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Bayesian Sequential Estimation of the Parameter of a Class of Life Distributions

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Abstract

A Bayesian sequential procedure for estimating the parameter of a class of life distributions under the squared loss is proposed. It is assumed that the cost per observation is one unit and that the parameter is the value of a random variable having the Inverse Gamma distribution with known parameters. The procedure has two components, a stopping time and the Bayes estimator based on the observations taken up to the stopping time. An upper bound is obtained for the Bayes regret, that is used as a measure of the performance of the proposed. It is found that the proposed procedure performs better than the best fixed-sample-size procedure, asymptotically.

Keywords: Bayes estimator, Bayes regret, Jensen's inequality, Fatou's Lemma, martingale, posterior distribution, sequential procedure, stopping time.

1. Introduction

Consider the class of life distributions with probability density function of the form

$$f_\theta(x) = \begin{cases} \frac{g'(x)}{\theta} e^{-\frac{g(x)}{\theta}} & \text{if } x > x_0 \\ 0 & \text{if not,} \end{cases} \tag{1}$$

where θ is an unknown positive number, g is a differentiable real-valued function such that $g(x_0) = 0$, $g'(x) > 0$ for $x > x_0$ and $x_0 \geq 0$ is a known number. This class includes distributions that have been widely used in the analysis of data arising from life-testing experiments. In fact, this class contains the following probability distributions:

a) The Exponential distribution with p.d.f. $f_\theta(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$ for $x > 0$ ($g(x) = x$ and $x_0 = 0$);

b) The Weibull distribution with p.d.f. $f_\theta(x) = \frac{\gamma}{\theta} x^{\gamma-1} e^{-\frac{x^\gamma}{\theta}}$ for $x > 0$ ($g(x) = x^\gamma$ with γ known and $x_0 = 0$);

c) The Pareto distribution with p.d.f. $f_\theta(x) = \frac{1}{\theta} x^{-\frac{1}{\theta}-1}$ for $x > 1$ ($g(x) = \ln x$ and $x_0 = 1$)

d) The Rayleigh distribution with p.d.f. $f_\theta(x) = \frac{2x}{\theta} e^{-\frac{x^2}{\theta}}$ for $x > 0$ $(g(x) = x^2$ and $x_0 = 0)$

e) The Burr distribution with p.d.f. $f_\theta(x) = \frac{1}{\theta}\left(1 + x^\beta\right)^{-\frac{1}{\theta}-1}$ for $x > 0$ $(g(x) = \ln\left(1 + x^\beta\right)$ with β known and $x_0 = 0)$.

Let $X_1,\ X_2, \ldots$ be independent observations to be taken one at a time from the population with probability density function given by (1). The sampling process will stop after N observations have been taken, at which time the parameter θ is estimated by $\Theta_{N,}$ subject to the loss function

$$L_a(\theta, \Theta_N) = a(\Theta_N - \theta)^2 + N, \tag{2}$$

where a is a known positive and Θ_N is the Bayes estimator of θ obtained under the assumption that θ is a value of a random variable Θ whose prior distribution is the Inverse Gamma distribution with parameters α_0 and β_0; that is Θ has prior density function

$$\xi(\theta) = \begin{cases} \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)\theta^{\alpha_0+1}} e^{-\frac{\beta_0}{\theta}} & \text{if } \theta > 0 \\ 0 & \text{if not,} \end{cases} \tag{3}$$

where α_0 and β_0 are known positive numbers and $\Gamma(\alpha_0) = \int_0^\infty t^{\alpha_0 - 1} e^{-t} dt$. The loss function in (2) includes the loss due to estimation error, $(\Theta_N - \theta)^2$, the sampling cost, which is based on one unit per observation and the number $a,$ which is determined by the importance of estimation error relative to the cost of sampling. The sample size N is not chosen in advance. It is determined as a stopping time, which is a random variable determined by $X_1,\ X_2, \ldots$ This is accomplished in Section 2. An upper bound for the Bayes regret incurred by the sequential procedure (N, Θ_N) is provided in Section 3. Finally, some asymptotic results, as $a \to \infty,$ are presented in Section 4. The advantage of using a sequential estimation procedure is that it can be constructed with a substantially smaller number of observations compared to any equally reliable procedure based on a random sample with a predetermined size.

