Abstract

This paper is a continuation of [3]. In this paper we introduce a new type of crisp set viz., \( \alpha^e \)-closed set which inherits \( \alpha^e \)-almost compactness [3] of a fuzzy topological space. In the last section we introduce \( \alpha^e \)-continuous function between two fuzzy topological spaces under which \( \alpha^e \)-almost compactness for crisp subsets remains invariant.

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1. Introduction

After introducing fuzzy topology given by Chang [4], different types of closed sets are introduced in fuzzy set theory. But after introducing \( \alpha \)-shading (where \( 0 < \alpha < 1 \)) by Gantner et al. [6] in 1978, new types of closed sets which are crisp subsets of a space \( X \) where the underlying structure is fuzzy topology are introduced and studied. Here we introduce a new type of crisp subset with the help of \( \alpha \)-shading, viz., \( \alpha^e \)-closed set. Using the idea of \( \alpha \)-shading in [3] \( \alpha^e \)-almost compactness for crisp set is introduced and studied.

2. Preliminaries

Throughout the paper by \((X, \tau)\) or simply by \(X\), we mean a fuzzy topological space (fts, for short) in the sense of Chang [4]. A crisp set \( A \) in an fts \( X \) means an ordinary subset of the set \( X \) where the underlying structure of the set \( X \) being a fuzzy topology \( \tau \). A fuzzy set [8] \( A \) is a mapping from a nonempty set \( X \) into the closed interval \( I = [0, 1] \) of the real line, i.e., \( A \in I^X \). For a fuzzy set \( A \), the fuzzy closure [4] and fuzzy interior [4] of \( A \) in \( X \) are denoted by \( cl A \)
and \( \text{int}A \) respectively. The support [8] of a fuzzy set \( A \) in \( X \) will be denoted by \( \text{supp}A \) and is defined by \( \text{supp}A = \{x \in X : A(x) \neq 0\} \). A fuzzy point [7] in \( X \) with the singleton support \( \{x\} \subseteq X \) and the value \( \alpha \) \((0 < \alpha \leq 1)\) at \( x \) will be denoted by \( x_\alpha \). For a fuzzy set \( A \), the complement [8] of \( A \) in \( X \) will be denoted by \( 1_X \setminus A \) and is defined by \((1_X \setminus A)(x) = 1 - A(x)\), for each \( x \in X \). For any two fuzzy sets \( U, V \) in \( X \) we write \( A \leq B \) if \( A(x) \leq B(x) \), for each \( x \in X \) [8] while we write \( AqB \) if \( A \) is quasi-coincident \((q\text{-coincident}, \text{for short})\) with \( B \) [7], i.e., if there exists \( x \in X \) such that \( A(x) + B(x) > 1 \); the negation of these statements are written as \( A \nsucceq B \) and \( A \notqB \) respectively. A fuzzy set \( A \) is called fuzzy regular open [1] if \( A = \text{int}(\deltaclA) \). A fuzzy set \( B \) is called a quasi-neighbourhood \((q\text{-nbd}, \text{for short})\) [7] of a fuzzy point \( x_t \) if there is a fuzzy open set \( U \) in \( X \) such that \( x_tqU \subseteq B \). If, in addition, \( B \) is fuzzy open \((\text{resp.}, \text{fuzzy regular open})\), then \( B \) is called a fuzzy open [7] \((\text{resp.}, \text{fuzzy regular open})\) [1] \( q\text{-nbd} \) of \( x_t \). A fuzzy point \( x_\alpha \) is said to be a fuzzy \( \delta \)-cluster point of a fuzzy set \( A \) in an fts \( X \) if every fuzzy regular open \( q\text{-nbd} \) \( U \) of \( x_\alpha \) is \( q\text{-coincident} \) with \( A \) [5]. The union of all fuzzy \( \delta \)-cluster points of \( A \) is called the fuzzy \( \delta \)-closure of \( A \) and is denoted by \( \deltaclA \) [5]. A fuzzy set \( A \) is fuzzy \( \delta \)-closed iff \( A = \deltaclA \) [5]. The complement of a fuzzy \( \delta \)-closed set in an fts \( X \) is called fuzzy \( \delta \)-open [5]. A fuzzy set \( A \) is fuzzy \( \delta \)-open iff \( A = \delta\text{int}A \) [5].

