n},$$

$$\mathfrak{g}_{\mathbb{R}^n} = \overbrace{\mathfrak{g}_\mathbb{R}|_{s=s_1}ds_1 \oplus \cdots \oplus \mathfrak{g}_\mathbb{R}|_{s=s_n}ds_n}^{n}.$$

Hence definitions and results in single variable case are extended straight forwardly to several variable case. But since $\mathrm{Exp}(\mathbb{C})_\delta|_{x=x_1}dx_1 \oplus \cdots \oplus \mathrm{Exp}(\mathbb{C})_\delta|_{x=x_n}dx_n$ is only a subspace of $\Lambda^1\mathrm{Exp}(\mathbb{C}^n)_\delta$, the space of 1-forms with coefficients are $\mathrm{Exp}(\mathbb{C}^n)$, the space of finite exponential functions on $\mathbb{C}^n$. Hence we need fix the coordinate system of $\mathbb{C}^n$. To overcome this restriction will be a problem.

2. The spaces $\mathrm{Exp}(\mathbb{C})$ and $\mathrm{Ent}(\mathbb{C})$ and their dual spaces

We denote the space of finite exponential type functions and entire functions on $\mathbb{C}$ by $\mathrm{Exp}(\mathbb{C})$ and $\mathrm{Ent}(\mathbb{C})$. Their topologies are given by the following neighborhood systems:

$\mathrm{Exp}(\mathbb{C})$: We take $U(f;K,r;C,M)$, where K is a compact set of $\mathbb{C}$, r,C,M are positive numbers, and $U(f;K,r,C,M)$ is

$$\{g \in \mathrm{Exp}(\mathbb{C}) | |f(z)-g(z)| < r, z \in K, |f(z)-g(z)| \leq Ce^{|Mz|}, z \in \mathbb{C}\}.$$

$\mathrm{Ent}(\mathbb{C})$: We take $U(f,K,r) = \{g \in \mathrm{Ent}(\mathbb{C}) | |f(z)-g(z)| < r, z \in K\}$, where K is a compact set of $\mathbb{C}$.

A series $\{f_n\}$ of $\mathrm{Ent}(\mathbb{C})$ converges to a function f by this topology, if and only if $\{f_n\}$ converges uniformly in wider sense to f. Hence f is an entire function. While a series $\{f_n\}$ of $\mathrm{Exp}(\mathbb{C})$ to f by this topology, if and only if the estimate $|f_n(z)| \leq Ce^{|Mz|}$ holds for some positive C, M, for all n and converges uniformly in wider sense to f. Hence $f \in \mathrm{Exp}(\mathbb{C})$.

Entire functions allow Taylor expansions on $\mathbb{C}$. We set an entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and $f_N(x) = \sum_{m=0}^{N} c_n z^n$. Then we have $\lim_{N\to\infty} f_N(z) = f(z)$. where convergence is in the above sense. Hence we obtain

Lemma 1. The polynomial algebra $\mathbb{C}[z]$ is dense both in $\mathrm{Exp}(\mathbb{C})$ and $\mathrm{Ent}(\mathbb{C})$ by the above topologies.

Let $f(z) = \sum_n c_n z^n$. Then we use the notation

$$f(\delta) = \sum_n c_n \delta^{(n)}.$$

In general, $f(\delta)$ is not a distribution in the sense of Schwartz. But if $f \in \mathrm{Exp}(\mathbb{C})$, or $f \in \mathrm{Ent}(\mathbb{C})$, $f(\delta)$ can be considered as a generalized function by the following Lemma.

Lemma 2. Let $f(z) \in \mathrm{Exp}(\mathbb{C})$ and $g(z) \in \mathrm{Ent}(\mathbb{C})$ with Taylor expansions $f(z) = \sum_n a_n z^n$ and $g(z) = \sum_n b_n z^n$. Then $\sum_n n! a_n b_n$ converges.

Proof. Since $f \in \mathrm{Exp}(\mathbb{C})$, there exist $C > 0, M > 0$ such that $|a_n| \le C\dfrac{M^n}{n!}$ for all n Hence we have

$$|\sum_{n=N}^{n=L} n! a_n b_n| \le C(\sum_{n=N}^{n=L} M^n |b_n|).$$

Then since $\sum_n |b_n| z^n$ is also an entire function, we have

$$\lim_{N\to\infty, L\to\infty} \sum_{n=N}^{n=L} n! a_n b_n = 0.$$

This proves Lemma.

We use notations $\mathbf{a} = (a_1, a_2, \ldots)$,*etc.*, $\mathbf{ab} = (a_1 b_1, a_2 b_2, \ldots)$ and

$$\rho(\mathbf{a}) = \limsup_{n\to\infty} |a_n|^{1/n}, \quad r(\mathbf{a}) = \frac{1}{\rho(\mathbf{a})}$$

Here, we set $r(\mathbf{a}) = \infty$ if $\rho(\mathbf{a}) = 0$, and $r(\mathbf{a}) = 0$ if $\rho(\mathbf{a}) = \infty$, $r(\mathbf{a})$ is the convergence radius of the power series $\sum_n a^n z^n$.

Lemma 3. The followings hold.

1. $\rho(\mathbf{b}) = 0$ if $\mathbf{ab}$ converges for all $\mathbf{a}$ with $\rho(\mathbf{a}) < \infty$.
2. $\rho(\mathbf{a}) < \infty$ if $\mathbf{ab}$ converges for all $\mathbf{b}$ with $\rho(\mathbf{b}) = 0$.

Proof. To show 1, we use $\rho(\mathbf{a}) = |\xi|$, if $\mathbf{a} = (\xi, \xi^2, \ldots)$, where ξ is an arbitrary complex number. Then set $f_{\mathbf{b}}(z) = \sum_n b_n z^n$, $f_{\mathbf{b}}(\xi) = \mathbf{ab}$ converges if 1 is hold. Therefore convergence radius of $f_{\mathbf{b}}(z)$ is ∞. This proves 1.

