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The relative error bounds of fuzzy linear systems

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Abstract
In this paper, the perturbation analysis of fully fuzzy linear systems is presented. The relative error bounds of fully fuzzy linear systems for perturbation of both right hand and coefficient matrix are derived. The results are illustrated by numerical example.

1. Introduction

The topic of fuzzy linear systems, which attracted increasing interest for some time, in particular in relation to fuzzy neural network, has been rapidly grown in recent years [1,2].
A general model for solving a fuzzy linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy vector, first proposed by Friedman et al. [2]. Dehghan [3] extended some iterative methods on the same system and discussed [4] the case in which all parameters in a fuzzy linear system are fuzzy numbers, which is called a fully fuzzy linear system (FFLS).
As claimed by Skalna et al.[5], it is necessary to analyze sensitivity of the computed solutions of fuzzy linear systems to the coefficient and the right hand vector. Tang [6] considered the perturbation problem of a fuzzy matrix equation. Wang et al. [7] analyzed the perturbation of fuzzy linear system. However, they focused their discussion only on fuzzy linear systems with the right hand side being triangular fuzzy numbers and their results were only valid for numerical approaches based on the embedding method.

2. Preliminaries

Definition 2.1. [10,11] Let X denote a universal set. Then a fuzzy subset $\tilde{A}$ of X is defined by its membership function
$\mu_{\tilde{A}} : X \to [0,1]$
Which assigns a real number $\mu_{\tilde{A}}(x)$ in the interval [0,1], to each element $x \in X$, where the value of $\mu_{\tilde{A}}(x)$ at x shows the grade of membership of x in $\tilde{A}$.
A fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\tilde{A}}(x)$ and is often written
$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$.
Definition 2.2. The r-level set of a fuzzy set $\tilde{A}$ is defined as an ordinary set $[\tilde{A}]_r$ for which the degree of its membership function exceeds the level r
$[\tilde{A}]_r = \{x | \mu_{\tilde{A}}(x) \geq r, r \in (0,1]\}$.

Definition 2.3. A fuzzy set $\tilde{A}$ in $X = R^n$ is said to be a convex fuzzy set if and only if its r-level sets are convex.

Definition 2.4. A fuzzy set $\tilde{A}$ in X is said be normal if there exist $x \in X$ such that $\mu_{\tilde{A}}(x)=1$.

Definition 2.5. A fuzzy number is a convex normalized fuzzy set of the real line R whose membership function is piecewise continuous.

Definition 2.6. A fuzzy number $\tilde{m}$ is called positive(negative), denoted by $\tilde{m}>0$ ($\tilde{m}<0$), if its membership function $\mu_{\tilde{m}}(x)$ satisfies $\mu_{\tilde{m}}(x)=0 \quad \forall x<0 (\forall x>0)$.

Remark 2.7. A fuzzy number could be neither positive, nor negative.

Definition 2.8. A fuzzy number $\tilde{m}$ is said to be an LR fuzzy number if

$$\mu_{\tilde{m}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & x \le m, \alpha > 0 \\ R(\frac{x-m}{\beta}) & x \ge m, \beta > 0 \end{cases}$$

Where m is the mean value of $\tilde{m}$ and α and β are left and right spreads, respectively, and a function L(x), the left shape function, satisfying

(1) L(x)=L(-x),
(2) L(0)=1 and L(1)=0,
(3) L(x)is non-increasing on $[0,\infty)$.

Naturally, a right shape function, R(x) is similarly defined as L(x).

Remark 2.9. LR fuzzy number $\tilde{m}$ is symbolically written $\tilde{m} = (m, \alpha, \beta)_{LR}$.

Clearly, $\tilde{m} = (m, \alpha, \beta)_{LR}$ is positive, if and only if $m - \alpha > 0$ (note that L(1)=0).

