fuzzy minimal structure was introduced by Alimohammady and Roohi [1] . In [8], Bhattacharyya introduced fuzzy upper (lower) M-continuous multifunctions between a set having m-structure and a set having fuzzy minimal structure. In this paper we introduce a fuzzy multifunction between a set having m-structure and a fuzzy topological space.

2. Preliminaries

Let Y be a non-empty set and $I = [0, 1]$. Then a fuzzy set [23] A in Y is a mapping from Y into I. The set of all fuzzy sets in Y is denoted by I^Y. For a fuzzy set A in Y, the support of A, denoted by $suppA$ [23] and is defined by $suppA = \{y \in Y : A(y) \neq 0\}$. A fuzzy point [21] with the singleton support $y \in Y$ and the value t $(0 < t \leq 1)$ at y will be denoted by y_t. 0_Y and 1_Y are the constant fuzzy sets taking respectively the constant values 0 and 1 on Y. The complement of a fuzzy set A in Y will be denoted by $1_Y \setminus A$ [23] and is defined by $(1_Y \setminus A)(y) = 1 - A(y)$, for all $y \in Y$. For two fuzzy sets A and B in Y, we write $A \leq B$ iff $A(y) \leq B(y)$, for each $y \in Y$, while we write AqB to mean A is quasi-coincident (q-coincident, for short) with B [21] if there exists $y \in Y$ such that $A(y) + B(y) > 1$; the negation of AqB is written as $A \not{q} B$. clA and $intA$ of a set A in X (respectively, a fuzzy set A [23] in Y) respectively stand for the closure and interior of A in X (respectively, fuzzy closure and fuzzy interior of A in Y). A fuzzy set A in Y is called fuzzy regular open [3] (resp., fuzzy semiopen [3], fuzzy β-open [4], fuzzy α-open [10], fuzzy preopen [16], fuzzy γ-open [9]) if $intclA = A$ (resp., $A \leq clintA$, $A \leq clintclA$, $A \leq intclintA$, $A \leq intclA$, $A \leq (cl(intA)) \bigvee (int(clA))$). The complement of a fuzzy semiopen (resp., fuzzy β-open, fuzzy α-open, fuzzy preopen, fuzzy γ-open) set is called fuzzy semiclosed [3] (resp., fuzzy β-closed [4], fuzzy α-closed [10], fuzzy preclosed [16], fuzzy γ-closed [9]). The intersection of all fuzzy semiclosed (resp., fuzzy β-closed, fuzzy α-closed, fuzzy preclosed, fuzzy γ-closed) sets containing a fuzzy set A in Y is called fuzzy semiclosure [3] (resp., fuzzy β-closure [4], fuzzy α-closure [10], fuzzy preclosure [16], fuzzy γ-closure [9]) of A and is denoted by $sclA$ (resp., βclA, αclA, $pclA$, γclA). A fuzzy set A in Y is called a fuzzy neighbourhood (nbd, for short) [21] of a fuzzy set B in Y if there is a fuzzy open set U in Y such that $B \leq U \leq A$. A fuzzy set B is called a quasi neighbourhood (q-nbd, for short) [21] of a fuzzy set A if there is a fuzzy open set U in Y such that $AqU \leq B$. If, in addition, B is fuzzy regular open, then B is called a fuzzy regular open q-nbd of A. A fuzzy point x_α is said to be a fuzzy δ-cluster point of a fuzzy set A in an fts Y if every fuzzy regular open q-nbd U of x_α is q-coincident with

A [12]. The union of all fuzzy δ-cluster points of a fuzzy set A is called the fuzzy δ-closure of A and is denoted by δclA [12]. A fuzzy set A in an fts Y is called fuzzy δ-preopen [5] if $A \leq int(\delta clA)$. The complement of a fuzzy δ-preopen set is called fuzzy δ-preclosed [5]. The intersection of all fuzzy δ-preclosed sets containing a fuzzy set A in an fts Y is called fuzzy δ-preclosure of A and is denoted by $\delta pclA$ [5]. A subset A of an ordinary topological space X is called γ-open [9] (formerly known as b-open [2]) if $A \subseteq (cl(intA)) \bigcup (int(clA))$.

3. Some Well Known Definitions, Lemmas and Theorems

In this section, we first recall some definitions, lemmas and theorems for ready references.

Definition 3.1 [19, 20]. A subfamily m of the power set $\mathcal{P}(X)$ of a non empty set X is called a minimal structure (m-structure, for short) on X if $\emptyset \in m$ and $X \in m$. (X, m) is called an m-space. The members of m are called m-open and the complement of an m-open set is called m-closed.

Definition 3.2 [13]. Let (X, m) be an m-space. For a subset A of X, the m-closure and m-interior of A are defined as follows :

$$mClA = \bigcap\{F : F \supseteq A, X \setminus F \in m\}$$

$$mIntA = \bigcup\{U : U \subseteq A, U \in m\}$$

. **Remark 3.3**. From Definition 3.1 and Definition 3.2, it is to be noted that $mIntA$ (resp., $mClA$) may not be m-open (resp., m-closed) in an m-space (X, m). But if we assume that m is closed under arbitrary union (this condition is known as Maki condition [13]), then immediately, we have that $mIntA$ is an element of m and hence $A \subseteq X$ is m-open if and only if $mIntA = A$ and m-closed if and only if $mClA = A$.

Lemma 3.4 [13]. Let (X, m) be an m-space. For two subsets A, B of X, the following properties hold :
(i) $mCl(X \setminus A) = X \setminus mIntA$, $mInt(X \setminus A) = X \setminus mClA$,
(ii) If $X \setminus A \in m$, then $mClA = A$ and if $A \in m$, then $mIntA = A$,

(iii) $mCl(\emptyset) = \emptyset$, $mInt(\emptyset) = \emptyset$, $mCl(X) = X$, $mInt(X) = X$,

(iv) If $A \subset B$, then $mCl(A) \subset mCl(B)$ and $mInt(A) \subset mInt(B)$,

(v) $A \subset mCl(A)$ and $mInt(A) \subset A$

(vi) $mCl(mClA) = mClA$ and $mInt(mIntA) = mIntA$.

Lemma 3.5 [19]. Let (X, m) be an m-space and A, a subset of X. Then $x \in mClA$ if and only if $U \bigcap A \neq \emptyset$ for every $U \in m$ containing x.

Definition 3.6 [22]. Let (X, m) be an m-space. A subset A of X is said to be

(i) m-regular if $A = mInt(mClA)$,

(ii) m-semiopen if $A \subseteq mCl(mIntA)$,

(iii) m-α-open if $A \subseteq mInt(mCl(mIntA))$,

(iv) m-preopen if $A \subseteq mInt(mClA)$.