If $\rho(\mathbf{a}) = \infty$, there is a subseries $(a_{n_1}, a_{n_2}, \ldots)$ of $(a_1.a_2, \ldots)$ such that

$$\lim_{k\to\infty} |a_{n_k}|^{1/n_k} = \infty.$$

Then $\lim_{k\to\infty} |a_{n_k}^{\epsilon}|^{1/n_k} = \infty$, if $0 < \epsilon < 1$. We define a series $\mathbf{b} = (b_1, b_2, \ldots)$ by $\begin{cases} b_{n_k} = |a_{n_k}|^{-\epsilon}, \\ b_n = 0, \text{otherwise} \end{cases}$. Then $\rho(\mathbf{b}) = 0$ and $\mathbf{ab}$ diverges, because $\limsup_{n\to\infty} |a_n b_n| = \infty$. Hence we have 2.

Definition 1. We define the spaces $\mathrm{Exp}(\mathbb{C})_\delta$ and $\mathrm{Ent}(\mathbb{C})_\delta$ by

$$\mathrm{Exp}(\mathbb{C})_\delta = \{f(\delta) | f \in \mathrm{Exp}(\mathbb{C})\}, \tag{1}$$
$$\mathrm{Ent}(\mathbb{C})_\delta = \{f(\delta) | f \in \mathrm{Ent}(\mathbb{C})\}. \tag{2}$$

By using these notations, we have

Theorem 1. The followings hold.

$$\mathrm{Exp}(\mathbb{C})_\delta = \mathrm{Ent}(\mathbb{C})^\dagger,\ \mathrm{Ent}(\mathbb{C})_\delta = \mathrm{Exp}(\mathbb{C})^\dagger. \tag{3}$$

Proof. Let $f(z) = \sum_n a_n z^n \in \mathrm{Exp}(\mathbb{C})$ and $g(z) \in \mathrm{Ent}(\mathbb{C})$. Then we have

$$f(\delta)g(z) = \sum_n (-1)^n n! a_n b_n, \quad g(\delta)f(z) = \sum_n (-1)^n n! a_n b_n. \tag{4}$$

These right hand sides converges by Lemma 2. Hence we have

$$\mathrm{Exp}(\mathbb{C})_\delta \subset \mathrm{Ent}(\mathbb{C})^\dagger,\ \mathrm{Ent}(\mathbb{C})_\delta \subset \mathrm{Exp}(\mathbb{C})^\dagger. \tag{5}$$

Since $C[z]$ is dense both in $\mathrm{Exp}(\mathbb{C})$ and $\mathrm{Ent}(\mathbb{C})$, linear functional T on these spaces is determined by $Tz^n, n = 0.1, 2, \ldots$. If $Tz^n = c_n$, we have

$$Tz^n = (-1)^n \frac{c_n}{n!} \delta^{(n)} z^n.$$

Hence to set $h(z) = \sum_n (-1)^n \frac{c_n}{n!} z^n$, we may write $T = h(\delta)$.Then Lemma 3,1 show $\mathrm{Exp}(\mathbb{C})^\dagger \subset \mathrm{Ent}(\mathbb{C})_\delta$ and 2 shows $\mathrm{Ent}(\mathbb{C})^\dagger \subset \mathrm{Exp}(\mathbb{C})_\delta$. Therefore we have Theorem.

Note. Theorem 1 is essentially reinterpretation of the duality of O, the algebra of germs of holomorphic functions at the origin and $\mathrm{Ent}(\mathbb{C})$ given by

$$(f(x), g(x)) = \frac{1}{2\pi i} \oint f(x) g(\frac{1}{x}) \frac{dx}{x}, \quad f(x) \in \mathsf{O}, g(x) \in \mathrm{Ent}(\mathbb{C}).$$

Precisely, since

$$\frac{1}{2\pi i} \oint \frac{1}{x^{n+1}} f(x) dx, = \frac{1}{n!} \frac{d^n f(x)}{dx^n}|_{x=0},$$

if $\phi(x)$ is hlomorphic at $x = 0$, we get

$$\phi(\delta)[f(x)] = \frac{1}{2\pi i} \oint \mathcal{B}[\phi](-\frac{1}{x}) f(x) dx,$$

if $f(x) \in \mathrm{Ent}(\mathbb{C})$. Hence we obtain Th.2 from the duality of O and $\mathrm{Ent}(\mathbb{C})$.

3. Taylor expansion of δ_a

If $f(z)$ is an entire function, we have

$$f(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}(0), \quad a \in \mathbb{C}.$$

Therefore we obtain

$$f(a) = \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{n!} \delta^{(n)} f(z), \quad a \in \mathbb{C}. \tag{6}$$

This means taking $\mathrm{Ent}(\mathbb{C})$ or $\mathrm{Exp}(\mathbb{C})$ as the space of testing functions, the following Taylor expansion of $\delta_a = \delta(z-a)$ holds

$$\delta_a = \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{n!} \delta^{(n)}. \tag{7}$$

Let $\mathsf{D}^d_{\mathbb{C};\mathbb{C}}$ be the $\mathbb{C}$-vector space spanned by $\{\delta_a^{(k)} | a \in \mathbb{C}, k = 0, 1.2, \ldots\}$. Then by (7), we have

Proposition 1. By the map

$$\delta_a^{(k)} \to \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{n!} \delta^{(n+k)}, \tag{8}$$

$\mathsf{D}^d_{\mathbb{C};\mathbb{C}}$ is regarded to be a subspace of $\mathrm{Exp}(\mathbb{C})_\delta$.

Note. This Proposition and Theorem 1 are already proved in [4] and [5].

We can extend $\mu_{-x,\Psi}$ on $\mathrm{Exp}(\mathbb{C})_\delta$. Then we obtain the map

$$\vartheta \circ \mu_{-x,\Psi} : \mathrm{Exp}(\mathbb{C})_\delta \to \bar{\mathfrak{a}}.$$

This map becomes an isomorphism if we define $\bar{\mathfrak{a}}$ to be

$$\{\sum_n c_n \Psi^{(n)}(1+s) | \sum_n c_n z^n \in \mathrm{Exp}(\mathbb{C})\} \tag{9}$$

Definition 2. We set

$$\mathfrak{a}_{\exp} = \{\sum_{n=0}^{\infty} c_n \Psi^{(n)}(1+s) | \sum_{n=0}^{\infty} c_n z^n \in \mathrm{Exp}(\mathbb{C})\}. \tag{10}$$

The Lie algebra $\mathbb{C}\frac{d}{ds} \oplus \mathfrak{a}_{\exp}$, with the brackets

$$[\frac{d}{ds}, \Psi^{(n)}(1+s)] = \Psi^{(n+1)}(1+s),\ n = 0, 1, 2, \dots,$$

is denoted by $\mathfrak{g}_{\mathbb{R};\exp}$.