Bayesian sequential estimation problems were studied by several authors, notably Bickel and Yahav (1969), Alvo (1977), Rasmussen (1980), Shapiro and Wardrop (1980), Woodroofe (1985), Tahir (1989), Woodroofe and Hardwick (1990) and Hwang (1997) and Rekab, Tahir (2004) and Jokiel-Rokita (2008). Recently, Tahir (2016) considered Bayesian sequential estimation of the inverse of the Pareto shape parameter. He proposed a sequential procedure and obtained a second-order asymptotic expansion for the Bayes regret incurred by the proposed procedure, under the assumption that the prior density function has a compact support.

2. The Proposed Stopping Time

Suppose that N is chosen in advance. If $x_1 > x_0, \dots, x_N > x_0$ are observed values of $X_1, \dots, X_N$, respectively, then the likelihood function is given by

$$h_N(\theta) = \left[\prod_{i=1}^{N} g'(x_i)\right] exp\left\{-N \ln\theta - \frac{N}{\theta}\bar{y}_N\right\} \text{ for } \theta > 0, \text{where } \bar{y}_N = \frac{1}{N}\sum_{i=1}^{N} g(x_i).$$

It follows that the posterior distribution of Θ, given that $X_1 = x_1, \dots, X_N = x_N$, based on (3), is the Inverse Gamma distribution with parameters $\alpha_N = \alpha_0 + N$ and $\beta_N = \beta_0 + N\bar{y}_N$. Thus, the Bayes estimate of θ is the mean of the posterior distribution of Θ; that is

$$\theta_N = E\{\Theta | X_1 = x_1, \dots, X_N = x_N\} = \frac{\beta_N}{\alpha_N - 1}.$$

Definition 2.1 The risk incurred by estimating θ by Θ_N under the loss (2) is defined as

$$R_a(N,\theta) = E_\theta[L_a(\theta, \Theta_N)] = aE_\theta[(\Theta_N - \theta)^2] + N$$

for any value of $a > 0$, where E_θ denotes conditional expectation, given $\Theta = \theta$ and

$$\Theta_n = E\{\Theta | X_1, \dots, X_n\} = \frac{\beta_0 + n\bar{Y}_n}{n + \alpha_0 - 1} \text{ with } \bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} g(X_i) \qquad (4)$$

for $n \geq 1$.

Using (4) with $n = N$ and the fact that, given $\Theta = \theta$, $Y_1, \dots, Y_N$ are independent random variables with common distribution the Exponential distribution with mean θ yields

$$R_a(N,\theta) = aE_\theta[(\bar{Y}_N - \theta + U_N)^2] + N = aE_\theta[(\bar{Y}_N - \theta)^2] + N = \frac{a\theta^2}{N} + ar_N + N$$

for any value of $a > 0$, where

$$U_N = \frac{\beta_0}{N + \alpha_0 - 1} + \frac{1 - \alpha_0}{N + \alpha_0 - 1}\bar{Y}_N \text{ and } r_N = 2E_\theta[(\bar{Y}_N - \theta)U_N] + E_\theta[U_N^2].$$

Next,

$$E_\theta[(\bar{Y}_N - \theta)U_N] = \frac{\beta_0 E_\theta[(\bar{Y}_N - \theta)]}{N + \alpha_0 - 1} + \frac{(1 - \alpha_0)E_\theta[(\bar{Y}_N - \theta)^2]}{N + \alpha_0 - 1} = \frac{(1 - \alpha_0)\theta^2}{N(N + \alpha_0 - 1)} = o\left(\frac{1}{N}\right)$$

and

$$E_\theta[U_N^2] \leq \frac{2[\beta_0 + (1-\alpha_0)\theta]^2}{(N+\alpha_0-1)^2} + \frac{2(1-\alpha_0)^2 E_\theta[(\bar{Y}_N - \theta)^2]}{(N+\alpha_0-1)^2} = o\left(\frac{1}{N}\right)$$

as $N \to \infty$. Thus, if N is large, then $R_a(N,\theta) \approx \frac{a\theta^2}{N} + N$ for any value of $a > 0$. It follows that the approximate risk is minimized with respect to N by choosing N adjacent to $N_a = \theta\sqrt{a}$. The minimum risk or the risk incurred by the best fixed-sample-size procedure is $R_a(N_a,\theta) \approx 2\theta\sqrt{a}$ for large values of a. The sample size N_a cannot be used in practice since it depends on the unknown value of θ. Thus, there is no fixed-sample-size procedure that attains the minimum risk in practice. To overcome this problem, we use the stopping time N, defined as follows: Observe the values of $X_1, \ldots, X_n$ and compute the value of Θ_n using (4) for $n \geq m$, where $m \geq 2$ is an initial sample size chosen in advance. Stop the sampling process after taking N observations, where

$$N = N_a = \text{the smallest integer } n \geq m \text{ such that } n > \sqrt{a}\Theta_n. \quad (5)$$