The complement of the above mentioned sets are called their respective closed sets.

Definition 3.7 [22]. Let (X, m) be an m-space and $A \subseteq X$. The m-δ-closure and the m-δ-interior of the set A, are defined, respectively as :

$$m\delta clA = \{x \in X : A \bigcap mInt(mClU) \neq \phi, \ \textit{for all}\ U \in m, x \in U\}$$

$$m\delta intA = \bigcup \{W : W \subseteq A, W \ \textit{is}\ m - \textit{regular open set in}\ X\}$$

.

Definition 3.8 [22]. A subset A of an m-space (X, m) is called

(i) m-δ-open if $A = m\delta intA$,

(ii) m-δ-preopen if $A \subseteq mInt(m\delta clA)$.

The complement of the above mentioned sets are called their respective closed sets.

Definition 3.9 [22]. An m-space (X, m) is said to be m-extremally disconnected if the m-closure of all m-open sets of X is m-open.

Definition 3.10 [11]. Let A be a fuzzy set in an fts Y. A collection $\mathcal{U}$ of fuzzy sets in Y is called a fuzzy cover of A if $sup\{U(x) : U \in \mathcal{U}\} = 1$ for each $x \in suppA$. If, in addition, the members of $\mathcal{U}$ are fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of A. In particular, if

$A = 1_Y$, we get the definition of fuzzy cover (resp., fuzzy open cover) of the fts Y.

Definition 3.11 [11]. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in an fts Y is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \geq A$. Clearly, if $A = 1_Y$, in particular, then the requirements on $\mathcal{U}_0$ is $\bigcup \mathcal{U}_0 = 1_Y$.

Definition 3.12 [11]. An fts Y is said to be fuzzy compact if every fuzzy open cover of Y has a finite subcover.

Definition 3.13 [18]. Let (X, τ) and (Y, τ_Y) be respectively an ordinary topological space and an fts. We say that $F : X \to Y$ is a fuzzy multifunction if corresponding to each $x \in X$, $F(x)$ is a unique fuzzy set in Y.
Henceforth by $F : X \to Y$ we shall mean a fuzzy multifunction in the above sense.

Definition 3.14 [18, 14]. For a fuzzy multifunction $F : X \to Y$, the upper inverse F^+ and lower inverse F^- are defined as follows :
For any fuzzy set A in Y, $F^+(A) = \{x \in X : F(x) \leq A\}$ and $F^-(A) = \{x \in X : F(x)qA\}$.

There is a following relationship between the upper and the lower inverses of a fuzzy multifunction.

Theorem 3.15 [14]. For a fuzzy multifunction $F : X \to Y$, we have $F^-(1_Y \setminus A) = X \setminus F^+(A)$, for any fuzzy set A in Y.

Definition 3.16 [9]. A fuzzy multifunction $F : X \to Y$ is called fuzzy
(i) upper γ-continuous at a point $x \in X$ if for each fuzzy open set V in Y with $F(x) \leq V$, there exists a γ-open set U in X containing x such that $F(U) \leq V$,
(ii) lower γ-continuous at a point $x \in X$ if for each fuzzy open set V in Y with $F(x)qV$, there exists a γ-open set U in X containing x such that $F(u)qV$, for all $u \in U$,
(iii) upper (lower) γ-continuous if F has this property at each point of X.

Definition 3.17 [8]. A fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is said to be fuzzy
(i) upper m-continuous (resp., upper m-quasi continuous, upper m-precontinuous, upper

m-δ-precontinuous, upper m-α-continuous) if for each $x \in X$ and each fuzzy open set V of Y with $F(x) \leq V$, there exists an m-open (resp., m-semiopen, m-preopen, m-δ-preopen, m-α-open) set U of X containing x such that $F(U) \leq V$,
(ii) lower m-continuous (resp., lower m-quasi continuous, lower m-precontinuous, lower m-δ-precontinuous, lower m-α-continuous) if for each $x \in X$ and each fuzzy open set V of Y with $F(x)qV$, there exists an m-open (resp., m-semiopen, m-preopen, m-δ-preopen, m-α-open) set U of X containing x such that $F(u)qV$, for all $u \in U$.

Definition 3.18 [8]. A fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is said to be fuzzy
(i) upper m-irresolute (resp., upper m-preirresolute, upper m-δ-preirresolute, upper m-α-irresolute) if for each $x \in X$ and each fuzzy semiopen (resp., fuzzy preopen, fuzzy δ-preopen, fuzzy α-open) set V of Y with $F(x) \leq V$, there exists an m-semiopen (resp., m-preopen, m-δ-preopen, m-α-open) set U of X containing x such that $F(U) \leq V$,
(ii) lower m-irresolute (resp., lower m-preirresolute, lower m-δ-preirresolute, lower m-α-irresolute) if for each $x \in X$ and each fuzzy semiopen (resp., fuzzy preopen, fuzzy δ-preopen, fuzzy α-open) set V of Y with $F(x)qV$, there exists an m-semiopen (resp., m-preopen, m-δ-preopen, m-α-open) set U of X containing x such that $F(u)qV$, for all $u \in U$.

Theorem 3.19 [8]. A fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is said to be fuzzy
(i) upper m-continuous (resp., upper m-quasi continuous, upper m-precontinuous, upper m-δ-precontinuous, upper m-α-continuous) iff $F^+(G)$ is m-open (resp., m-semiopen, m-preopen, m-δ-preopen, m-α-open) set in X for every fuzzy open set G of Y.

Theorem 3.20 [8]. A fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is said to be fuzzy
(i) lower m-continuous (resp., lower m-quasi continuous, lower m-precontinuous, lower m-δ-precontinuous, lower m-α-continuous) iff $F^-(G)$ is m-open (resp., m-semiopen, m-preopen, m-δ-preopen, m-α-open) set in X for every fuzzy open set G of Y.

Theorem 3.21 [8]. A fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is said to be fuzzy
(i) upper m-irresolute (resp., upper m-preirresolute, upper m-δ-preirresolute, upper m-α-irresolute) iff $F^+(G)$ is m-semiopen (resp., m-preopen, m-δ-preopen, m-α-open) set in X for every fuzzy semiopen (resp., fuzzy preopen, fuzzy δ-preopen, fuzzy α-open) set G of Y.

Theorem 3.22 [8]. A fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is said to be fuzzy
(i) lower m-irresolute (resp., lower m-preirresolute, lower m-δ-preirresolute, lower m-α-irresolute) iff $F^{-}(G)$ is m-semiopen (resp., m-preopen, m-δ-preopen, m-α-open) set in X for every fuzzy semiopen (resp., fuzzy preopen, fuzzy δ-preopen, fuzzy α-open) set G of Y.