Definition 2.10. Two LR fuzzy numbers $\tilde{m} = (m, \alpha, \beta)_{LR}$ and $\tilde{n} = (n, \gamma, \delta)_{LR}$ are said to be equal, if and only if $m = n$, $\alpha = \gamma$ and $\beta = \delta$.

Dubois and Prade show exact formulas for $\oplus$ and $\ominus$, [11]

$\tilde{m} \ominus \tilde{n} = (m, \alpha, \beta)_{LR} \ominus (n, \gamma, \delta)_{LR} = (m-n, \alpha+\gamma, \beta+\delta)_{LR}$.

$\tilde{m} \oplus \tilde{n} = (m, \alpha, \beta)_{LR} \oplus (n, \gamma, \delta)_{LR} = (m+n, \alpha+\gamma, \beta+\delta)_{LR}$.

Scalar multiplication:

$$h \otimes \tilde{m} = \begin{cases} (hm, h\alpha, h\beta) & h > 0 \\ (hm, -h\beta, -h\alpha) & h < 0 \end{cases}$$

if $\tilde{m}>0$ and $\tilde{n}>0$ then approximate formula for $\otimes$ is

$(m, \alpha, \beta)_{LR} \otimes (n, \gamma, \delta)_{LR} \cong (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR}$.

Up to rest of this paper, equal is considered instead of approximation [4].

Definition 2.11. A matrix $\tilde{A}$ is called a fuzzy matrix, if each element of $\tilde{A}$ is a fuzzy number and will be positive (negative) if each element of $\tilde{A}$ be positive(negative).

We may represent fuzzy number matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, that $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, with new notation $\tilde{A} = (A, M, N)$, where $A = (a_{ij})_{n\times n}$, $M = (\alpha_{ij})_{n\times n}$ and $N = (\beta_{ij})_{n\times n}$ are three crisp matrices.

Definition 2.12. A norm of matrix $\tilde{A}=(A,M,N)$ is $\|\tilde{A}\|_p=\|A\|_p+\|M\|_p+\|N\|_p$ where $\|.\|_p$ is p-norm of crisp matrix.

Definition 2.13. Absolute value of a LR number $\tilde{m}=(m,\alpha,\beta)_{LR}$ is $|\tilde{m}|=|m|+\alpha+\beta.$

Definition 2.14. A distance of $\tilde{m}=(m,\alpha,\beta)_{LR}$ and $\tilde{n}=(n,\gamma,\delta)_{LR}$ is defined as
$d(\tilde{m},\tilde{n})=|m-n|+|\alpha-\gamma|+|\beta-\delta|.$

Definition 2.15. A distance of fuzzy vectors $\tilde{X}=(X_1,Y_1,Z_1)$ and $\tilde{Y}=(X_2,Y_2,Z_2)$ is defined as
$d(\tilde{X},\tilde{Y})=\|X_1-X_2\|_p+\|Y_1-Y_2\|_p+\|Z_1-Z_2\|_p$ where $\|.\|_p$ is p-norm of crisp vector.

Definition 2. 16. Consider the $n\times n$ fully fuzzy linear system of equation:

$$\begin{cases}(\tilde{a}_{11}\otimes\tilde{x}_1)\oplus(\tilde{a}_{12}\otimes\tilde{x}_2)\oplus\ldots\oplus(\tilde{a}_{1n}\otimes\tilde{x}_n)=\tilde{b}_1\\(\tilde{a}_{21}\otimes\tilde{x}_1)\oplus(\tilde{a}_{22}\otimes\tilde{x}_2)\oplus\ldots\oplus(\tilde{a}_{2n}\otimes\tilde{x}_n)=\tilde{b}_2\\\vdots\\(\tilde{a}_{n1}\otimes\tilde{x}_1)\oplus(\tilde{a}_{n2}\otimes\tilde{x}_2)\oplus\ldots\oplus(\tilde{a}_{nn}\otimes\tilde{x}_n)=\tilde{b}_n\end{cases}$$

The matrix form of the above equation is $\tilde{A}\otimes\tilde{X}=\tilde{B}$ where $\tilde{A}=(A,M,N)$ and $\tilde{X}=(X\ ,Y\ ,Z)$ and $\tilde{B}=(b,h,g)$.