### 3. Some Known Definitions and Results

**Definition 3.1**[2]. A fuzzy set \( A \) in an fts \( X \) is said to be fuzzy \( e \)-open if \( A \leq cl(\delta\text{int}A) \lor \text{int}(\deltaclA) \). The complement of a fuzzy \( e \)-open set is called fuzzy \( e \)-closed.

**Definition 3.2**[2]. The intersection of all fuzzy \( e \)-closed sets containing a fuzzy set \( A \) in an fts \( X \) is called fuzzy \( e \)-closure of \( A \), to be denoted by \( eclA \).

**Result 3.3**[2]. A fuzzy set \( A \) in an fts \( X \) is fuzzy \( e \)-closed iff \( A = eclA \).

**Definition 3.4**[2]. The union of all fuzzy \( e \)-open sets contained in a fuzzy set \( A \) in \( X \) is called fuzzy \( e \)-interior of \( A \), to be denoted by \( e\text{int}A \).

**Result 3.5**[2]. A fuzzy set \( A \) is fuzzy \( e \)-open iff \( A = e\text{int}A \).

**Result 3.6**[2]. (i) For any fuzzy set \( A \) in \( X \), \( x_t \in eclA \iff UqA \) for any fuzzy \( e \)-open set \( U \) in \( X \) with \( x_tqU \).

(ii) For any two fuzzy sets \( U, V \) in \( X \) where \( V \) is fuzzy \( e \)-open set, \( U \notqV \Rightarrow eclU \notqV \).

**Definition 3.7.** Let \( X \) be an fts and \( A \), a crisp subset of \( X \). A collection \( \mathcal{U} \) of fuzzy sets in \( X \) is called an \( \alpha \)-shading \((\text{where } 0 < \alpha < 1)\) of \( A \) if for each \( x \in A \), there is some \( U_x \in \mathcal{U} \) such that \( U_x(x) > \alpha \) [6]. If, in addition, the members of \( \mathcal{U} \) are fuzzy \( e \)-open sets, then \( \mathcal{U} \) is...
called a fuzzy $e$-open [3] $\alpha$-shading of $A$.

**Definition 3.8**[3]. Let $X$ be an fts and $A$, a crisp subset of $X$. $A$ is said to be $\alpha$-$e$-almost compact if each fuzzy $e$-open $\alpha$-shading $U$ of $A$ has a finite $e$-proximate $\alpha$-subshading, i.e., there exists a finite subcollection $U_0$ of $U$ such that $\{eclU : U \in U_0\}$ is again an $\alpha$-shading of $A$. If, in particular $A = X$, we get the definition of $\alpha$-$e$-almost compact space $X$.

### 4. $\alpha^e$-Closed Set: Some Properties

**Definition 4.1.** Let $(X, \tau)$ be an fts and $A \subseteq X$. A point $x \in X$ is said to be an $\alpha^e$-limit point of $A$ if for every fuzzy $e$-open set $U$ in $X$ with $U(x) > \alpha$, there exists $y \in A \setminus \{x\}$ such that $(eclU)(y) > \alpha$. The set of all $\alpha^e$-limit points of $A$ will be denoted by $[A]^\alpha_e$.

The $\alpha^e$-closure of $A$, to be denoted by $\alpha^e-clA$, is defined by $\alpha^e-clA = A \cup [A]^\alpha_e$.

**Definition 4.2.** A crisp subset $A$ of an fts $X$ is said to be $\alpha^e$-closed if it contains all its $\alpha^e$-limit points. Any subset $B$ of $X$ is called $\alpha^e$-open if $X \setminus B$ is $\alpha^e$-closed.

**Remark 4.3.** It is clear from Definition 4.1 that for any set $A \subseteq X$, $A \subseteq \alpha^e-clA$ and $\alpha^e-clA = A$ if and only if $[A]^\alpha_e \subseteq A$. Again it follows from Definition 4.1 that $A$ is $\alpha^e$-closed if and only if $\alpha^e-clA = A$. It is also clear that $A \subseteq B \subseteq X \Rightarrow [A]^\alpha_e \subseteq [B]^\alpha_e$.

**Theorem 4.4.** An $\alpha^e$-closed subset $A$ of an $\alpha$-$e$-almost compact space $X$ is $\alpha$-$e$-almost compact.