4. Fuzzy Upper (Lower) *m*-*γ*-Continuous Multifunction: Characterizations

In this section we first define m-γ-open set in an m-space. Afterwards, fuzzy upper and fuzzy lower m-γ-continuous multifunctions are introduced and studied.

Definition 4.1. A subset A in an m-space (X, m) is said to be m-γ-open if $A \subseteq (mCl(mIntA)) \bigcup (mInt(mClA))$.

The complement of an m-γ-open set in an m-space is called m-γ-closed. The union (intersection) of all m-γ-open (resp., m-γ-closed) sets contained in (resp., containing) a subset A in an m-space (X, m) is called m-γ-interior (m-γ-closure) of A, denoted by $m\gamma intA$ (resp., $m\gamma clA$). $m\gamma intA$ (resp., $m\gamma clA$) is not m-γ-open (resp., m-γ-closed), in general, but if m satisfies Maki condition, then $m\gamma intA = A$ (resp., $m\gamma clA = A$) if A is m-γ-open (resp., m-γ-closed).

The collection of all m-γ-open (resp., m-γ-closed) sets in an m-space (X, m) is denoted by $m\gamma O(X)$ (resp., $m\gamma C(X)$).

If we put $m = \tau$, we get the definition of γ-open set [9].

Definition 4.2. A subset A of an m-space (X, m) is called an m-γ-nbd of a point $x \in X$ if there exists an m-γ-open set U in X such that $x \in U \subseteq A$.

Result 4.3. Let (X, m) be an m-space and $A \subseteq X$. Then $x \in m\gamma clA$ iff $U \bigcap A \neq \phi$ for every m-γ-open set U containing x.

Proof. Let $x \in m\gamma clA$ and U be any m-γ-open set of X containing x. If possible, let $U \bigcap A = \phi$. Then $A \subseteq X \setminus U$ where $X \setminus U$ is m-γ-closed set of X and $x \notin X \setminus U$ and so by definition, $x \notin m\gamma clA$, a contradiction.

Conversely, let $U \bigcap A \neq \phi$, for every m-γ-open set U containing x. Let V be an m-γ-

closed set of X containing A. We have to show that $x \in V$. If possible, let $x \notin V$. Then $x \in X \setminus V$ which is m-γ-open set of X. By assumption, $(X \setminus V) \bigcap A \neq \phi \Rightarrow A \nsubseteq V$, a contradiction.

Remark 4.4. It is clear from definition that m-open, m-semiopen, m-preopen, m-α-open sets are m-γ-open, but not conversely follow from next examples. Also m-γ-open set and m-δ-preopen set are independent concepts follow from next examples.

Example 4.5. m-γ-open set $\not\Rightarrow$ m-open, m-semiopen, m-α-open set
Let $X = \{a, b, c\}$, $m = \{\phi, X\}$. Then (X, m) is an m-space. Now $\{a\}$ is clearly m-γ-open in X, but $\{a\} \notin m \Rightarrow \{a\}$ is not m-open in X. Again, $mCl(mInt(\{a\})) = \phi \Rightarrow \{a\}$ is not m-semiopen. Also, $mInt(mCl(mInt(\{a\}))) = \phi \Rightarrow \{a\}$ is not m-α-open in X.

Example 4.6. m-γ-open set $\not\Rightarrow$ m-preopen, m-δ-preopen
Let $X = \{a, b, c\}$, $m = \{\phi, X, \{a\}, \{b\}\}$. Then (X, m) is an m-space. Then $\{b, c\}$ is m-γ-open as $mCl(mInt(\{b, c\})) = mCl(\{b\}) = \{b, c\}$. But $\{b, c\}$ is not m-preopen as $\{b, c\} \nsubseteq mInt(mCl(\{b, c\})) = mInt(\{b, c\}) = \{b\}$.
Again $\{b, c\} \nsubseteq mInt(m\delta cl(\{b, c\})) = mInt(\{b, c\}) = \{b\} \Rightarrow \{b, c\}$ is not m-δ-preopen in X.

Example 4.7. m-δ-preopen set $\not\Rightarrow$ m-γ-open set
Let $X = \{a, b, c\}$, $m = \{\phi, X, \{b\}\}$. Then (X, m) is an m-space. Consider the set $\{a, c\}$. Now $mCl(mInt(\{a, c\})) = \phi$ and $mInt(mCl(\{a, c\})) = \phi \Rightarrow (mCl(mInt(\{a, c\}))) \bigcup (mInt(mCl(\{a, c\})))$ $= \phi \Rightarrow \{a, c\}$ is not m-γ-open in X. But $\{a, c\} \subset X = mIntX = mInt(m\delta cl(\{a, c\})) \Rightarrow$ $\{a, c\}$ is m-δ-preopen in X.

Note 4.8. Let (X, m) be an m-space where m satisfies Maki condition. If X is m-extremally disconnected, then m-γ-open set is m-preopen and m-δ-preopen.

Definition 4.9. A fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is called fuzzy
(i) upper m-γ-continuous at a point $x \in X$ if for each fuzzy open set V in Y with $F(x) \leq V$, there exists an m-γ-open set U in X containing x such that $F(U) \leq V$,
(ii) lower m-γ-continuous at a point $x \in X$ if for each fuzzy open set V in Y with $F(x)qV$, there exists a m-γ-open set U in X containing x such that $F(u)qV$, for all $u \in U$,

(iii) upper (lower) m-γ-continuous if F has this property at each point of X.

Theorem 4.10. For a fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ where m satisfies Maki condition, the following statements are equivalent :

(a) F is fuzzy upper m-γ-continuous,

(b) $F^+(V) \in m\gamma O(X)$, for any fuzzy open set V of Y,

(c) $F^-(V) \in m\gamma C(X)$, for any fuzzy closed set V of Y,

(d) $m\gamma cl(F^-(B)) \subseteq F^-(clB)$, for any $B \in I^Y$,

(e) for each point $x \in X$ and each fuzzy nbd V of $F(x)$, $F^+(V)$ is an m-$\gamma-$nbd of x,

(f) for each point $x \in X$ and each fuzzy nbd V of $F(x)$, there exists an m-γ-nbd U of x such that $F(U) \leq V$,

(g) $mCl(mInt(F^-(B))) \bigcap mInt(mCl(F^-(B))) \subseteq F^-(clB)$, for any $B \in I^Y$,

(h) $F^+(intB) \subseteq mInt(mCl(F^+(B))) \bigcup mCl(mInt(F^+(B)))$, for any $B \in I^Y$.