The sum $\sum_{n=0}^{\infty} c_n \Psi^{(n)}(1+s)$ is formal. We ask can we give meanings to these formal sum?

We also denote $\mathsf{A}^{\natural}_{\mathbb{C};\exp^{\natural}}$ the multiplicative group $\mu_{-x;\Psi}\mathrm{Exp}(\mathbb{C})_{\delta}$. Then we have

$$\mathsf{A}^{\natural}_{\mathbb{C};\exp} = \{\prod_{n=0}^{\infty} (e^{c_n \Psi^{(n-1)}(1+s)}) | \sum_{n=0}^{\infty} c_n z^n \in \mathrm{Exp}(\mathbb{C})\}. \tag{11}$$

Here $\Psi^{-1}(1+s) = \log(\Gamma(1+s))$. We denote $\mathsf{A}^{\natural}_{\mathbb{R}:\exp}$ the subgroup $\mathsf{A}^{\natural}_{\mathbb{R};\exp}$ consisted those elements whose coefficients c_n in (11) are real numbers.

Lemma 4. We can define the action τ_a; $\tau_a f(x) = f(x+a)$ on $\mathfrak{a}_{\exp}$

Proof. First we note that $\Psi^{(k)}(1+s+a), a \neq 0$ does not appear directly in $\mathfrak{a}_{\exp}$. It appears as the sum $\sum_{n=0}^{\infty} \frac{a^n}{n!} \Psi\ (k+n)(1+s)$. Hence to handle $\sum_k c_k \Psi^{(k)}(1+s+a)$ in $\mathfrak{a}_{\exp}$, we need to rewrite

$$\begin{aligned}\sum_{k=0}^{\infty} c_k \Psi(1+s+a) &= \sum_{k=0}^{\infty} c_k (\sum_{n=0}^{\infty} \frac{a^n}{n!} \Psi^{k+n}(1+s) \\ &= \sum_{m=0}^{\infty} (\sum_{k+n=m} c_k \frac{a^n}{n!}) \Psi^{(m)}(1+s).\end{aligned}$$

Since $\sum_k c_k z^k$ is a finite exponential type function, $|c_k|$ satisfies estimate $|c_k| \le C\dfrac{L^n}{n!}$. Hence we have

$$|\sum_{k+n=m} c_k \frac{a^n}{n!}| \le \sum_{k+n=m} C\frac{L^k}{k!}\frac{|a|^n}{n!} = C\frac{(|a|+L)^m}{m!}.$$

Hence $\sum_{m=0}^{\infty}(\sum_{k+n=m} c_k \frac{a^n}{n!})z^m$ is a finite exponential type function. Therefore we obtain Lemma.

By this Lemma and (11), τ_a acts on $\mathsf{A}^{\natural}_{\mathbb{R};\mathrm{exp}}$.

Definition 3. We denote $G_{\mathbb{R};\mathrm{exp}}$ the extension of $\mathsf{A}^{\natural}_{\mathbb{R};\mathrm{exp}}$ by $\mathbb{R} = \{\tau_a | a \in \mathbb{R}\}$. The group $G_{\mathbb{C};\mathrm{exp}}$ is similarly defined.

Note. In $\mathrm{Exp}(\mathbb{C})_\delta$, we identify δ_a and $\sum_{n=0}^{\infty}(-1)^n\dfrac{a^n}{n!}\delta^{(n)}$. Then by the same computations and estimate as in the proof of Lemma 4, τ_a; $\tau_a\delta^{(n)} = \delta_a^{(n)}$, acts on $\mathrm{Exp}(\mathbb{C})_\delta$.But since

$$\delta_a^{(n)}[f(z)] = (-1)^n f^{(n)}(a) = \delta^{(n)}[\tau_a f(z)],$$

we obtain this conclusion more simply by Theorem 1, not only for $\mathrm{Exp}(\mathbb{C})_\delta$ but also for $\mathrm{Ent}(\mathbb{C})_\delta$..

Then we have

$$\mathbb{R} \ltimes \mathrm{Exp}(\mathbb{C})_\delta \cong G_{\mathbb{C};\mathrm{exp}}. \tag{12}$$

We also note by previous discussions, the following isomorphisms hold.

$$\mu_{-x;\Psi} : \mathrm{Exp}(\mathbb{C})_\delta \cong \mathsf{A}^{\natural}_{\mathbb{C}:\mathrm{exp}}, \quad \vartheta : \mathsf{A}^{\natural}_{\mathbb{C};\mathrm{exp}} \cong \mathfrak{a}_{\mathrm{exp}}. \tag{13}$$

Proposal. We take $\mathfrak{a}_{\mathrm{exp}}$, *etc.*, as appropriate completions of $\mathfrak{a}$, *etc.*.

Note. To define $\vartheta : \mathsf{A}^{\natural}_{\mathbb{R}} \to \bar{\mathfrak{a}}$, $\sum_n \dfrac{a^n}{n!}\Psi^{(n)}(1+s)$ should belong to $\bar{\mathfrak{a}}$. Therefore, to define ϑ, it seems $\mathfrak{a}_{\mathrm{exp}}$ is the minimal completion of $\mathfrak{a}$ (or $\mathfrak{a} \otimes \mathbb{C}$).

4. The map $\iota_\Psi = \vartheta \circ \mu_{-x,\Psi}$

We denote $\vartheta \circ \mu_{-x,\Psi}$ by ι_Ψ. Let T be a (generalized) function (or distribution), then we have by the definitions of ϑ and $\mu_{-x;\Psi}$

$$\iota_\Psi T = T_t \Psi(1 + x + t). \tag{14}$$

Strictly saying, $\Psi(1 + x + t)$ needs to belong the domain of T_t. Otherwise, we need some modification to give meanings of (14). For example, to get

$$\iota_\Psi \delta^{(n)} = (-1)^n \Psi^{(n)}(1 + x), \quad n = 0, 1, 2, \ldots, \tag{15}$$

we need to assume $|x + t| < 1$. But since $\Psi^{(n)}(1 + x)$ is a single valued meromorphic function on $\mathbb{C}$, we justify (14) for all $x \in \mathbb{C}$, by analytic continuation.