**Proof.** Let $A(\subseteq X)$ be $\alpha^e$-closed in an $\alpha$-$e$-almost compact space $X$. Then for any $x \notin A$, there is a fuzzy $e$-open set $U_x$ in $X$ such that $U_x(x) > \alpha$, and $(eclU_x)(y) \leq \alpha$, for every $y \in A$. Consider the collection $U = \{U_x : x \notin A\}$. For proving $A$ to be $\alpha$-$e$-almost compact, consider a fuzzy $e$-open $\alpha$-shading $V$ of $A$. Clearly $U \cup V$ is a fuzzy $e$-open $\alpha$-shading of $X$. Since $X$ is $\alpha$-$e$-almost compact, there exists a finite subcollection $\{V_1, V_2, ..., V_n\}$ of $U \cup V$ such that for every $t \in X$, there exists $V_i(1 \leq i \leq n)$ such that $(eclV_i)(t) > \alpha$. For every member $U_x$ of $U$, $(eclU_x)(y) \leq \alpha$, for every $y \in A$. So if this subcollection contains any member of $U$, we omit it and hence we get the result.

To achieve the converse of Theorem 4.4, we define the following.

**Definition 4.5.** An fts $(X, \tau)$ is said to be $\alpha$-$e$-Urysohn if for any two distinct points $x, y$ of $X$, there exist a fuzzy open set $U$ and a fuzzy $e$-open set $V$ in $X$ with $U(x) > \alpha$, $V(y) > \alpha$ and $min((eclU)(z), (eclV)(z)) \leq \alpha$, for each $z \in X$.

**Theorem 4.6.** An $\alpha$-$e$-almost compact set in an $\alpha$-$e$-Urysohn space $X$ is $\alpha^e$-closed.

**Proof.** Let $A$ be an $\alpha$-$e$-almost compact set and $x \in X \setminus A$. Then for each $y \in A$, $x \neq y$. 

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As $X$ is $\alpha$-$e$-Urysohn, there exist a fuzzy open set $U_y$ and a fuzzy $e$-open set $V_y$ in $X$ such that $U_y(x) > \alpha, V_y(y) > \alpha$ and $\min((eclU_y)(z),(eclV_y)(z)) < \alpha$, for all $z \in X$ ... (1).

Then $\mathcal{U} = \{V_y : y \in A\}$ is a fuzzy $e$-open $\alpha$-shading of $A$ and so by $\alpha$-$e$-almost compactness of $A$, there exist finitely many points $y_1, y_2, ..., y_n$ of $A$ such that $\mathcal{U}_0 = \{eclV_{y_1}, eclV_{y_2}, ..., eclV_{y_n}\}$ is again an $\alpha$-shading of $A$. Now $U = U_{y_1} \cap ... \cap U_{y_n}$ being a fuzzy open set is a fuzzy $e$-open set in $X$ such that $U(x) > \alpha$. In order to show that $A$ to be $\alpha$-$e$-closed, it now suffices to show that $(eclU)(y) < \alpha$, for each $y \in A$. In fact, if for some $z \in A$, we assume $(eclU)(z) > \alpha$, then as $z \in A$, we have $(eclV_{y_k})(z) > \alpha$, for some $k (1 \leq k \leq n)$. Also $(eclU_{y_k})(z) > \alpha$. Hence $\min((eclU_{y_k})(z),(eclV_{y_k})(z)) > \alpha$, contradicting (1).

**Corollary 4.7.** In an $\alpha$-$e$-almost compact, $\alpha$-$e$-Urysohn space $X$, a subset $A$ of $X$ is $\alpha$-$e$-almost compact if and only if it is $\alpha e$-closed.

**Theorem 4.8.** In an $\alpha$-$e$-almost compact space $X$, every cover of $X$ by $\alpha e$-open sets has a finite subcover.

**Proof.** Let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be a cover of $X$ by $\alpha e$-open sets. Then for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since $X \setminus U_x$ is $\alpha e$-closed, there exists a fuzzy $e$-open set $V_x$ in $X$ such that $V_x(x) > \alpha$ and $(eclV_x)(y) < \alpha$, for each $y \in X \setminus U_x$ ... (1).

Then $\{V_x : x \in X\}$ forms a fuzzy $e$-open $\alpha$-shading of the $\alpha$-$e$-almost compact space $X$.