Proof. (a) $\Rightarrow$ (b) Let V be a fuzzy open set of Y and $x \in F^+(V)$. Then $F(x) \leq V$. By(a), there exists an m-γ-open set U containing x such that $F(U) \leq V$. Therefore, we obtain, $x \in U \subseteq (mCl(mIntU)) \bigcup (mInt(mClU)) \subseteq (mCl(mInt(F^+(V)))) \bigcup (mInt(mCl(F^+(V))))$ and so we have $F^+(V) \subseteq (mCl(mInt(F^+(V)))) \bigcup (mInt(mCl(F^+(V)))) \Rightarrow F^+(V) \in m\gamma O(X)$.

(b)$\Leftrightarrow$ (c) Follows from Theorem 3.14.

(c) $\Rightarrow$ (d) Let $B \in I^Y$. Then clB is fuzzy closed set in Y and so by (c), $F^-(clB) \in m\gamma C(X)$ and so $m\gamma cl(F^-(clB)) \subseteq F^-(clB) \Rightarrow m\gamma cl(F^-(B)) \subseteq m\gamma cl(F^-(clB)) \subseteq F^-(clB)$.

(d) $\Rightarrow$ (c) Let V be a fuzzy closed set of Y. Then $clV = V$ and so by (d), $m\gamma cl(F^-(V)) \subseteq F^-(clV) = F^-(V) \Rightarrow F^-(V) \in m\gamma C(X)$.

(b) $\Rightarrow$ (e) Let $x \in X$ and V be a fuzzy nbd of $F(x)$. Then there exists a fuzzy open set G of Y such that $F(x) \leq G \leq V \Rightarrow x \in F^+(G) \subseteq F^+(V)$. Since by (b), $F^+(G)$ is m-γ-open in X, $F^+(V)$ is an m-γ-nbd of x.

(e)$\Rightarrow$ (f) Let $x \in X$ and V be a fuzzy nbd of $F(x)$. Put $U = F^+(V)$. By (e), U is an m-γ-nbd of x and $F(U) \leq V$.

(f) $\Rightarrow$ (a) Let $x \in X$ and V be a fuzzy open set of Y with $F(x) \leq V$. Then V is a fuzzy nbd of $F(x)$. By (f), there exists an m-γ-nbd U of x such that $F(U) \leq V$. Therefore, there exists $W \in m\gamma O(X)$ such that $x \in W \subseteq U$ and so $F(W) \leq F(U) \leq V \Rightarrow F(W) \leq V$.

(c) $\Rightarrow$ (g) Let $B \in I^Y$. Then clB is fuzzy closed in Y and so by (c), $F^-(clB) \in m\gamma C(X) \Rightarrow (mInt(mCl(F^-(B)))) \bigcap (mCl(mInt(F^-(B)))) \subseteq (mInt(mCl(F^-(clB)))) \bigcap (mCl(mInt(F^-(clB)))) \subseteq F^-(clB)$.

(g) $\Rightarrow$ (h) Let $B \in I^Y$. Then $1_Y \backslash B \in I^Y$. By (g), $(mCl(mInt(F^-(1_Y \backslash B)))) \bigcap (mInt(mCl(F^-(1_Y \backslash B)))) \subseteq F^-(cl(1_Y \setminus B)) \Rightarrow (mCl(mInt(X \setminus F^+(B)))) \bigcap (mInt(mCl(X \setminus F^+(B)))) \subseteq F^-(1_Y \setminus intB) \Rightarrow X \backslash ((mInt(mCl(F^+(B)))) \bigcup (mCl(mInt(F^+(B))))) = (X \backslash (mInt(mCl(F^+(B))))) \bigcap (X \backslash (mCl(mInt(F^+(B))))) \subseteq X \backslash F^+(intB) \Rightarrow F^+(intB) \subseteq (mInt(mCl(F^+(B)))) \bigcup (mCl(mInt(F^+(B))))$.

(h) $\Rightarrow$ (b) Let V be a fuzzy open set of Y. By (h), $F^+(intV) = F^+(V) \subseteq (mInt(mCl(F^+(V)))) \bigcup (mCl(mInt(F^+(V)))) \Rightarrow F^+(V) \in m\gamma O(X)$.

Theorem 4.11. For a fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ where m satisfies Maki condition, the following statements are equivalent :

(a) F is fuzzy lower m-γ-continuous,

(b) $F^-(V) \in m\gamma O(X)$, for any fuzzy open set V of Y,

(c) $F^+(V) \in m\gamma C(X)$, for any fuzzy closed set V of Y,

(d) $m\gamma cl(F^+(B)) \subseteq F^+(clB)$, for any $B \in I^Y$,

(e) $F(m\gamma clA) \leq cl(F(A))$, for any subset A of X,

(f) $mCl(mInt(F^+(B))) \bigcap mInt(mCl(F^+(B))) \subseteq F^+(clB)$, for any $B \in I^Y$,

(g) $F^-(intB) \subseteq mInt(mCl(F^-(B))) \bigcup mCl(mInt(F^-(B)))$, for any $B \in I^Y$,

(h) for each point $x \in X$ and each fuzzy q-nbd V of $F(x)$, $F^-(V)$ is an m-$\gamma-$nbd of x,

(i) for each point $x \in X$ and each fuzzy q-nbd V of $F(x)$, there exists an m-γ-nbd U of x such that $F(u)qV$, for all $u \in U$.

Proof (a) $\Rightarrow$ (b) Let $x \in X$ and V be a fuzzy open set of Y such that $x \in F^-(V)$. Then $F(x)qV$. By (a), there exists $U \in m\gamma O(X)$ containing x such that $F(u)qV$, for all $u \in U \Rightarrow U \subseteq F^-(V)$. Thus we have $x \in U \subseteq (mCl(mIntU)) \bigcup (mInt(mClU)) \subseteq (mCl(mInt(F^-(V)))) \bigcup (mInt(mCl(F^-(V)))) \Rightarrow F^-(V) \subseteq (mCl(mInt(F^-(V)))) \bigcup (mInt(mCl(F^-(V)))) \Rightarrow F^-(V) \in m\gamma O(X)$.

(b) $\Leftrightarrow$ (c) Follows from Theorem 3.14.

(c) $\Rightarrow$ (d) Let $B \in I^Y$. Then clB is fuzzy closed set of Y. By (c), $F^+(clB) \in m\gamma C(X) \Rightarrow m\gamma cl(F^+(B)) \subseteq m\gamma cl(F^+(clB)) \subseteq F^+(clB)$.

(d) $\Rightarrow$ (c) Let V be a fuzzy closed set of Y. Then $clV = V$. By (d) , $m\gamma cl(F^+(V)) = m\gamma cl(F^+(clV)) \subseteq F^+(clV) = F^+(V) \Rightarrow F^+(V) \in m\gamma C(X)$.