The proposed sequential procedure is the pair (N, Θ_N), where $\Theta_N = E\{\Theta | X_1, \ldots, X_N\}$ and is defined by $\Theta_N(\omega) = E\{\Theta | X_1, \ldots, X_{N(\omega)}\}$ for each $\omega \in \Omega$. Note that $N(\omega)$ is an integer.

3. Analysis of the Bayes regret

The performance of the sequential procedure (N, Θ_N) is measured by the Bayes regret, $\bar{r}_a(\alpha_0, \beta_0)$.

Definition 3.1 The regret, $r_a(\theta)$, incurred by the procedure (N, Θ_N) is the difference between the risk incurred by the sequential procedure (N, Θ_N) and the risk incurred by the best fixed-sample-size procedure; that is

$$r_a(\theta) = E_\theta[L_a(\theta, \Theta_N)] - 2\theta\sqrt{a} = E_\theta[a(\Theta_N - \theta)^2 + N] - 2\theta\sqrt{a}$$

for $a > 0$.

Definition 3.2 The Bayes regret incurred by the procedure (N, Θ_N) is defined by

$$\bar{r}_a(\alpha_0, \beta_0) = \int_0^\infty r_a(\theta)\xi(\theta)d\theta,$$

for $a > 0$.

For $n \geq 1$, let $\mathcal{D}_n$ denote the σ-algebra generated by $X_1, \ldots, X_n$. Then, the conditional variance of Θ, given $\mathcal{D}_n$, is

$$E\{(\Theta - E\{\Theta|\mathcal{D}_n\})^2|\mathcal{D}_n\} = \frac{(\beta_0+n\bar{Y}_n)^2}{(\alpha_0+n-1)^2(\alpha_0+n-2)} = \frac{\Theta_n^2}{\alpha_0+n-2}. \quad (6)$$

Next, let $(\Omega, \mathfrak{F}, P)$ denote the probability space on which $X_1, \dots, X_n$ are defined and let N be defined by (5). Then, $\mathcal{D}_N = \{D \in \mathfrak{F}: D \cap \{N \le n\} \in \mathcal{D}_n\}$ represents the information available up to the sampling stage N.

Lemma 3.1. For any positives values of α_0 and β_0,

$$E\{\Theta|\mathcal{D}_N\} = \Theta_N \; w.p.1 \; (P)$$

and

$$E\{(\Theta - \Theta_N)^2|\mathcal{D}_N\} = \frac{\Theta_N^2}{N+\alpha_0-2} \; w.p.1 \; (P).$$

Proof. For any $D \in \mathfrak{F}$,

$$\int_D E\{\Theta|\mathcal{D}_N\}\, dP = \sum_{n=1}^{\infty} \int_{D\cap\{N=n\}} E\{\Theta|\mathcal{D}_n\}\, dP = \sum_{n=1}^{\infty} \int_{D\cap\{N=n\}} \Theta_n dP = \int_D \Theta_N dP,$$

since $\Theta_n = E\{\Theta|\mathcal{D}_n\}$ for $n \ge 1$. Likewise, for any $D \in \mathfrak{F}$,

$$\int_D E\{(\Theta - \Theta_N)^2|\mathcal{D}_N\} dP = \sum_{n=1}^{\infty} \int_{D\cap\{N=n\}} E\{(\Theta - \Theta_n)^2|\mathcal{D}_n\} dP$$

$$= \sum_{n=1}^{\infty} \int_{D\cap\{N=n\}} \frac{\Theta_n^2}{\alpha_0+n-2} dP = \int_D \frac{\Theta_N^2}{N+\alpha_0-2}\, dP,$$

by using (6), to complete the proof.