Thus there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of $X$ such that $\{eclV_{x_i} : i = 1, 2, ..., n\}$ is an $\alpha$-shading of $X$ ... (2).

We claim that $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$ is a finite subcover of $\mathcal{U}$. If not, then there exists $y \in X \setminus \bigcup_{i=1}^{n} U_{x_i} = \bigcap_{i=1}^{n} (X \setminus U_{x_i})$. Then by (1), $(eclV_{x_i})(y) < \alpha$, for $i = 1, 2, ..., n$ and so $(\bigcup_{i=1}^{n} eclV_{x_i})(y) < \alpha$, contradicting (2).

**Theorem 4.9.** Let $(X, \tau)$ be an fts. If $X$ is $\alpha$-$e$-almost compact, then every collection of $\alpha e$-closed sets in $X$ with finite intersection property has non-empty intersection.

**Proof.** Let $\mathcal{F} = \{F_i : i \in \Lambda\}$ be a collection of $\alpha e$-closed sets in an $\alpha$-$e$-almost compact space $X$ having finite intersection property. If possible, let $\bigcap_{i \in \Lambda} F_i = \emptyset$. Then $X \setminus \bigcap_{i \in \Lambda} F_i = \bigcup_{i \in \Lambda} (X \setminus F_i) = X \Rightarrow \mathcal{U} = \{X \setminus F_i : i \in \Lambda\}$ is an $\alpha e$-open cover of $X$. Then by Theorem 4.8, there is a finite subset $\Lambda_0$ of $\Lambda$ such that $\bigcup_{i \in \Lambda_0} (X \setminus F_i) = X \Rightarrow \bigcap_{i \in \Lambda_0} F_i = \emptyset$, a contradiction.

### 5. $\alpha e$-Continuity

**Definition 5.1.** Let $X, Y$ be fts’s. A function $f : X \to Y$ is said to be $\alpha e$-continuous if for each point $x \in X$ and each fuzzy $e$-open set $V$ in $Y$ with $V(f(x)) > \alpha$, there exists a fuzzy
We now introduce a function under which $\alpha$ exists that $(\alpha)$ follows.

**Theorem 5.2.** If $f : X \to Y$ is $\alpha$-continuous (where $X, Y$ are, as usual, fts’s), then the following are true:

(a) $f([A]_e^\alpha) \subseteq [f(A)]_e^\alpha$, for every $A \subseteq X$.

(b) $[f^{-1}(A)]_e^\alpha \subseteq f^{-1}([A]_e^\alpha)$, for every $A \subseteq Y$.

(c) For each $\alpha$-closed set $A$ in $Y$, $f^{-1}(A)$ is $\alpha$-closed in $X$.

(d) For each $\alpha$-open set $A$ in $Y$, $f^{-1}(A)$ is $\alpha$-open in $X$.

**Proof** (a). Let $x \in [A]_e^\alpha$ and $U$ be any fuzzy $e$-open set in $Y$ with $U(f(x)) > \alpha$. Then there is a fuzzy $e$-open set $V$ in $X$ with $V(x) > \alpha$ and $eclV \leq f^{-1}(eclU)$. Now $x \in [A]_e^\alpha$ and $V$ is a fuzzy $e$-open set in $X$ with $V(x) > \alpha \Rightarrow \ eclV(x_0) > \alpha$, for some $x_0 \in A \setminus \{x\} \Rightarrow \alpha < eclV(x_0) \leq (f^{-1}(eclU))(x_0) = (eclU)(f(x_0))$ where $f(x_0) \in f(A) \setminus \{f(x)\} \Rightarrow f(x) \in [f(A)]_e^\alpha$. Thus (a) follows.

(b) By (a), $f([f^{-1}(A)]_e^\alpha) \subseteq [ff^{-1}(A)]_e^\alpha \subseteq [A]_e^\alpha \Rightarrow [f^{-1}(A)]_e^\alpha \subseteq f^{-1}([A]_e^\alpha)$.

(c) We have $[A]_e^\alpha = A$. By (b), $[f^{-1}(A)]_e^\alpha \subseteq f^{-1}([A]_e^\alpha) = f^{-1}(A) \Rightarrow [f^{-1}(A)]_e^\alpha = f^{-1}(A) \Rightarrow f^{-1}(A)$ is $\alpha$-closed in $X$.