(c) $\Rightarrow$ (e) Let A be a subset of X. Then $cl(F(A))$ is fuzzy closed set of Y. By (c), $F^+(cl(F(A))) \in m\gamma C(X) \Rightarrow m\gamma cl(F^+(cl(F(A)))) \subseteq F^+(cl(F(A))) \Rightarrow F(m\gamma cl(F^+(cl(F(A))))) \leq F(F^+(cl(F(A)))) \leq cl(F(A)) \Rightarrow cl(F(A)) \geq F(m\gamma cl(F^+(F(A)))) \geq F(m\gamma clA)$.

(e) $\Rightarrow$ (d) Let $B \in I^Y$. Then $F^+(B) \subseteq X$. By (e), $F(m\gamma cl(F^+(B))) \leq cl(F(F^+(B))) \leq$

$clB \Rightarrow m\gamma cl(F^+(B)) \subseteq F^+(clB)$.

(c) $\Rightarrow$ (f) Let $B \in I^Y$. Then clB is fuzzy closed set of Y. By (c), $F^+(clB) \in m\gamma C(X) \Rightarrow F^+(clB) \supseteq (mInt(mCl(F^+(clB)))) \bigcap (mCl(mInt(F^+(clB)))) \supseteq (mInt(mCl(F^+(B)))) \bigcap (mCl(mInt(F^+(B))))$.

(f) $\Rightarrow$ (g) Let $B \in I^Y$. Then $1_Y \setminus B \in I^Y$. By (f), $F^+(cl(1_Y \setminus B)) \supseteq (mCl(mInt(F^+(1_Y \setminus B)))) \bigcap (mInt(mCl(F^+(1_Y \setminus B)))) \Rightarrow F^+(1_Y \setminus intB) \supseteq (mCl(mInt(X \setminus F^-(B)))) \bigcap (mInt(mCl(X \setminus F^-(B)))) \Rightarrow X \setminus F^-(intB) \supseteq (X \setminus (mInt(mCl(F^-(B))))) \bigcap (X \setminus (mCl(mInt(F^-(B))))) = X \setminus ((mInt(mCl(F^-(B)))) \bigcup (mCl(mInt(F^-(B))))) \Rightarrow F^-(intB) \subseteq (mInt(mCl(F^-(B)))) \bigcup (mCl(mInt(F^-(B))))$.

(g) $\Rightarrow$ (b) Let V be a fuzzy open set of Y. Then $F^-(V) = F^-(intV) \subseteq mInt(mCl(F^-(V))) \bigcup mCl(mInt(F^-(V)))$ (by (g)) $\Rightarrow F^-(V) \in m\gamma O(X)$.

(b) $\Rightarrow$ (h) Let $x \in X$ and V be a fuzzy q-nbd of $F(x)$. Then there exists a fuzzy open set G of Y such that $F(x)qG \leq V$. Then $x \in F^-(G) \subseteq F^-(V)$. By (b), $F^-(G) \in m\gamma O(X)$ and so $F^-(V)$ is an m-γ-nbd of x.

(h) $\Rightarrow$ (i) Let $x \in X$ and V be a fuzzy q-nbd of $F(x)$. Put $U = F^-(V)$. By (h), U is an m-γ-nbd of x and $F(u)qV$, for all $u \in U$.

(i) $\Rightarrow$ (a) Let $x \in X$ and V be a fuzzy open set of Y such that $F(x)qV$. Then V is a fuzzy q-nbd of $F(x)$. By (i), there exists an m-γ-nbd U of x such that $F(u)qV$, for all $u \in U \Rightarrow U \subseteq F^-(V)$. Therefore, there exists $W \in m\gamma O(X)$ containing x such that $x \in W \subseteq U$ and so $W \subseteq F^-(V) \Rightarrow F(w)qV$, for all $w \in W$.

If we take $m = \tau$, we get fuzzy upper (lower) γ-continuous multifunction.

Definition 4.12. For a fuzzy multifunction $F : X \to Y$, fuzzy multifunction $\gamma clF : X \to Y$ [9], $\alpha clF : X \to Y$ [9], $\beta clF : X \to Y$ [9], $clF : X \to Y$ [6], $sclF : X \to Y$ [6], $pclF : X \to Y$ [9], $\delta pclF : X \to Y$ [7] are defined by $(\gamma clF)(x) = \gamma clF(x)$, $(\alpha clF)(x) = \alpha clF(x)$, $(\beta clF)(x) = \beta clF(x)$, $(clF)(x) = clF(x)$, $(sclF)(x) = sclF(x)$, $(pclF)(x) = pclF(x)$, $(\delta pclF)(x) = \delta pclF(x)$, for all $x \in X$.

Lemma 4.13 [9]. Let $F : X \to Y$ be a fuzzy multifunction. Then we have $(\gamma clF)^-(G) = F^-(G)$, $(\alpha clF)^-(G) = F^-(G)$, $(\beta clF)^-(G) = F^-(G)$, $(clF)^-(G) = F^-(G)$, $(sclF)^-(G) = F^-(G)$, $(pclF)^-(G) = F^-(G)$, $(\delta pclF)^-(G) = F^-(G)$, for each fuzzy open set G of Y.

Using Lemma 4.13, we can easily state the following theorem

Theorem 4.14. For a fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$, the following statements are equivalent :

(i) F is fuzzy lower m-γ-continuous,

(ii) γclF is fuzzy lower m-γ-continuous,

(iii) αclF is fuzzy lower m-γ-continuous,

(iv) βclF is fuzzy lower m-γ-continuous,

(v) $sclF$ is fuzzy lower m-γ-continuous,

(vi) clF is fuzzy lower m-γ-continuous,

(vii) $pclF$ is fuzzy lower m-γ-continuous,

(viii) $\delta pclF$ is fuzzy lower m-γ-continuous.

5. Mutual Relationship

In this section, the mutual relationship between fuzzy upper (lower) m-γ-continuous multifunction and fuzzy multifunctions in Section 3 are established.

Remark 5.1. Using Remark 4.4, we have from Theorem 3.19 and Theorem 3.20 that fuzzy upper (lower) m-continuous, fuzzy upper (lower) m-quasi continuous, fuzzy upper (lower) m-precontinuous, fuzzy upper (lower) m-α-continuous multifunctions are fuzzy upper (lower) m-γ-continuous multifunction. But the converses are not true, in general, as shown from the following examples.

