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α^e-Closed Set, α^e-Continuity and α-e-Almost Compactness For Crisp Subsets of a Fuzzy Topological Space

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Abstract

This paper is a continuation of [3]. In this paper we introduce a new type of crisp set viz., α^e-closed set which inherits α-e-almost compactness [3] of a fuzzy topological space. In the last section we introduce α^e-continuous function between two fuzzy topological spaces under which α-e-almost compactness for crisp subsets remains invariant.

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1. Introduction

After introducing fuzzy topology given by Chang [4], different types of closed sets are introduced in fuzzy set theory. But after introducing α-shading (where $0 < \alpha < 1$) by Gantner et al. [6] in 1978, new types of closed sets which are crisp subsets of a space X where the underlying structure is fuzzy topology are introduced and studied. Here we introduce a new type of crisp subset with the help of α-shading, viz., α^e-closed set. Using the idea of α-shading in [3] α-e-almost compactness for crisp set is introduced and studied.

2. Preliminaries

Throughout the paper by (X, τ) or simply by X, we mean a fuzzy topological space (fts, for short) in the sense of Chang [4]. A crisp set A in an fts X means an ordinary subset of the set X where the underlying structure of the set X being a fuzzy topology τ. A fuzzy set [8] A is a mapping from a nonempty set X into the closed interval $I = [0, 1]$ of the real line, i.e., $A \in I^X$. For a fuzzy set A, the fuzzy closure [4] and fuzzy interior [4] of A in X are denoted by clA

and $intA$ respectively. The support [8] of a fuzzy set A in X will be denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [7] in X with the singleton support $\{x\} \subseteq X$ and the value α $(0 < \alpha \leq 1)$ at x will be denoted by x_α. For a fuzzy set A , the complement [8] of A in X will be denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A and B in X, we write $A \leq B$ if $A(x) \leq B(x)$, for each $x \in X$ [8] while we write AqB if A is quasi-coincident (q-coincident, for short) with B [7], i.e., if there exists $x \in X$ such that $A(x) + B(x) > 1$; the negation of these statements are written as $A \not\leq B$ and $A \not{q} B$ respectively. A fuzzy set A is called fuzzy regular open [1] if $A = int(clA)$. A fuzzy set B is called a quasi-neighbourhood (q-nbd, for short) [7] of a fuzzy point x_t if there is a fuzzy open set U in X such that $x_t qU \leq B$. If, in addition, B is fuzzy open (resp., fuzzy regular open), then B is called a fuzzy open [7] (resp., fuzzy regular open [1]) q-nbd of x_t. A fuzzy point x_α is said to be a fuzzy δ-cluster point of a fuzzy set A in an fts X if every fuzzy regular open q-nbd U of x_α is q-coincident with A [5]. The union of all fuzzy δ-cluster points of A is called the fuzzy δ-closure of A and is denoted by δclA [5]. A fuzzy set A is fuzzy δ-closed iff $A = \delta clA$ [5]. The complement of a fuzzy δ-closed set in an fts X is called fuzzy δ-open [5]. A fuzzy set A is fuzzy δ-open iff $A = \delta intA$ [5].

3. Some Known Definitions and Results

Definition 3.1[2]. A fuzzy set A in an fts X is said to be fuzzy e-open if $A \leq cl(\delta intA) \bigvee int(\delta clA)$. The complement of a fuzzy e-open set is called fuzzy e-closed.

Definition 3.2[2]. The intersection of all fuzzy e-closed sets containing a fuzzy set A in an fts X is called fuzzy e-closure of A, to be denoted by $eclA$.

Result 3.3[2]. A fuzzy set A in an fts X is fuzzy e-closed iff $A = eclA$.

Definition 3.4[2]. The union of all fuzzy e-open sets contained in a fuzzy set A in X is called fuzzy e-interior of A, to be denoted by $eintA$.

Result 3.5[2]. A fuzzy set A is fuzzy e-open iff $A = eintA$.

Result 3.6[2]. (i) For any fuzzy set A in X, $x_t \in eclA \Leftrightarrow UqA$ for any fuzzy e-open set U in X with $x_t qU$.

(ii) for any two fuzzy sets U, V in X where V is fuzzy e-open set, $U \not{q} V \Rightarrow eclU \not{q} V$.