(d) Follows from (c).

**Theorem 5.3.** Let $X, Y$ be fts’s and $f : X \to Y$ be fuzzy $\alpha$-continuous function. If $A(\subseteq X)$ is $\alpha$-e-almost compact, then so is $f(A)$ in $Y$.

**Proof.** Let $\mathcal{V} = \{V_i : i \in \Lambda\}$ be a fuzzy $e$-open $\alpha$-shading of $f(A)$, where $A$ is $\alpha$-e-almost compact set in $X$. For each $x \in A$, $f(x) \in f(A)$ and so there exists $V_x \in \mathcal{V}$ such that $V_x(f(x)) > \alpha$. As $f$ is fuzzy $\alpha$-continuous, there exists a fuzzy $e$-open set $U_x$ in $X$ such that $U_x(x) > \alpha$ and $f(eclU_x) \leq eclV_x$. Then $\{U_x : x \in A\}$ is a fuzzy $e$-open $\alpha$-shading of $A$. By $\alpha$-e-almost compactness of $A$, there are finitely many points $a_1, a_2, \ldots, a_n$ in $A$ such that $\{eclU_{a_i} : i = 1, 2, \ldots, n\}$ is again an $\alpha$-shading of $A$.

We claim that $\{eclU_{a_i} : i = 1, 2, \ldots, n\}$ is an $\alpha$-shading of $f(A)$. In fact, $y \in f(A) \Rightarrow$ there exists $x \in A$ such that $y = f(x)$. Now there is an $U_{a_j}$ (for some $j, 1 \leq j \leq n$) such that $(eclU_{a_j})(x) > \alpha$ and hence $(eclU_{a_j})(y) \geq f(eclU_{a_j})(y) \geq eclU_{a_j}(x) > \alpha$.

We now introduce a function under which $\alpha$-closedness of a set remains invariant.

**Definition 5.4.** Let $X, Y$ be fts’s. A function $f : X \to Y$ is said to be fuzzy $e$-open if $f(A)$ is fuzzy $e$-open in $Y$ whenever $A$ is fuzzy $e$-open in $X$.

**Remark 5.5.** For a fuzzy $e$-open function $f : X \to Y$, for every fuzzy $e$-closed set $A$ in $X$, $f(A)$ is fuzzy $e$-closed in $Y$.

**Theorem 5.6.** If $f : (X, \tau) \to (Y, \tau_1)$ is a bijective fuzzy $e$-open function, then the image
of a $\alpha^e$-closed set in $(X, \tau)$ is $\alpha^e$-closed in $(Y, \tau_1)$.

**Proof.** Let $A$ be an $\alpha^e$-closed set in $(X, \tau)$ and let $y \in Y \setminus f(A)$. Then there exists a unique $z \in X$ such that $f(z) = y$. As $y \notin f(A)$, $z \notin A$. Now, $A$ being $\alpha^e$-closed in $X$, there exists a fuzzy $e$-open set $V$ in $X$ such that $V(z) > \alpha$ and $eclV(p) \leq \alpha$, for each $p \in A$ ... (1).

As $f$ is fuzzy $e$-open, $f(V)$ is a fuzzy $e$-open set in $Y$, and also $(f(V))(y) = V(z) > \alpha$. Let $t \in f(A)$. Then there is a unique $t_0 \in A$ such that $f(t_0) = t$. As $f$ is bijective and fuzzy $e$-open, by Remark 5.5, $ecl f(V) \leq f(ecl V)$. Then $(ecl f(V))(t) \leq f(ecl V)(t) = ecl V(t_0) \leq \alpha$, by (1). Thus $y$ is not an $\alpha^e$-limit point of $f(A)$. Hence the proof.

From Theorem 5.2 (c) and Theorem 5.6, it follows that

**Corollary 5.7.** Let $f : X \to Y$ be a fuzzy $\alpha^e$-continuous, bijective and fuzzy $e$-open function. Then $A$ is $\alpha^e$-closed in $Y$ if and only if $f^{-1}(A)$ is $\alpha^e$-closed in $X$.

**References**


[3] Bhattacharyya, Anjana; *$\alpha$-e-almost compact crisp subsets of a fuzzy topological space*, Journal of Mathematics and Statistical Science (Accepted for publication).