Example 5.2. Fuzzy upper m-γ-continuity $\not\Rightarrow$ fuzzy upper m-continuity
Let $X = \{a, b, c\}$, $m = \{\phi, X\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.35, B(y) = 0.4$, for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy multifunction defined by $F(a) = A, F(b) = B, F(c) = C$ where $C(y) = 0.6$ for all $y \in Y$. Now $F^+(A) = \{x \in X : F(x) \leq A\} = \{a\} \notin m$ and so F is not fuzzy upper m-continuous multifunction. But $F^+(A) = \{a\} \Rightarrow int(cl(\{a\}) = X \Rightarrow F^+(A)$ is m-γ-open in X. Again $F^+(B) = \{a, b\} \Rightarrow int(cl(\{a, b\}) = X \Rightarrow F^+(B)$ is m-γ-open in $X \Rightarrow F$ is fuzzy upper m-γ-continuous multifunction.

Example 5.3. Fuzzy lower m-γ-continuity $\not\Rightarrow$ fuzzy lower m-continuity
Let $X = \{a, b, c\}$, $m = \{\phi, X\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.35, B(y) =$

0.5, for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy multifunction defined by $F(a) = A, F(b) = B, F(c) = C$ where $C(y) = 0.6$ for all $y \in Y$. Now $F^-(A) = \{x \in X : F(x)qA\} = \phi \in m \Rightarrow F^-(A) \in m\gamma O(X)$, $F^-(B) = \{x \in X : F(x)qB\} = \{c\}$. Now $int(cl(\{c\})) = X \Rightarrow F^-(B)$ is m-γ-open in $X \Rightarrow F$ is fuzzy lower m-γ-continuous multifunction. But $F^-(B) \notin m \Rightarrow F$ is not fuzzy lower m-continuous multifunction.

Example 5.4. Fuzzy upper m-γ-continuity $\not\Rightarrow$ fuzzy upper m-quasi continuity
Consider Example 5.2. Here $F^+(A) = \{a\} \not\subseteq mCl(mInt(\{a\})) = \phi \Rightarrow F$ is not fuzzy upper m-quasi continuous multifunction though it is fuzzy upper m-γ-continuous multifunction.

Example 5.5. Fuzzy lower m-γ-continuity $\not\Rightarrow$ fuzzy lower m-quasi continuity
Consider Example 5.3. Here $F^-(B) = \{c\} \not\subseteq mCl(mInt(\{c\})) = \phi \Rightarrow F^-(B)$ is not m-semiopen in $X \Rightarrow F$ is not fuzzy lower m-quasi continuous multifunction though it is fuzzy lower m-γ-continuous multifunction.

Example 5.6. Fuzzy upper m-γ-continuity $\not\Rightarrow$ fuzzy upper m-α-continuity
Consider Example 5.2. Here $F^+(A) = \{a\} \not\subseteq mInt(mCl(mInt(\{a\}))) = \phi \Rightarrow F^+(A)$ is not m-α-open in $X \Rightarrow F$ is not fuzzy upper m-α-continuous multifunction though it is fuzzy upper m-γ-continuous multifunction.

Example 5.7. Fuzzy lower m-γ-continuity $\not\Rightarrow$ fuzzy lower m-α-continuity
Consider Example 5.3. Here $F^-(B) = \{c\} \not\subseteq mInt(mCl(mInt(\{c\}))) = \phi \Rightarrow F^-(B)$ is not m-α-open $X \Rightarrow F$ is not fuzzy lower m-α-continuous multifunction though it is fuzzy lower m-γ-continuous multifunction.

Example 5.8. Fuzzy upper m-γ-continuity $\not\Rightarrow$ fuzzy upper m-precontinuity
Let $X = \{a, b, c\}$, $m = \{\phi, X, \{b\}, \{c\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.35, B(y) = 0.4$, for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy multifunction defined by $F(a) = F(c) = A, F(b) = B$. Now $F^+(A) = \{a, c\}$. Now $mInt(mCl(\{a, c\})) = mInt(\{a, c\}) = \{c\} \not\supseteq \{a, c\} \Rightarrow F^+(A)$ is not m-preopen in $X \Rightarrow F$ is not fuzzy upper m-precontinuous multifunction though it is fuzzy upper m-γ-continuous multifunction.

Example 5.9. Fuzzy lower m-γ-continuity $\not\Rightarrow$ fuzzy lower m-precontinuity
Let $X = \{a, b, c\}$, $m = \{\phi, X, \{a\}, \{c\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A\}$ where $A(y) = 0.7$, for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy multifunction defined by $F(a) = A, F(b) = B, F(c) = C$ where $B(y) = 0.4, C(y) = 0.01$ for all $y \in Y$. Now $F^-(A) = \{a, b\}$. Then $mCl(mInt(\{a, b\})) = mCl(\{a\}) = \{a, b\} \Rightarrow F^-(A) \in m\gamma O(X) \Rightarrow F$ is lower m-γ-continuous multifunction. But $mInt(mCl(\{a, b\})) = mInt(\{a, b\}) = \{a\} \not\supseteq \{a, b\} \Rightarrow F^-(A)$ is not m-preopen in $X \Rightarrow F$ is not fuzzy lower m-precontinuous multifunction.

Remark 5.10. Fuzzy upper (lower) m-γ-continuity and fuzzy upper (lower) m-δ-precontinuity are independent concepts follow from next examples.

Example 5.11. Fuzzy upper m-γ-continuity $\not\Rightarrow$ fuzzy upper m-δ-precontinuity
Consider Example 5.2. Here $F^+(A) = \{a\}$. Now $m\delta cl(\{a\}) = \{x \in X : \{a\} \bigcap mInt(mClU) \neq \phi, U \in m, x \in U\} = \{a\}$, $mInt(m\delta cl(\{a\})) = \phi \not\supseteq \{a\} \Rightarrow \{a\}$ is not m-δ-preopen in $X \Rightarrow F$ is not fuzzy upper m-δ-precontinuous multifunction though it is fuzzy upper m-γ-continuous multifunction.

Example 5.12. Fuzzy lower m-γ-continuity $\not\Rightarrow$ fuzzy lower m-δ-precontinuity
Consider Example 5.3. Here $F^-(B) = \{c\}$. Now $m\delta cl(\{c\}) = \{c\} \Rightarrow mInt(m\delta cl(\{c\})) = \phi \not\supseteq \{c\} \Rightarrow F$ is not fuzzy lower m-δ-precontinuous multifunction though it is fuzzy lower m-γ-continuous multifunction.

Example 5.13. Fuzzy upper m-δ-precontinuity $\not\Rightarrow$ fuzzy upper m-γ-continuity
Let $X = \{a, b, c\}$, $m = \{\phi, X, \{b\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A\}$ where $A(y) = 0.5$ for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = F(c) = A, F(b) = B$, where $B(y) = 0.6$ for all $y \in Y$. Now $F^+(A) = \{a, c\}$. Then $mInt(m\delta cl(\{a, c\})) = X \supset \{a, c\} \Rightarrow \{a, c\}$ is m-δ-preopen in $X \Rightarrow F$ is fuzzy upper m-δ-precontinuous multifunction. But $mInt(mCl(\{a, c\})) = mInt(\{a, c\}) = \phi \not\supseteq \{a, c\} \Rightarrow F^+(A)$ is not m-γ-open in $X \Rightarrow F$ is not fuzzy upper m-γ-continuous multifunction.