Definition 3.7. Let X be an fts and A, a crisp subset of X. A collection $\mathcal{U}$ of fuzzy sets in X is called an α-shading (where $0 < \alpha < 1$) of A if for each $x \in A$, there is some $U_x \in \mathcal{U}$ such that $U_x(x) > \alpha$ [6]. If, in addition, the members of $\mathcal{U}$ are fuzzy e-open sets, then $\mathcal{U}$ is called a fuzzy e-open [3] α-shading of A.

Definition 3.8[3]. Let X be an fts and A, a crisp subset of X. A is said to be α-e-almost compact if each fuzzy e-open α-shading $\mathcal{U}$ of A has a finite e-proximate α-subshading, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{eclU : U \in \mathcal{U}_0\}$ is again an α-shading of A. If, in particular $A = X$, we get the definition of α-e-almost compact space X.

4. α^e-Closed Set : Some Properties

Definition 4.1. Let (X, τ) be an fts and $A \subseteq X$. A point $x \in X$ is said to be an α^e-limit point of A if for every fuzzy e-open set U in X with $U(x) > \alpha$, there exists $y \in A \setminus \{x\}$ such that $(eclU)(y) > \alpha$. The set of all α^e-limit points of A will be denoted by $[A]_e^\alpha$.
The α^e-closure of A, to be denoted by α^e-clA, is defined by α^e-cl$A = A \bigcup [A]_e^\alpha$.

Definition 4.2. A crisp subset A of an fts X is said to be α^e-closed if it contains all its α^e-limit points. Any subset B of X is called α^e-open if $X \setminus B$ is α^e-closed.

Remark 4.3. It is clear from Definition 4.1 that for any set $A \subseteq X$, $A \subseteq \alpha^e$-clA and α^e-$clA = A$ if and only if $[A]_e^\alpha \subseteq A$. Again it follows from Definition 4.1 that A is α^e-closed if and only if α^e-$clA = A$. It is also clear that $A \subseteq B \subseteq X \Rightarrow [A]_e^\alpha \subseteq [B]_e^\alpha$.

Theorem 4.4. An α^e-closed subset A of an α-e-almost compact space X is α-e-almost compact.

Proof. Let $A(\subseteq X)$ be α^e-closed in an α-e-almost compact space X. Then for any $x \notin A$, there is a fuzzy e-open set U_x in X such that $U_x(x) > \alpha$, and $(eclU_x)(y) \leq \alpha$, for every $y \in A$. Consider the collection $\mathcal{U} = \{U_x : x \notin A\}$. For proving A to be α-e-almost compact, consider a fuzzy e-open α-shading $\mathcal{V}$ of A. Clearly $\mathcal{U} \bigcup \mathcal{V}$ is a fuzzy e-open α-shading of X. Since X is α-e-almost compact, there exists a finite subcollection $\{V_1, V_2, ..., V_n\}$ of $\mathcal{U} \bigcup \mathcal{V}$ such that for every $t \in X$, there exists $V_i(1 \leq i \leq n)$ such that $(eclV_i)(t) > \alpha$. For every member U_x of $\mathcal{U}$, $(eclU_x)(y) \leq \alpha$, for every $y \in A$. So if this subcollection contains any member of $\mathcal{U}$, we omit it and hence we get the result.
To achieve the converse of Theorem 4.4, we define the following.

Definition 4.5. An fts (X, τ) is said to be α-e-Urysohn if for any two distinct points x, y of X, there exist a fuzzy open set U and a fuzzy e-open set V in X with $U(x) > \alpha$, $V(y) > \alpha$ and $min((eclU)(z), (eclV(z)) \leq \alpha$, for each $z \in X$.

Theorem 4.6. An α-e-almost compact set in an α-e-Urysohn space X is α^e-closed.

Proof. Let A be an α-e-almost compact set and $x \in X \setminus A$. Then for each $y \in A$, $x \neq y$.