Adopting this argument, we justify

$$\iota_\Psi : \mathrm{Exp}(\mathbb{C})_\delta \cong \mathfrak{a}_{\exp}, \tag{16}$$

as an adaptation of (15). Note that since $\Psi(1+x+t) \notin \mathrm{Ent}(\mathbb{C})$, we need take care to use discussions in §1 to the study of ι_Ψ.

The topology of $\mathfrak{a}_{\exp}$ is transfered from the topology of $\mathrm{Exp}(\mathbb{C})$ via the map (16). For the simplicity, we denote $\sum_n c_n \Psi^{(n)}(1 + x)$ by $f^\sharp(\Psi(1 + x))$, where $f(z) = \sum_n c_n z^n \in \mathrm{Exp}(\mathbb{C})$, then we define

Definition 4. A series $\{f^\sharp_m(\Psi(1 + x))\}$ of $\mathfrak{a}_{\exp}$ is said to be $f^\sharp(\Psi(1 + x))$ if and only if the series $f_n(x)$ converges to $f(x) \in \mathrm{Exp}(\mathbb{C})$ by the topology of $\mathrm{Exp}(\mathbb{C})$.

Since $\Psi(1 + x)$ is holomorphic on the unit disc $D = \{x||x| < 1\}$, if $a \in D$ then

$$|\frac{d^n \Phi(1 + x)}{dx^n}| < n! C_{a,\epsilon} M^{n+1}_{a,\epsilon},$$

where $C_{a,\epsilon} = \max_{x \in S_{a,\epsilon}} |\Phi(1 + x)|$ and $M^{-1} = \min_{x \in S_{a,\epsilon}, y \in S} |x - y|$. Here $S_{a,\epsilon} = \{y||y - a| = \epsilon\}$ is assumed to be contained in D and $S = \{x||x| = 1\}$. Hence if $f(x) = \mathcal{B}[g(s)](x)$, where $g(s)$ is an entire function, $f^\sharp(\Psi(1 + x))$ converges on $\{y||y - a| < \epsilon\}$. Because set $g(x) = \sum_n c_n x^n$, we have

$$|f^\sharp(\Psi(1 + x))(s)| \leq \sum_n C_{a,\epsilon} M |c_n| M^n_{a,\epsilon}, \quad |y - a| < \epsilon.$$

Hence we have

Proposition 2. If $f(x) = \mathcal{B}[g(s)](x)$, $g(s)$ is an entire function, then $f^\sharp(\Psi(1+x))$ converges and holomorphic on D.

Note. We denote formal series $\sum_n c_n \Phi^{(n)}(x)$ by $f^\sharp(\Phi(x))$, where $\sum_n c_n x^n = f(x)$. Then denoting $\check{f}(x) = \sum_n (-1)^n c_n x^n (= f(-x))$, we have

$$f^\sharp(\Phi(x)) = \check{f}_\delta[\Phi(x)], \tag{17}$$

provided $f^\sharp(\Phi(x))$ has a meaning.

Since $\sum_n \dfrac{a^n}{n!}\delta^{(n)}$ and δ_a are identified and δ_a is regarded as an element of $\mathrm{Exp}(\mathbb{C})_\delta$, we can identify $\sum_n \dfrac{a^n}{n!}\Psi^{(n)}(1+x)$ and $\Psi(1+x+a)$, so that we may regard $\Psi(1+x+a)$ to be an element of $\mathfrak{a}_{;\exp}$.

To search meanings of $f^\sharp(\Psi(1+x))$ is a problem. For example, denoting $f_s(t) = f(st)$, we have $f_s(t) = \sum_n (c_n s^n)t^n$. if $f(t) = \sum_n c_n t^n$. Hence, since

$$\Psi(1+x) = -\gamma + \sum_{n=1}^{\infty} (-1)^{n+1}\zeta(n+1)x^n,$$

$f_s^\sharp(\Psi(1+x))$ converges if $|sx|$ is sufficiently small. This suggests we may give meanings of $f^\sharp(\Psi(1+x))$ by using analytic continuation of $f_s^\sharp(\Psi(1+x))$ to $s=1$. When $f(t) = e^{at}$, originally, it should be

$$f_t^\sharp(\Psi(1+tx)) = \Psi(1+ts+ta), \quad |ts+ta| < 1.$$

But by analytic continuation, we regard this wright hand side to be a (single valued) meromorphic function on the (t,s)-space $\mathbb{C}^2$. Then put $t=1$, we get $f^\sharp(\Psi(1+x)) = \Psi(1+s+a)$ on the whole x-space $\mathbb{C}$. We note if $f_a(t) = e^{at}$, we have allowing to use analytic continuation

$$f_a^\sharp(\Psi(1+x)) = \delta_t e^{a\frac{d}{dt}}\Psi(1+t+x) = \delta_t \tau_a \Psi(1+t+x+a) = \Psi(1+x+a),$$

which gives another justification of $f_a^\sharp(\Psi(1+x)) = \Psi(1+x+a)$ if we allow to use analytic continuation.

In these discussions, we use the following properties of $\Psi(1+x)$;

1. $\Psi(1+x), \Psi'(1+x), \Psi^{(2)}(1+x), \ldots$ are linear independent over $\mathbb{C}$.

2. $\Psi(1+x)$ is a single valued meromorphic function on $\mathbb{C}$.

Other properties of $\Psi(1+x)$ are not used above discussions. We ask to search the reason why $\Psi(1+x)$ is selected in the definition of ι_Ψ in one hand and are there exist a function ϕ which satisfy the above 1, 2, and $\iota_\phi : \iota_\phi T = T_t \phi(x+t)$ is useful to the study $\mathrm{Exp}(\mathbb{C})_\delta$ or similar space of generalized functions, on the other hand.

In general, if $T = f(\delta) \in \mathrm{Exp}(\mathbb{C})_\delta$, where $f(z) = \sum_n c_n e^{a_n z}$, then

$$\iota_\Psi T = \sum_n c_n \Psi(1 + x + a_n). \tag{18}$$

We also note we can regard $T = \sum_n c_n \tau_{a_n}$ as a linear operator on $\mathrm{Ent}(\mathbb{C})$. We ask can we extend these arguments for more general f. For example, if f allows Fourier expansion $f = \sum_n c_n e^{2n\pi i z}$, are the followings have meanings ?;

$$\mu_\Psi T = \sum_n c_n \Psi(1 + x + 2n\pi i), \tag{19}$$

Here $T = f(\delta)$ and $f(x) = \sum_n c_n e^{2n\pi i x}$.