To analyze the Bayes regret, observe first that $\bar{r}_a(\alpha_0, \beta_0)$ can be rewritten as

$$\bar{r}_a(\alpha_0, \beta_0) = a^2 E[(\Theta_N - \Theta)^2] + E[N] - 2aE[\Theta] \quad (7)$$

for $a > 0$. Next, condition on $\mathcal{D}_N$ in the first and third expectations in (8) and apply Lemma 3.1 to obtain

$$\bar{r}_a(\alpha_0, \beta_0) = aE\left[\frac{\Theta_N^2}{N+\alpha_0-2}\right] + E[N] - 2\sqrt{a}\mathrm{E}[\Theta_N]$$

$$= E\left[\frac{a}{\alpha_0+N-2}\Theta_N^2 + N + \alpha_0 - 2 - 2\sqrt{a}\Theta_N\right] + 2 - \alpha_0$$

$$= E\left[\frac{a}{N+\alpha_0-2}\left(\frac{N+\alpha_0-2}{\sqrt{a}} - \Theta_N\right)^2\right] + 2 - \alpha_0. \quad (8)$$

Theorem 3.1. If $\alpha_0 > 2$, then

$$\bar{r}_a(\alpha_0, \beta_0) \le \frac{\alpha_0^3}{\beta_0\sqrt{a}} + 2 - \alpha_0$$

for $a > 0$.

Proof. Since $N > \sqrt{a}\Theta_N$, by definition of N, it follows that

$$\frac{N + \alpha_0 - 2}{\sqrt{a}} - \Theta_N = \frac{\alpha_0 - 2}{\sqrt{a}} + \frac{N}{\sqrt{a}} - \Theta_N > \frac{\alpha_0 - 2}{\sqrt{a}} > 0$$

if $\alpha_0 > 2$. Next,

$$0 < \frac{N + \alpha_0 - 2}{\sqrt{a}} - \Theta_N = \frac{N - 1}{\sqrt{a}} + \frac{\alpha_0 - 1}{\sqrt{a}} - \Theta_N \le \Theta_{N-1} - \Theta_N + \frac{\alpha_0 - 1}{\sqrt{a}}$$

$$\le \frac{\Theta_N}{N + \alpha_0 - 2} + \frac{\alpha_0 - 1}{\sqrt{a}} \le \frac{1}{\sqrt{a}} + \frac{\alpha_0 - 1}{\sqrt{a}} = \frac{\alpha_0}{\sqrt{a}} \qquad (9)$$

if $\alpha_0 > 2$, since $N - 1 \le a\Theta_{N-1}$, by definition of N and

$$\Theta_{N-1} \le \Theta_N \frac{N + \alpha_0 - 1}{N + \alpha_0 - 2} = \Theta_N + \frac{\Theta_N}{N + \alpha_0 - 2},$$

by (4). Thus, if $\alpha_0 > 2$, then

$$E\left[\frac{a}{N + \alpha_0 - 2}\left(\frac{N + \alpha_0 - 2}{\sqrt{a}} - \Theta_N\right)^2\right] \le E\left[\frac{\alpha_0^2}{N}\right] \le \frac{\alpha_0^2}{\sqrt{a}} E\left[\frac{1}{\Theta_N}\right]$$

by using (9) and the fact that $N > \sqrt{a}\Theta_N$. Since

$$\frac{1}{\Theta_N} = \frac{1}{E\{\Theta|\mathcal{D}_N\}} \le E\left\{\frac{1}{\Theta}\middle|\mathcal{D}_N\right\} \ w.p.1\ (P)$$

by Jensen's inequality, it follows that

$$E\left[\frac{a}{\alpha_0 + N - 2}\left(\frac{\alpha_0 + N - 2}{\sqrt{a}} - \Theta_N\right)^2\right] \le \frac{\alpha_0^2}{\sqrt{a}} E\left[\frac{1}{\Theta}\right] = \frac{\alpha_0^2}{\sqrt{a}} \int_0^\infty \frac{1}{\theta} \xi(\theta) d\theta = \frac{\alpha_0^3}{\beta_0\sqrt{a}} \qquad (10)$$

if $\alpha_0 > 2$. Now substitute (10) in (8) to complete the proof.

Corollary 3.1. If $\alpha_0 > 2$, then

$$\lim_{a\to\infty} \bar{r}_a(\alpha_0, \beta_0) = 2 - \alpha_0$$

for any value of $\beta_0 > 0$.