As X is α-e-Urysohn, there exist a fuzzy open set U_y and a fuzzy e-open set V_y in X such that $U_y(x) > \alpha, V_y(y) > \alpha$ and $min((eclU_y)(z), (eclV_y)(z)) \leq \alpha$, for all $z \in X$ ... (1).
Then $\mathcal{U} = \{V_y : y \in A\}$ is a fuzzy e-open α-shading of A and so by α-e-almost compactness of A, there exist finitely many points $y_1, y_2, ..., y_n$ of A such that $\mathcal{U}_0 = \{eclV_{y_1}, eclV_{y_2}, ..., eclV_{y_n}\}$ is again an α-shading of A. Now $U = U_{y_1} \bigcap ... \bigcap U_{y_n}$ being a fuzzy open set is a fuzzy e-open set in X such that $U(x) > \alpha$. In order to show that A to be α^e-closed, it now suffices to show that $(eclU)(y) \leq \alpha$, for each $y \in A$. In fact, if for some $z \in A$, we assume $(eclU)(z) > \alpha$, then as $z \in A$, we have $(eclV_{y_k})(z) > \alpha$, for some k $(1 \leq k \leq n)$. Also $(eclU_{y_k})(z) > \alpha$. Hence $min((eclU_{y_k})(z), (eclV_{y_k})(z)) > \alpha$, contradicting (1).

Corollary 4.7. In an α-e-almost compact, α-e-Urysohn space X, a subset A of X is α-e-almost compact if and only if it is α^e-closed.

Theorem 4.8. In an α-e-almost compact space X, every cover of X by α^e-open sets has a finite subcover.

Proof. Let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be a cover of X by α^e-open sets. Then for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since $X \setminus U_x$ is α^e-closed, there exists a fuzzy e-open set V_x in X such that $V_x(x) > \alpha$ and $(eclV_x)(y) \leq \alpha$, for each $y \in X \setminus U_x$ ... (1).
Then $\{V_x : x \in X\}$ forms a fuzzy e-open α-shading of the α-e-almost compact space X. Thus there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of X such that $\{eclV_{x_i} : i = 1, 2, ..., n\}$ is an α-shading of X ... (2).
We claim that $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$ is a finite subcover of $\mathcal{U}$. If not, then there exists $y \in X \setminus \bigcup\limits_{i=1}^{n} U_{x_i} = \bigcap\limits_{i=1}^{n}(X \setminus U_{x_i})$. Then by (1), $(eclV_{x_i})(y) \leq \alpha$, for $i = 1, 2, ..., n$ and so $(\bigcup\limits_{i=1}^{n} eclV_{x_i})(y) \leq \alpha$, contradicting (2).

Theorem 4.9. Let (X, τ) be an fts. If X is α-e-almost compact, then every collection of α^e-closed sets in X with finite intersection property has non-empty intersection.

Proof. Let $\mathcal{F} = \{F_i : i \in \Lambda\}$ be a collection of α^e-closed sets in an α-e-almost compact space X having finite intersection property. If possible, let $\bigcap\limits_{i \in \Lambda} F_i = \phi$. Then $X \setminus \bigcap\limits_{i \in \Lambda} F_i = \bigcup\limits_{i \in \Lambda}(X \setminus F_i) = X \Rightarrow \mathcal{U} = \{X \setminus F_i : i \in \Lambda\}$ is an α^e-open cover of X. Then by Theorem 4.8, there is a finite subset Λ_0 of Λ such that $\bigcup\limits_{i \in \Lambda_0} (X \setminus F_i) = X \Rightarrow \bigcap\limits_{i \in \Lambda_0} F_i = \phi$, a contradiction.

5. α^e-Continuity

Definition 5.1. Let X, Y be fts's. A function $f : X \to Y$ is said to be α^e-continuous if for each point $x \in X$ and each fuzzy e-open set V in Y with $V(f(x)) > \alpha$, there exists a fuzzy

e-open set U in X with $U(x) > \alpha$ such that $eclU \leq f^{-1}(eclV)$.

Theorem 5.2. If $f : X \to Y$ is α^e-continuous (where X, Y are, as usual, fts's), then the following are true :

(a) $f([A]_e^\alpha) \subseteq [f(A)]_e^\alpha$, for every $A \subseteq X$.

(b) $[f^{-1}(A)]_e^\alpha \subseteq f^{-1}([A]_e^\alpha)$, for every $A \subseteq Y$.

(c) For each α^e-closed set A in Y, $f^{-1}(A)$ is α^e-closed in X.

(d) For each α^e-open set A in Y, $f^{-1}(A)$ is α^e-open in X.

Proof (a). Let $x \in [A]_e^\alpha$ and U be any fuzzy e-open set in Y with $U(f(x)) > \alpha$. Then there is a fuzzy e-open set V in X with $V(x) > \alpha$ and $eclV \leq f^{-1}(eclU)$. Now $x \in [A]_e^\alpha$ and V is a fuzzy e-open set in X with $V(x) > \alpha \Rightarrow eclV(x_0) > \alpha$, for some $x_0 \in A \setminus \{x\} \Rightarrow \alpha < eclV(x_0) \leq (f^{-1}(eclU))(x_0) = (eclU)(f(x_0))$ where $f(x_0) \in f(A) \setminus \{f(x)\} \Rightarrow f(x) \in [f(A)]_e^\alpha$. Thus (a) follows.