Example 5.14. Fuzzy lower m-δ-precontinuity $\not\Rightarrow$ fuzzy lower m-γ-continuity
Let $X = \{a, b, c\}$, $m = \{\phi, X, \{b\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A\}$ where $A(y) = 0.6$ for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = F(c) = A, F(b) = B$, where $B(y) = 0.3$ for all $y \in Y$. Now $F^-(A) = \{a, c\}$. Then $mInt(m\delta cl(\{a, c\})) = X \supset \{a, c\} \Rightarrow \{a, c\}$ is m-δ-preopen in $X \Rightarrow F$ is fuzzy lower m-δ-precontinuous multifunction. Now $mInt(mCl(\{a, c\})) = mInt(\{a, c\}) = \phi$, $mCl(mInt(\{a, c\})) = \phi \Rightarrow mInt(mCl(\{a, c\})) \bigcup mCl(mInt(\{a, c\})) = \phi \Rightarrow \{a, c\}$ is not m-γ-open in $X \Rightarrow F$ is not fuzzy lower m-γ-continuous multifunction.

Remark 5.15. It is clear from Theorem 3.21 and Theorem 3.22 that fuzzy upper (lower) m-irresolute, fuzzy upper (lower) m-preirresolute, fuzzy upper (lower) m-α-irresolute multifunctions are fuzzy upper (lower) m-γ-continuous multifunction. But the converses are not true, in general, follow from next examples. Also fuzzy upper (lower) m-γ-continuous multifunction and fuzzy upper (lower) m-δ-preirresolute multifunction are independent concepts follow from next examples.

Example 5.16. Fuzzy upper m-γ-continuous multifunction $\not\Rightarrow$ fuzzy upper m-irresolute multifunction
Consider Example 5.2. Here the fuzzy set A being fuzzy open in Y is fuzzy semiopen in Y. Now $F^+(A) = \{a\} \not\subseteq mCl(mInt(\{a\})) = \phi \Rightarrow F^+(A)$ is not m-semiopen in $X \Rightarrow F$ is not fuzzy upper m-irresolute multifunction though it is fuzzy upper m-γ-continuous multifunction.

Example 5.17. Fuzzy lower m-γ-continuous multifunction $\not\Rightarrow$ fuzzy lower m-irresolute multifunction
Consider Example 5.3. Here the fuzzy set B is fuzzy semiopen in Y. Now $F^-(B) = \{c\} \not\subseteq mCl(mInt(\{c\})) = \phi \Rightarrow F^-(B)$ is not m-semiopen in $X \Rightarrow F$ is not fuzzy lower m-irresolute multifunction though it is fuzzy lower m-γ-continuous multifunction.

Example 5.18. Fuzzy upper m-γ-continuous multifunction $\not\Rightarrow$ fuzzy upper m-preirresolute multifunction
Let $X = \{a, b, c\}$, $m = \{\phi, X, \{b\}, \{c\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.35, B(y) = 0.4$, for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively.

Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = F(c) = A, F(b) = B$. Now $F^+(A) = \{a, c\}$. Then $mCl(mInt(\{a, c\})) = mCl(\{c\}) = \{a, c\} \Rightarrow F^+(A) \subseteq (mCl(mInt(\{a, c\}))) \cup (mInt(mCl(\{a, c\}))) \Rightarrow F^+(A) \in m\gamma O(X)$, $F^+(B) = \{b\} \in m$ and so $F^+(B) \in m\gamma O(X) \Rightarrow F$ is fuzzy upper m-γ-continuous multifunction. Consider the fuzzy set D defined by $D(y) = 0.37$ for all $y \in Y$. Then D is fuzzy preopen in Y. Now $F^+(D) = \{a, c\}$. Now $mInt(mCl(\{a, c\})) = mInt(\{a, c\}) = \{c\} \not\supseteq \{a, c\} \Rightarrow F^+(D)$ is not m-preopen in $X \Rightarrow F$ is not fuzzy upper m-preirresolute.

Example 5.19. Fuzzy lower m-γ-continuous multifunction $\not\Rightarrow$ fuzzy lower m-preirresolute multifunction

Let $X = \{a, b, c\}$, $m = \{\phi, X, \{c\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.4, B(y) = 0.44$ for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = A, F(b) = B, F(c) = C$ where $C(y) = 0.39$ for all $y \in Y$. Here $F^-(A) = F^-(B) = \phi \in m \Rightarrow F$ is fuzzy lower m-γ-continuous multifunction. Consider the fuzzy set D defined by $D(y) = 0.61$ for all $y \in Y$. Then $int(clD) = 1_Y > D \Rightarrow D$ is fuzzy prepen in Y. Now $F^-(D) = \{a, b\}$. Then $mInt(mCl(\{a, b\})) = mInt(\{a, b\}) = \phi \not\supseteq \{a, b\} \Rightarrow F^-(D)$ is not m-preopen in $X \Rightarrow F$ is not fuzzy lower m-preirresolute multifunction.

Example 5.20. Fuzzy upper m-γ-continuous multifunction $\not\Rightarrow$ fuzzy upper m-α-irresolute multifunction

Consider Example 5.18. Here D is fuzzy α-open in Y. Now $F^+(D) = \{a, c\}$. Then $mInt(mCl(mInt(\{a, c\})) = mInt(mCl(\{c\})) = mInt(\{a, c\}) = \{c\} \not\supseteq \{a, c\} \Rightarrow F^+(D)$ is not m-α-open in $X \Rightarrow F$ is not fuzzy upper m-α-irresolute multifunction though it is fuzzy upper m-γ-continuous multifunction.

Example 5.21. Fuzzy lower m-γ-continuous multifunction $\not\Rightarrow$ fuzzy lower m-α-irresolute multifunction

Let $X = \{a, b, c\}$, $m = \{\phi, X, \{a\}, \{c\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A\}$ where $A(y) = 0.7$ for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = A, F(b) = B, F(c) = C$ where $B(y) = 0.2, C(y) = 0.01$ for all $y \in Y$. Now $F^-(A) = \{a\} \in m \Rightarrow F^-(A) \in m\gamma O(X) \Rightarrow F$ is fuzzy lower m-γ-continuous multifunction. Consider the fuzzy set D defined by $D(y) = 0.81$, for all $y \in Y$.