Proof. (8) implies that $\bar{r}_a(\alpha_0, \beta_0) \geq 2 - \alpha_0$ for $a > 0$. Also, if $\alpha_0 > 2$,

$$\bar{r}_a(\alpha_0, \beta_0) \leq \frac{\alpha_0^3}{\beta_0\sqrt{a}} + 2 - \alpha_0 \to 2 - \alpha_0 \text{ as } a \to \infty,$$

by Theorem 4.1. Thus,

$$2 - \alpha_0 \leq \liminf_{a\to\infty} \bar{r}_a(\alpha_0, \beta_0) \leq \limsup_{a\to\infty} \bar{r}_a(\alpha_0, \beta_0) \leq 2 - \alpha_0$$

if $\alpha_0 > 2$. The corollary follows.

If a is large and $\alpha_0 > 2$, then $\bar{r}_a(\alpha_0, \beta_0) < 0$, by Corollary 3.1. This means that the sequential procedure (N, Θ_N) performs better than the best fixed-sample-size procedure, in the sense that it has a smaller Bayes risk. Table 1 below shows that the upper bound r_a^*, say, for the Bayes regret in Theorem 3.1 gets very close to $2 - \alpha_0$ for very large values of a. Table 2, on the other hand, shows that when β_0 is very large and α_0 is not large, the upper bound r_a^* gets very close to $2 - \alpha_0$ for values of a starting at 30.

Table 1

a	30	50	100	500	1000	5000	10000	50000	100000	500000
α_0	3	3	3	3	3	3	3	3	3	3
β_0	5	5	5	5	5	5	5	5	5	5
r_a^*	-0.014	-0.236	-0.46	-0.758	-0.829	-0.924	-0.946	-0.976	-0.983	-0.992
$2 - \alpha_0$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

Table 2

a	30	50	100	500	2500	30	50	100	500	2500
α_0	3	3	3	3	3	5	5	5	5	5
β_0	500	500	500	500	500	500	500	500	500	500
r_a^*	-0.990	-0.992	-0.995	-0.998	-0.999	-2.954	-2.965	-2.975	-2.989	-2.995
$2 - \alpha_0$	-1	-1	-1	-1	-1	-3	-3	-3	-3	-3

4. Asymptotic results

Lemma 4.1. Let N be defined by (5).

(i) If $\alpha_0 > 1$, then $N + \alpha_0 - 1 > \sqrt{\beta_0} a^{\frac{1}{4}}$ w. p. 1 for any value of $a > 0$.

(ii) $\Theta_N = E\{\Theta | X_1, \dots, X_N\} \to \theta$ w. p. 1 as $a \to \infty$.

(iii) $\frac{N}{\sqrt{a}} \to \theta$ w. p. 1 as $a \to \infty$.

Proof. If $\alpha_0 > 1$, then

$$N + \alpha_0 - 1 > N > \sqrt{a}\Theta_N = \sqrt{a}\frac{\beta_0 + N\bar{Y}_N}{N + \alpha_0 - 1} > \frac{\beta_0\sqrt{a}}{N + \alpha_0 - 1}$$

for any value of $a > 0$, by using the definition of N and (4) with $n = N$. Thus,

$$(N + \alpha_0 - 1)^2 > \beta_0\sqrt{a} \Longrightarrow N + \alpha_0 - 1 > \sqrt{\beta_0} a^{\frac{1}{4}}$$

for $a > 0$. For (ii), observe that $E\{\Theta | X_1, \dots, X_n\}$, $n \geq 1$, is a sequence of martingales such that $\Theta_n = E\{\Theta | X_1, \dots, X_n\} \longrightarrow \Theta$ w. p. 1 as $n \to \infty$, by the Martingale Convergence Theorem. This implies that $\Theta_N \longrightarrow \Theta$ w. p. 1 as $a \to \infty$, since $N \to \infty$ w. p. 1 as $a \to \infty$, by (i). For (iii), use the fact that

$$\Theta_N \leq \frac{N}{\sqrt{a}} \leq \Theta_N + \frac{m-1}{\sqrt{a}}$$

for any value of $a > 0$, by definition of N to obtain that

$$\liminf_{a\to\infty} \Theta_N \leq \liminf_{a\to\infty} \frac{N}{\sqrt{a}} \leq \limsup_{a\to\infty} \frac{N}{\sqrt{a}} \leq \limsup_{a\to\infty} \left(\Theta_N + \frac{m-1}{\sqrt{a}}\right);$$

so that

$$\Theta \leq \liminf_{a\to\infty} \frac{N}{\sqrt{a}} \leq \limsup_{a\to\infty} \frac{N}{\sqrt{a}} \leq \theta \quad w.p.1$$

by (ii). Thus, $\frac{N}{\sqrt{a}} \to \theta$ w. p. 1 as $a \to \infty$.