In [4] finding positive solution of FFLS $\tilde{A}\otimes\tilde{X}=\tilde{B}$ where $\tilde{A}$ and $\tilde{B}$ are positive, was lead to

$$(AX,AY+MX,AZ+NX)=(b,h,g) \qquad (1)$$

$AX=b$, $AY+MX=h$ and $AZ+NX=g$.

By assuming A is nonsingular crisp matrix, then

$$X=A^{-1}b,\ Y=A^{-1}h-A^{-1}MX \text{ and } Z=A^{-1}g-A^{-1}NX. \qquad (2)$$

Theorem 2.17. [4] let $\tilde{A}=(A,M,N)$ be a positive fuzzy matrix and $\tilde{B}=(b,h,g)$ be a positive fuzzy vector. also, assume A is the product of a permutation matrix by a diagonal matrix with positive diagonal entries. Moreover, let $h\geq MA^{-1}B,$ $g\geq NA^{-1}B$ and $(MA^{-1}+I)B\geq h$, Then the system $\tilde{A}\otimes\tilde{X}=\tilde{B}$ has a positive fuzzy solution.

3. Perturbation analysis of the FFLS

Both the coefficient fuzzy matrix and the right hand side perturbed

In the final case, we assume that both the coefficient fuzzy matrix $\tilde{A}$ and the right hand side $\tilde{B}$ in the FFLS are perturbed.

Theorem 3.1

Let $\tilde{X}_1=(X_1,Y_1,Z_1)$ and $\tilde{X}_2=(X_2,Y_2,Z_2)$ be the solution vectors to $\tilde{A}\otimes\tilde{X}=\tilde{B}_1$ and $\tilde{A}_2\otimes\tilde{X}=\tilde{B}_2$, respectively, where $\tilde{A}_2=\tilde{A}\oplus\delta\tilde{A}$, then by hypotheses of theorem 2.17, we can find coefficients $k_1,k_2,\ldots,k_6$ that

$$d(\tilde{X}_1,\tilde{X}_2) \le k_1.\|\delta A\|_p + k_2.\|\delta M\|_p + k_3.\|\delta N\|_p + k_4.\|b_1 - b_2\|_p + k_5.\|h_1 - h_2\|_p + k_6.\|g_1 - g_2\|_p$$

Proof:

We have

$$A^{-1}b_1 - A_2^{-1}b_2 = A_2^{-1}(A_2A^{-1}b_1 - b_2) = A_2^{-1}((b_1 - b_2) + \delta A.A^{-1}.b_1) \quad (3)$$

And

$$\begin{aligned} &A^{-1}MA^{-1}.b_1 - A_2^{-1}M_2A_2^{-1}.b_2 = A_2^{-1}.(A_2A^{-1}MA^{-1}.b_1 - M_2A_2^{-1}.b_2) \\ &= A_2^{-1}.(M(A^{-1}b_1 - A_2^{-1}b_2) + \delta A.A^{-1}.M.A^{-1}.b_1 - \delta M.A_2^{-1}.b_2) \end{aligned} \quad (4)$$

And too

$$A^{-1}NA^{-1}.b_1 - A_2^{-1}N_2A_2^{-1}.b_2 = A_2^{-1}.(N(A^{-1}b_1 - A_2^{-1}b_2) + \delta A.A^{-1}.N.A^{-1}.b_1 - \delta N.A_2^{-1}.b_2) \quad (5)$$