Note. If $f(x) = \int_{-\infty}^{\infty} g(t) e^{ctx} dt$, we may have

$$\mu_\Psi f(\delta) = \int_{-\infty}^{\infty} g(t)(e^{ctx}|_{t\to\delta})dt = \int_{-\infty}^{\infty} g(t)\Psi(1 + x + ct)dt.$$

Since $\Psi(1+t)$ has poles on real axis, if c is an imaginary number, $\Psi(1+x+ct)$ is smooth on $\mathbb{R} \setminus \{0\}$. Hence we need not to take care on singularities when $f = \mathcal{F}[g]$, $\mathcal{F}$ is the Fourier transform. For example, by the Paley-Wiener Theorem ([12], [15], Chap.VI), if $f(x)$ ia a finite exponential type function and rapidly decreasing at $|x| \to \pm\infty$, on real axis, then $\mu_\Psi f(\delta)$ is smooth except $x = 0$.

On the other hand, when $f = \mathcal{L}[g]$, we need to take care of singularities on real axis.

Similarly if $f(x)$ is a finite exponential type function, then $f(x) = \mathcal{B}[\phi(t)](x)$,where $\phi(t)$ is holomorphic at the origin ([10]). This suggests the following formula holds for any finite exponential type function f.

$$\mu_\Psi f(\delta) = \frac{1}{2\pi i} \int_{|\xi|=\epsilon} \frac{\phi(\xi)}{\xi} \Psi(1 + x + \frac{1}{\xi}) d\xi. \tag{20}$$

Since $\Psi(1+x+\dfrac{1}{\xi})$ has poles at $\xi = -x - \frac{1}{n}$, $n = 1, 2, \ldots$, this right hand side depends on the choice of ϵ If $f(x)$ is an entire function, then we can take ϵ sufficiently large to include all poles of $\Psi(1+x+\dfrac{1}{\xi})$ inside $\{x||x| < \epsilon\}$ which should be the appropriate choice of ϵ. But otherwise, how to choose ϵ in (20) should be a problem,

If (20) is valid, we can give concrete meaning of the map μ_Φ : $\mathrm{Exp}(\mathbb{C}) \to \mathfrak{a}_{\exp}$.

5. Interpretation of elements of $\mathfrak{a}_{\exp}$ as operators

Taking suitable function space as the space of testing functions, we have

$$x^a \mathcal{R}[f(s)](x) = \int_{\infty}^{\infty} \frac{x^{s+a}}{\Gamma(1+s)} f(s) ds$$
$$= \int_{-\infty}^{\infty} \frac{x^t}{\Gamma(1+t)} \frac{\Gamma(1+t)}{\Gamma(1+t-a)} f(t-a) dt = \mathcal{R}[\frac{\Gamma(1+t)}{\Gamma(1+t-a)} \tau_{-a} f(t)](x),$$

where $t = s + a$.Then since $\log x = \dfrac{dx^a}{da}|_{a=0}$ and $\dfrac{d\tau_{-a}}{da}|_{a=0} = -\dfrac{d}{dx}$, we obtain

$$\log x \mathcal{R}[f(s)](x) = \frac{dx^a}{da} \mathcal{R}[f(s)](x)|_{a=0}$$
$$= \int_{-\infty}^{\infty} \frac{d}{da} (\frac{\Gamma(1+s)}{\Gamma(1+s-a)} \tau_{-a} f(s)) ds|_{a=0} = \int_{-\infty}^{\infty} (\Psi(1+s) - \frac{d}{ds}) f(s) ds$$
$$= -\mathcal{R}[\Gamma(1+s) \frac{d}{ds} (\frac{1}{\Gamma(1+s)} f(s))](x).$$

Hence we have

Lemma 5. If $\mathcal{R}^{-1} \log x \mathcal{R}$ is defined as an operator on a function space, then

$$\mathcal{R}^{-1} \log x \mathcal{R} = \Gamma(1+s) \frac{d}{ds} \frac{1}{\Gamma(1+s)}. \tag{21}$$

Here $\log x$ and $\Gamma(1; x)$ are regarded as multiplication operators $\log x(f(x)) = \log x f(x)$ and $\Gamma(1+s)(f(s)) = \Gamma(1+s) f(s)$.

Corollary. We have

$$\mathcal{R}^{-1}(\log x)^n\mathcal{R} = (-1)^n\Gamma(1+s)\frac{d^n}{ds^n}\frac{1}{\Gamma(1+s)}. \tag{22}$$

By the above calculation and $\log(\frac{d}{dx})\mathcal{R}[f(s)](x) = \mathcal{R}[\frac{df(s)}{ds}])x)$, we also have

$$(\log x + \log(\frac{d}{dx}))\mathcal{R}[f(s)](x) = \mathcal{R}[\Psi(1+s)f(s)](x). \tag{23}$$

Adopting (21), we have

$$\begin{aligned}&[\log(\frac{d}{dx}), \log x + \log(\frac{d}{dx})]\mathcal{R}[f(s)](x)\\ =&\mathcal{R}[\frac{d}{ds}(\Psi(1+s)f(s)) - \Psi(1+s)\frac{df(s)}{ds}](x) = \mathcal{R}[\Psi'(1+s)f(s)](x)\end{aligned}$$

Repeating same calculations, we obtain

Proposition 3. If $\Psi^{(n)}(1+s)$ is defined as an operator of a function space, then

$$\begin{aligned}&\mathcal{R}^{-1}\overbrace{[\log(\frac{d}{dx}),\cdots[}^{n}\log(\frac{d}{dx}), \log x + \log(\frac{d}{dx})\overbrace{]\cdots]}^{n}\mathcal{R}\\ =&\Psi^{(n)}(1+s).\end{aligned} \tag{24}$$

Examples of spaces on which $\Psi^{(n)}(1+s); \Psi^{(n)}(1+s)[f(s)] = \Psi^{(n)}(1+s)$ is defined, are $\mathsf{H}_{-\mathbb{N};n+1}$ and $\mathsf{H}_{-\mathbb{N},\infty} = \cap_{n\geq 1}\mathsf{H}_{-\mathbb{N},n}$, where $\mathsf{H}_{-\mathbb{N};n}$ is the space of those smooth function f such that $f(-i) = 0, i \in \mathbb{N}$ with order at least n and $\frac{f(s)}{\Gamma(1+s)}f(s)$ is rapidly decreasing at $s \to \pm\infty$. We may use $\mathsf{H}^{\omega,+}_{-\mathbb{N},n}$ which is horomorophic on upper half plane and extended to the completion of the upper half plane whose restriction on the real axis belongs to $\mathsf{H}_{-\mathbb{N};n}$. The space $\mathsf{H}^{\omega,-}_{-\mathbb{N};n}$ is similarly defined. By definition, we have $\cap_{n\geq 1}\mathsf{H}^{\omega,\pm}_{-\mathbb{N},n} = \{0\}$, we can not consider $\mathsf{H}^{\omega,\pm}_{-\mathbb{N},\infty}$.