(b) By (a), $f([f^{-1}(A)]_e^\alpha) \subseteq [ff^{-1}(A)]_e^\alpha \subseteq [A]_e^\alpha \Rightarrow [f^{-1}(A)]_e^\alpha \subseteq f^{-1}([A]_e^\alpha)$.

(c) We have $[A]_e^\alpha = A$. By (b), $[f^{-1}(A)]_e^\alpha \subseteq f^{-1}([A]_e^\alpha) = f^{-1}(A) \Rightarrow [f^{-1}(A)]_e^\alpha = f^{-1}(A) \Rightarrow f^{-1}(A)$ is α^e-closed set in X.

(d) Follows from (c).

Theorem 5.3. Let X, Y be fts's and $f : X \to Y$ be fuzzy α^e-continuous function. If $A(\subseteq X)$ is α-e-almost compact, then so is $f(A)$ in Y.

Proof. Let $\mathcal{V} = \{V_i : i \in \Lambda\}$ be a fuzzy e-open α-shading of $f(A)$, where A is α-e-almost compact set in X. For each $x \in A$, $f(x) \in f(A)$ and so there exists $V_x \in \mathcal{V}$ such that $V_x(f(x)) > \alpha$. As f is fuzzy α^e-continuous, there exists a fuzzy e-open set U_x in X such that $U_x(x) > \alpha$ and $f(eclU_x) \leq eclV_x$. Then $\{U_x : x \in A\}$ is a fuzzy e-open α-shading of A. By α-e-almost compactness of A, there are finitely many points $a_1, a_2, ..., a_n$ in A such that $\{eclU_{a_i} : i = 1, 2, ..., n\}$ is again an α-shading of A.

We claim that $\{eclV_{a_i} : i = 1, 2, ..., n\}$ is an α-shading of $f(A)$. In fact, $y \in f(A) \Rightarrow$ there exists $x \in A$ such that $y = f(x)$. Now there is an U_{a_j} (for some $j, 1 \leq j \leq n$) such that $(eclU_{a_j})(x) > \alpha$ and hence $(eclV_{a_j})(y) \geq f(eclU_{a_j})(y) \geq eclU_{a_j}(x) > \alpha$.

We now introduce a function under which α^e-closedness of a set remains invariant.

Definition 5.4. Let X, Y be fts's. A function $f : X \to Y$ is said to be fuzzy e-open if $f(A)$ is fuzzy e-open in Y whenever A is fuzzy e-open in X.

Remark 5.5. For a fuzzy e-open function $f : X \to Y$, for every fuzzy e-closed set A in X, $f(A)$ is fuzzy e-closed in Y.

Theorem 5.6. If $f : (X, \tau) \to (Y, \tau_1)$ is a bijective fuzzy e-open function, then the image of a α^e-closed set in (X, τ) is α^e-closed in (Y, τ_1).

Proof. Let A be an α^e-closed set in (X, τ) and let $y \in Y \setminus f(A)$. Then there exists a unique $z \in X$ such that $f(z) = y$. As $y \notin f(A)$, $z \notin A$. Now, A being α^e-closed in X, there exists a fuzzy e-open set V in X such that $V(z) > \alpha$ and $eclV(p) \leq \alpha$, for each $p \in A$ ... (1). As f is fuzzy e-open, $f(V)$ is a fuzzy e-open set in Y, and also $(f(V))(y) = V(z) > \alpha$. Let $t \in f(A)$. Then there is a unique $t_0 \in A$ such that $f(t_0) = t$. As f is bijective and fuzzy e-open, by Remark 5.5, $eclf(V) \leq f(eclV)$. Then $(eclf(V))(t) \leq f(eclV)(t) = eclV(t_0) \leq \alpha$, by (1). Thus y is not an α^e-limit point of $f(A)$. Hence the proof.

From Theorem 5.2 (c) and Theorem 5.6, it follows that

Corollary 5.7. Let $f : X \to Y$ be a fuzzy α^e-continuous, bijective and fuzzy e-open function. Then A is α^e-closed in Yif and only if $f^{-1}(A)$ is α^e-closed in X.

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