The following proposition shows that $\frac{N}{\sqrt{a}} \to \theta$ as $a \to \infty$ in mean square or in L^2.

Proposition 4.1. Let N be defined by (6). Then,

$$\lim_{a\to\infty} E\left[\left(\frac{N}{\sqrt{a}} - \theta\right)^2\right] = 0.$$

Proof. Write

$$E\left[\left(\frac{N}{\sqrt{a}}-\Theta\right)^2\right]=E\left[\left(\frac{N}{\sqrt{a}}\right)^2\right]-2E\left[\frac{N}{\sqrt{a}}\Theta\right]+E[\Theta^2] \tag{11}$$

and observe that

$$E\left\{\frac{N}{\sqrt{a}}\Theta\middle|\mathcal{D}_N\right\}=\frac{N}{\sqrt{a}}E\{\Theta|\mathcal{D}_N\}=\frac{N}{\sqrt{a}}\Theta_N\to\Theta^2 as\ a\to\infty,$$

by Assertions (ii) and (iii) of Lemma 4.1. It follows that

$$\liminf_{a\to\infty}E\left[\frac{N}{\sqrt{a}}\Theta\right]=\liminf_{a\to\infty}E\left[\frac{N}{\sqrt{a}}\Theta_N\right]\geq E[\Theta^2]$$

by Fatou's Lemma. Moreover, $N\leq\sqrt{a}\Theta_N+m-1$, by definition of N. This implies that

$$\limsup_{a\to\infty}E\left[\frac{N}{\sqrt{a}}\Theta\right]\leq E\left[\Theta\limsup_{a\to\infty}\left(\Theta_N+\frac{m-1}{\sqrt{a}}\right)\right];$$

or

$$\limsup_{a\to\infty}E\left[\frac{N}{\sqrt{a}}\Theta\right]\leq E[\Theta^2]$$

by Fatou's Lemma and Assertion (ii) of Lemma 4.1. Thus,

$$\lim_{a\to\infty}E\left[\frac{N}{\sqrt{a}}\Theta\right]=E[\Theta^2]. \tag{12}$$

Next, use Fatou's Lemma and Assertion (ii) of Lemma 4.1 to obtain

$$\liminf_{a\to\infty}E\left[\left(\frac{N}{\sqrt{a}}\right)^2\right]\geq E[\Theta^2].$$

Also,

$$E\left[\left(\frac{N}{\sqrt{a}}\right)^2\right]\leq E[\Theta_N^2]+\frac{2(m-1)}{\sqrt{a}}E[\Theta]+\frac{(m-1)^2}{a}$$

since $N\leq\sqrt{a}\Theta_N+m-1$, by definition of N. It follows that

$$\limsup_{a\to\infty}E\left[\left(\frac{N}{\sqrt{a}}\right)^2\right]\leq E[\Theta^2]$$

by Fatou's Lemma and Assertion (ii) of Lemma 4.1. Therefore,

$$\lim_{a\to\infty} E\left[\left(\frac{N}{\sqrt{a}}\right)^2\right] = E[\Theta^2]. \tag{13}$$

Now take the limit as $a \to \infty$ in (11) and use (12) and (13) to obtain

$$\lim_{a\to\infty} E\left[\left(\frac{N}{\sqrt{a}} - \Theta\right)^2\right] = E[\Theta^2] - 2E[\Theta^2] + E[\Theta^2] = 0.$$

5. Conclusion

We proposed a Bayesian sequential procedure for estimating the parameter of a class of life distributions under the squared loss and using the Inverse Gamma distribution as the prior distribution. We obtained an upper bound for the Bayes regret that was used as a measure of the performance of the proposed procedure. We found that, asymptotically, the regret can be negative, which means that the proposed procedure performs better than the best fixed-sample-size procedure, asymptotically.

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