Note. Proposition 3 is essentially obtained in our previous study of the structure of $\mathfrak{g}_{\mathbb{R}}$ ([6]). But its explicit description (24) was not given.

6. $\sum_n c_n\delta_{a_n}$ as an element of $\mathrm{Exp}(\mathbb{C})_\delta$

Let $\mathbf{a} = (a_1, a_2, \ldots) \neq 0$ and $\mathbf{c} = (c_1, c_2, \ldots) \neq 0$. We have discussed under what conditions on $\mathbf{a}, \mathbf{c}$, the sum $\sum_n c_n\delta_{a_n}$ has a meaning ([7]). In this section, we treat same problem describing $\delta_a = \sum_n \frac{a^n}{n!}\delta^{(n)}$. By this description, formally we have

$$\sum_n c_n\delta_{a_n} = \sum_m \sum_n c_n \frac{a_m^n}{m!}\delta^{(m)}. \tag{25}$$

Hence if $|\sum_n c_n a_m^n| \leq CM^m$, $\sum_m c_m\delta_{a_m}$ belongs to $\mathrm{Exp}(\mathbb{C})_\delta$. To evaluate $\sum_n c_n a_n^m$, we assume $\mathbf{a} \in \ell^p(\mathbb{N})$, that is $\|\mathbf{a}\|_p < \infty$, $\|\mathbf{a}\|_P = (\sum_n |a_n|^p)^{1/p}$, $p \geq 1$. By assumption $\{|a_1|, |a_2|, \ldots\}$ is a bounded set. We set

$$\sup_n\{|a_1|, |a_2|, \ldots\} = u, \quad \mathbf{b} = \frac{1}{u}\mathbf{a}.$$

Then set $\mathbf{b} = (b_1.b_2, \ldots)$, we have $|b_n| \leq 1, n = 1, 2, \ldots$. Since $\mathbf{a} = u\mathbf{b}$, $\|\mathbf{b}\|_p = u^{-1}\|\mathbf{a}\|_p < \infty$. We denote $(b_1^m, b_2^m, \ldots) = \mathbf{b}^m$. Then, since $|b_n| \leq 1$ for all n, we have $\|\mathbf{b}^m\|_p \leq \|\mathbf{b}\|_p < \infty$.Hence we get

$$\|\mathbf{a}^m\|_p = u^m\|\mathbf{b}^m\|_p \leq u^m\|\mathbf{b}\|_p. \tag{26}$$

Therefore, if $p > 1$ and $\mathbf{c} \in \ell^q(\mathbb{N})$, $\frac{1}{p} + \frac{1}{q} = 1$, by the Hölder inequality ([15]), we obtain

$$|\sum_n c_n a_n^m| \leq \|\mathbf{a}\|_q\|\mathbf{a}^m\|_p \leq \|\mathbf{c}\|_q u^m\|\mathbf{b}^m\|_p = u^{m-1}\|\mathbf{c}\|_q\|\mathbf{a}\|_p. \tag{27}$$

For the convenience, we set $\|\mathbf{a}\|_\infty = \sup\{|a_1|, |a_2|, \ldots\}$ and set $\ell^\infty(\mathbb{N}) = \{\mathbf{a} | \|\mathbf{a}\|_\infty < \infty\}$. Then we have

$$|\sum_n c_n a_n^m| \leq u^{m-1}\|\mathbf{c}\|_\infty\|\mathbf{a}\|_1, \quad \mathbf{a} \in \ell^1(\mathbb{N}), \mathbf{c} \in \ell^\infty(\mathbb{N}), \tag{28}$$

$$|\sum_n c_n a_n^m| \leq \|\mathbf{a}\|_\infty^m\|\mathbf{c}\|_1, \qquad \mathbf{a} \in \ell^\infty(\mathbb{N}), \mathbf{c} \in \ell^1(\mathbb{N}). \tag{29}$$

By (27), (28) and (29), we obtain

Theorem 2([7]). Let (p, q) a pair of positive numbers including ∞, such that$p \geq 1, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, where $q = \infty$ if $p = 1$

and $q = 1$ if $p = \infty$. Then by the identification δ_a and $\sum_n \frac{a^n}{n!}\delta^{(n)}$, $\sum_n c_n\delta_{a_n}$ belongs to $\mathrm{Exp}(\mathbb{C})_\delta$, if $\mathbf{a} \in \ell^p(\mathbb{N})$ and $\mathbf{c} \in \ell^q(\mathbb{N})$.

Corollary. Under the same assumptions on (p, q), $\sum_n c_n\delta_{a_n}$ acts on $\mathrm{Ent}(\mathbb{C})$, if $\mathbf{a} \in \ell^p(\mathbb{N})$ and $\mathbf{c} \in \ell^q(\mathbb{N})$.

Note. We introduce Sobolev k-norm $\|\mathbf{a}\|_{2,k}$ by $\sqrt{\sum_n (n^k|a_n|)^2}$. Then $\sum_n c_n\delta_{a_n} \in \mathrm{Exp}(\mathbb{C})_\delta$, if $\mathbf{a} \in \ell\ 2, k(\mathbb{N})$, $k < 0$ and $\mathbf{c} \in \ell^2(\mathbb{N})$. But when $k > 0$, it seems there are no appropriate class of $\mathbf{c}$ to contain $\sum_m c_n\delta_{a_n}$ in $\mathrm{Exp}(\mathbb{C})_\delta$.