Like proof of theorem 3.4 we have

$$\begin{aligned} X_1 - X_2 &= A^{-1}b_1 - A_2^{-1}b_2 = A_2^{-1}((b_1 - b_2) + \delta A.A^{-1}.b_1) \\ Y_1 - Y_2 &= A_2^{-1}((h_1 - h_2) + \delta A.A^{-1}.h_1) - (A^{-1}MA^{-1}.b_1 - A_2^{-1}M_2A_2^{-1}.b_2) \\ Z_1 - Z_2 &= A_2^{-1}((g_1 - g_2) + \delta A.A^{-1}.g_1) - (A^{-1}NA^{-1}b_1 - A_2^{-1}N_2A_2^{-1}.b_2) \end{aligned} \quad (6)$$

By replacing, (3), (4) and (5) in (6), proof is finished. □

4. Numerical example

Consider the following FFLS [4],

$$\begin{cases} \tilde{5}\tilde{X}_1 + \tilde{6}\tilde{X}_2 = 5\tilde{0} \\ \tilde{7}\tilde{X}_1 + \tilde{4}\tilde{X}_2 = 4\tilde{8} \end{cases}$$

We mean,

$$\begin{cases} (5,1,1) \otimes (x_1, y_1, z_1) \oplus (6,1,2) \otimes (x_2, y_2, z_2) = (50,10,17) \\ (7,1,0) \otimes (x_1, y_1, z_1) \oplus (4,0,1) \otimes (x_2, y_2, z_2) = (48,5,7) \end{cases}$$

Where

we have

$\tilde{B}_2 = (b_2, h_2, g_2)$, that $b_2 = (50.001, 48.001)$, $h_2 = (10.001, 5.001)$ and $g_2 = (17.001, 7.001)$.

$\tilde{X}_1 = (X_1, Y_1, Z_1)$, that $X_1 = (x_1, x_2)^t = (4,5)^t$, $Y_1 = (1/11, 1/11)^t$ and $Z_1 = (0, 0.5)^t$

$\tilde{X}_2 = (X_2, Y_2, Z_2)$,that $X_2 = (x_1, x_2)^t = (4.001, 5.001)^t$, $Y_2 = (.0910, .0905)^t$ and $Z_2 = (.0003, .49993)^t$.

$d(\tilde{X}_1,\tilde{X}_2)/\|\tilde{X}_1\|_1$	$\|A\|_1$	$\|A^{-1}\|_1$	$\|M\|_1$	$\|N\|_1$	$d(\tilde{B}_1,\tilde{B}_2)$	$\|\tilde{B}_1\|_1$	*bound*($d(\tilde{X}_1,\tilde{X}_2)/\|\tilde{X}_1\|_1$)
2.964876e-4	**12**	**.5**	**2**	**3**	**.003**	**137**	**4.598540e-4**

we have

$$\delta A = \begin{bmatrix} .001 & .001 \\ .001 & .001 \end{bmatrix}, \ \delta M = \begin{bmatrix} .001 & .001 \\ .001 & .001 \end{bmatrix} \text{ and } \delta N = \begin{bmatrix} .001 & .001 \\ .001 & .001 \end{bmatrix}.$$

$\tilde{X}_1 = (X_1, Y_1, Z_1)$, that $X_1 = (x_1, x_2)^t = (4,5)^t$, $Y_1 = (1/11, 1/11)^t$ and $Z_1 = (0, 0.5)^t$

$\tilde{X}_2 = (X_2, Y_2, Z_2)$,that $X_2 = (x_1, x_2)^t = (3.9992, 4.9992)^t$, $Y_2 = (.0901, .0901)^t$ and $Z_2 = (-.0009, .49991)^t$.

$d(\tilde{X}_1, \tilde{X}_2)/\|\tilde{X}_1\|_1$	$\|A_2^{-1}\|_1$	$\|A^{-1}\|_1$	$bound(d(\tilde{X}_1, \tilde{X}_2)/\|\tilde{X}_1\|_1)$
.516475e-4	**.5**	**.5**	**.887226 e-2**

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