Let D be a positive elliptic operator whose spectral ζ-function $\zeta_D(s) = \sum_n \lambda_n^{-s}$ is defined and allow analytic continuation to whole $\mathbb{C}$. We set

$$T_{D,t}(s) = \sum_n \lambda_n^{-s}\delta_{\lambda_n^t}.$$

Describing $\delta_{t\lambda_n} = \sum_n \frac{\lambda_n^{tn}}{n!}\delta^{(n)}$, formally, we can set

$$T_{D,t}(s) = \sum_m \frac{1}{m!}(\sum_n \lambda_n^{-s+tm})\delta^{(m)} = \sum_m \frac{\zeta_D(s - tm)}{m!}\delta^{(m)}. \qquad (30)$$

Hence if $\Re t < 0$, $T_{D,t}$ belongs to $\mathrm{Exp}(\mathbb{C})_\delta$.In this case, we have $\lim_{n\to\infty} \lambda_n^t = 0$. So this conclusion seems not interesting. But as a function of t, it is analytic. Therefore we can consider analytic continuation of $T_{D,t}$ to $t = t_0$, $\Re t_0 > 0$ (cf.[3]).

Example We assume $\lambda_n = n$, that is $\zeta_D(s) = \zeta(s)$, the Riemann ζ-function. Since $\zeta(-n) = -\frac{B_{n+1}}{n+1}$, where B_n is the n-th Bernoulli number, $T_{D,2k}(2\ell)$ is a finite sum, hence belong to $\mathrm{Exp}(\mathbb{C})_\delta$. While if t is a odd positive integer and s is even, or t is a positive even integer and s is odd, $T_{D,t}(s)$ does not belong to $\mathrm{Exp}(\mathbb{C})_\delta$.

7. Remarks on several variables case

The group $G_{\mathbb{R}^n}$ generated by $\{\frac{\partial^{a_1}}{\partial x_1^{a_1}}, \ldots, \frac{\partial^{a_n}}{\partial x_n^{a_n}} | \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n\}$ and $\{x_1^{a_1}, \ldots, x_n^{a_n} | \mathbf{a} \in \mathbb{R}^n\}$ is isomorphic to the extension of the multiplicative free abelian group $\mathcal{R}^{-1}\mathsf{A}_{\mathbb{R}^n}\mathcal{R}$ by $\mathbb{R}^n$, where $\mathsf{A}_{\mathbb{R}^n}$

is generated by $\{\frac{\Gamma(1+x_1)}{\Gamma(1+x_1-a_1)},\ldots,\frac{\Gamma(1+x_n)}{\Gamma(1+x_n-a_n)}|\mathbf{a}\in\mathbb{R}^n\}$. Here $\mathcal{R}=\mathcal{R}_{\mathbf{x}}$ is defined by

$$\mathcal{R}_{\mathbf{x}}[f(\mathbf{s})[\mathbf{x})=\int_{\mathbb{R}^n}\prod_{i=1}^{n}\frac{x_i^{s_i}}{\Gamma(1+s_i)}f(\mathbf{s})d\mathbf{s},$$

Consequently, we have

$$G_{\mathbb{R}^n}\cong\overbrace{G_{\mathbb{R}}\times\cdots\times G_{\mathbb{R}}}^{n}.,\quad \mathsf{A}_{\mathbb{R}^n}\cong\overbrace{\mathsf{A}_{\mathbb{R}}\times\cdots\times\mathsf{A}_{\mathbb{R}}}^{n} \tag{31}$$

The groups $G^{\natural}_{\mathbb{R}^n}$ and $\mathsf{A}^{\natural}_{\mathbb{R}^n}$ are also n-products of $G^{\natural}_{\mathbb{R}}$ and $\mathsf{A}^{\natural}_{\mathbb{R}}$.

Similarly, the Lie algebra $\mathfrak{g}_{\mathbb{R}^n}$ generated by $\log(\frac{\partial}{\partial x_1}),\ldots,\log(\frac{\partial}{\partial x_n})$ and $\log x_1,\ldots,\log x_n$ is isomorphic to $\overbrace{\mathfrak{g}_{\mathbb{R}}\oplus\cdots\oplus\mathfrak{g}_{\mathbb{R}}}^{n}$.It is generated by $\frac{\partial}{\partial s_1},\ldots,\frac{\partial}{\partial s_n}$ and $\Psi(1+s_1)ds_1,\ldots,\Psi(1+s_n)ds_n$. Let $\mathfrak{a}_{\mathbb{R}^n}$ be the ideal of $\mathfrak{g}_{\mathbb{R}^n}$ generated by $\Psi(1+s_1)ds_1,\ldots,\Psi(1+s_n)ds_n$. Then we have

$$\mathfrak{a}_{\mathbb{R}^n}\cong\mathfrak{a}ds_1\oplus\cdots\oplus\mathfrak{a}ds_n.$$

Definition 5. We define $\mathfrak{a}_{\mathbb{R}^n;\exp}$ by

$$\mathfrak{a}_{\mathbb{R}^n;\exp}=\mathfrak{a}_{\exp}ds_1\oplus\cdots\oplus\mathfrak{a}_{\exp}ds_n. \tag{32}$$

Precisely, $\mathfrak{a}_{\exp}ds_i$ in (31) means

$$\mathfrak{a}_{\exp}ds_i=\{(\sum_m c_m\Psi^{(m)}(1+s_i))ds_i|\sum_m c_m x^m\in\mathrm{Exp}(\mathbb{C})\}.$$

We consider $\mathfrak{a}_{\mathbb{R}^n;\exp}$ to be a good completion of $\mathfrak{a}_{\mathbb{R}^n}$. We also define $\mathsf{A}^{\natural}_{\mathbb{R}^n;\exp}$ by

$$\mathsf{A}^{\natural}_{\mathbb{R}^n;\exp}=\overbrace{\mathsf{A}^{\natural}_{\mathbb{R};\exp}\times\cdots\times\mathsf{A}^{\natural}_{\mathbb{R};\exp}}^{n}. \tag{33}$$

Then by the map $\rho:\rho(g)=g^{-1}dg$, we have

$$\rho:\mathsf{A}^{\natural}_{\mathbb{R}^n;\exp}\cong\mathfrak{a}_{\mathbb{R}^n;\exp}. \tag{34}$$

For the simplicity, we set $\mathrm{Exp}(\mathbb{C})_\delta|_{x=x_1}dx_1\oplus\cdots\oplus\mathrm{Exp}(\mathbb{C})_\delta|_{x=x_n}dx_n = (\Lambda^1\mathrm{Exp}(\mathbb{C})_\delta)^n$. Then as vector spaces, we have

$$\mathfrak{a}_{\mathbb{C}^n;\exp} \cong (\Lambda^1\mathrm{Exp}(\mathbb{C})_\delta)^n.$$

Note. To regard elements of $\mathfrak{a}_{\mathbb{R}^n}$ to be 1-forms reflects to this isomorphism.

On the other hand, it seems we can not use $\Lambda^1\mathrm{Exp}(\mathbb{C}^n)$, the space of 1-forms having $\mathrm{Exp}(\mathbb{C}^n)$ as coefficients, and its subspaces as the origin of $\mathsf{A}^\natural_{\mathbb{R}^n;\exp}$. instead of $\mathrm{Exp}(\mathbb{C})^n$;

$$\mathrm{Exp}(\mathbb{C})^n = \mathrm{Exp}(\mathbb{C})|_{x=x_1}\oplus\cdots\oplus\mathrm{Exp}(\mathbb{C})|_{x=x_1}.$$

. In fact, take the submodule $\mathrm{Exp}(\mathbb{C}^n)_s$;

$$\mathrm{Exp}(\mathbb{C}^n)_s = \{\sum_i f_i(x_i)|f_i(x)\in\mathrm{Exp}(\mathbb{C})\},$$

there is a map $\sigma : \mathrm{Exp}(\mathbb{C})^n \to \mathrm{Exp}(\mathbb{C}^n)_s$;

$$\sigma(f_1(x_1)\oplus\cdots\oplus f_n(x_n)) = f_1(x_1)+\cdots+f_n(x_n).$$

Then we have

$$\sigma(\mathrm{Exp}(\mathbb{C})^n) = \mathrm{Exp}(\mathbb{C}^n)_s,$$
$$\mathrm{Ker}\sigma = \{x_1\oplus\cdots\oplus x_n|(x_1,\ldots,x_n)\in\mathbb{C}^n, x_1+\cdots+x_n=0\}$$

Therefore $\mathrm{Exp}(\mathbb{C})^n$ and $\mathrm{Exp}(\mathbb{C}^n)_s$ are different.

The isomorphism from $\mathrm{Exp}(\mathbb{C})^n_\delta$ is given by

$$\mu_{-x_1;\Psi}\times\cdots\times\mu_{-x_n;\Psi} : \mathrm{Exp}(\mathbb{C})^n_\delta \cong \mathsf{A}^\natural_{\mathbb{R}^n;\exp}. \tag{35}$$

Here, $T\in\mathrm{Exp}(\mathbb{C}^n)_{\delta;\mathbb{X}}$ is written as $T=\oplus_{i=1}^n T_i$, $T_i\in\mathrm{Exp}(\mathbb{C})_\delta|_{x=x_i}$. Hence Th.2 is automatically extended to the elements of $\mathrm{Exp}(\mathbb{C})^n_\delta$. But definitions of $\mathfrak{a}_{\mathbb{C}^n;\exp}$, *etc.* depend on the choice of coordinate of $\mathbb{C}^n$. Moreover, even a linear transform of $\mathbb{C}^n$, it does not acts on $\mathrm{Exp}(\mathbb{C})^n_\delta$, *etc. To extend* $\mathfrak{a}_{\mathbb{C}^n;\exp}$, *etc., to allow action by suitable class of linear transform of* $\mathbb{C}^n$ *.should be a problem.*

Note. To define $\mathrm{Exp}(\mathbb{C}^n)_\delta$, *etc.*, we introduce the notation $\delta_{i_1,\ldots,i_n}$;

$$\delta^{i_1,\ldots,i_n}(x_1^{k_1}\cdots x_n^{k_n}) = \begin{cases}(-1)^{k_1+\cdots+k_n}i_1!\cdots i_n!, & (k_1,\ldots,k_n)=(i_1,\ldots,i_n),\\ 0, & (k_1,\ldots,k_n)\neq(i_1,\ldots,i_n).\end{cases} \tag{36}$$

Then if $f(\mathbf{x}) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$ is holomorphic at the origin of $\mathbb{C}^n$, we set

$$f(\delta) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} \delta^{i_1,\ldots,i_n}. \tag{37}$$

By using this notation, we define $\mathrm{Exp}(\mathbb{C}^n)_\delta$, *etc.*, by

$$\mathrm{Exp}(\mathbb{C}^n)_\delta = \{f(\delta) | f(\mathbf{x}) \in \mathrm{Exp}(\mathbb{C}^n)\}.$$

Then we have

$$\mathrm{Exp}(\mathbb{C}^n)_\delta = \mathrm{Ent}(\mathbb{C}^n)^\dagger, \ \mathrm{Ent}(\mathbb{C}^n)_\delta = \mathrm{Exp}(\mathbb{C}^n)^\dagger. \tag{38}$$

As noted in §2, (39) is a reinterpretation of the duality of O^n, the algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$ and $\mathrm{Ent}(\mathbb{C}^n)$;

$$\mathsf{O}^n \cong \mathrm{Ent}(\mathbb{C}^n)^\dagger, \quad \mathrm{Ent}(\mathbb{C}^n) \cong (\mathsf{O}^n)^\dagger,$$

$$(f(\mathbf{x}), g(\mathbf{x})) = \frac{1}{(2\pi i)^n} \overbrace{\oint \cdots \oint}^{n} f(\mathbf{x}) \frac{g(\frac{1}{x_1}, \cdots, \frac{1}{x_n})}{x_1 \cdots x_n} d\mathbf{x},$$

via the isomorphism $\mathcal{B} : \mathsf{O}^n \cong \mathrm{Exp}(\mathbb{C}^n)$. Because e have

$$\begin{aligned} \delta^{i_1,\ldots,i_n}[f(\mathbf{x}] =& (-1)^{i_1+\cdots+i_n} \frac{\partial^{i_1+\cdots+i_n} f(\mathbf{x})}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \\ =& (-1)^{i_1+\cdots+i_n} \overbrace{\oint \cdots \oint}^{n} \frac{f(\mathbf{x})}{x_1^{i_1+1} \cdots x_n^{i_n+1}} d\mathbf{x}. \end{aligned}$$

Although $\mathrm{Exp}(\mathbb{C}^n)_\delta$ do not have direct relation between completions of $\mathfrak{a}_{\mathbb{R}^n}$ and $\mathsf{A}^\natural_{\mathbb{R}^n}$ as remarked above, we expect (38) may serve to the study of $\mathfrak{a}_{\mathbb{C}^n;\exp}$, *etc.*, near future.

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