Then D is fuzzy α-open in Y. Now $F^-(D) = \{a, b\}$. Then $mInt(mCl(mInt(\{a, b\}))) = mInt(mCl(\{a\})) = mInt(\{a, b\}) = \{a\} \not\supseteq \{a, b\} \Rightarrow F^-(D)$ is not m-α-open in $X \Rightarrow F$ is not fuzzy lower m-α-irresolute multifunction.

Example 5.22. Fuzzy upper m-γ-continuous multifunction $\not\Rightarrow$ fuzzy upper m-δ-preirresolute multifunction

Consider Example 5.18. Here D is fuzzy δ-preopen in Y. Now $F^+(D) = \{a, c\}$. Then $mInt(m\delta cl(\{a, c\})) = mInt(\{a, c\}) = \{c\} \not\supseteq \{a, c\} \Rightarrow F^+(D)$ is not m-δ-preopen in $X \Rightarrow F$ is not fuzzy upper m-δ-preirresolute multifunction though it is fuzzy upper m-γ-continuous multifunction.

Example 5.23. Fuzzy lower m-γ-continuous multifunction $\not\Rightarrow$ fuzzy lower m-δ-preirresolute multifunction

Let $X = \{a, b, c\}$, $m = \{\phi, X, \{b\}, \{c\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.4, B(y) = 0.44$, for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = A, F(b) = B, F(c) = C$ where $C(y) = 0.29$ for all $y \in Y$. Then $F^-(A) = F^-(B) = \phi \in m \Rightarrow F$ is fuzzy lower m-γ-continuous multifunction. Now consider the fuzzy set D defined by $D(y) = 0.61$ for all $y \in Y$. Then D is fuzzy δ-preopen in Y. Now $F^-(D) = \{a, b\}$. $mInt(m\delta cl(\{a, b\})) = mInt(\{a, b\}) = \{b\} \not\supseteq \{a, b\} \Rightarrow F^-(D)$ is not m-δ-preopen in $X \Rightarrow F$ is not fuzzy lower m-δ-preirresolute multifunction.

Example 5.24. Fuzzy upper m-δ-preirresolute multifunction $\not\Rightarrow$ fuzzy upper m-γ-continuous multifunction

Let$X = \{a, b, c\}$, $m = \{\phi, X, \{c\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A\}$ where $A(y) = 0.4$ for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = F(b) = B, F(c) = D$ where $B(y) = 0.3, D(y) = 0.7$ for all $y \in Y$. Now the collection of all fuzzy δ-preopen sets in Y is $\{0_Y, 1_Y, U, V\}$ where $U \leq A, V > 1_Y \setminus A$. Then $F^+(U) = \phi$, if $U < B$, $F^+(U) = \{a, b\}$, if $B \leq U < D$, $F^+(U) = X$, if $U \geq D$. Then ϕ, X are obviously m-δ-preopen in X. Now $mInt(m\delta cl(\{a, b\})) = mIntX = X \supset \{a, b\} \Rightarrow \{a, b\}$ is m-δ-preopen in $X \Rightarrow F^+(U)$ is m-δ-preopen in X for every fuzzy δ-preopen set U of Y. But $mInt(mCl(\{a, b\})) = mInt(\{a, b\}) = \phi$, $mCl(mInt(\{a, b\})) = \phi \Rightarrow mInt(mCl(\{a, b\})) \bigcup mCl(mInt(\{a, b\})) = \phi \not\supseteq \{a, b\} \Rightarrow \{a, b\}$ is not m-γ-open in

$X \Rightarrow F$ is not fuzzy upper m-γ-continuous multifunction.

Example 5.25. Fuzzy lower m-δ-preirresolute multifunction $\not\Rightarrow$ fuzzy lower m-γ-continuous multifunction

Let $X = \{a, b, c\}$, $m = \{\phi, X, \{c\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A\}$ where $A(y) = 0.5$ for all $y \in Y$. Then (X, m) and (Y, τ_Y) are m-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = F(b) = B$, $F(c) = C$ where $B(y) = 0.51, C(y) = 0.3$ for all $y \in Y$. Any fuzzy set in Y is fuzzy δ-preopen in Y. Now $F^-(U) = \phi$, if $U \leq 1_Y \setminus B$, $F^-(U) = \{a, b\}$, if $1_Y \setminus B < U \leq 1_Y \setminus C$, $F^-(U) = X$, if $U > 1_Y \setminus C$. Then as in Example 5.24, $F^-(U)$ is m-δ-preopen in $X \Rightarrow F$ is fuzzy lower m-δ-preirresolute multifunction. But $\{a, b\}$ is not m-γ-open in X as shown in Example 5.24. So F is not fuzzy lower m-γ-continuous multifunction.

6. Fuzzy Upper (Lower) *m*-*γ*-Continuous Multifunction: More Characterizations and Applications

In this section fuzzy upper (lower) m-γ-continuous multifunction is characterized by fuzzy upper (lower) nbd [9] of a fuzzy set and also some applications of these fuzzy multifunctions have been shown.

Definition 6.1 [9]. A fuzzy set A in an fts Y is said to be a fuzzy lower (upper) nbd of a fuzzy set B of Y if there exists a fuzzy open set V of Y such that BqV (resp., $B \leq V$) and $V \not{q} (1_Y \setminus A)$.

Theorem 6.2. A fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is fuzzy upper m-γ-continuous on X iff for each point $x_0 \in X$ and each fuzzy upper nbd M of $F(x_0)$, $F^+(M)$ is an m-γ-nbd of x_0.

Proof. Let F be fuzzy upper m-γ-continuous multifunction on X. Then for any $x_0 \in X$ and for any fuzzy upper nbd M of $F(x_0)$, there exists a fuzzy open set V of Y such that $F(x_0) \leq V$ and $V \not{q} (1_Y \setminus M) \Rightarrow V \leq M$. Since F is fuzzy upper m-γ-continuous multifunction, there exists $U \in m\gamma O(X)$ containing x_0 such that $U \subseteq F^+(V) \Rightarrow F(U) \leq V \leq M \Rightarrow U \subseteq F^+(M)$. Therefore, $x_0 \in U \subseteq F^+(M) \Rightarrow F^+(M)$ is an m-γ-nbd of x_0.

Conversely, let for any $x_0 \in X$ and any fuzzy open set V of Y with $F(x_0) \leq V$, we have $V \not q (1_Y \setminus V)$. Therefore, V is a fuzzy upper nbd of $F(x_0)$. Then by hypothesis, $F^+(V)$ is an m-γ-nbd of x_0. Then there exists $U \in m\gamma O(X)$ containing x_0 such that $x_0 \in U \subseteq F^+(V) \Rightarrow F(U) \leq V \Rightarrow F$ is fuzzy upper m-γ-continuous multifunction.

Theorem 6.3. A fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is fuzzy lower m-γ-continuous on X iff for each point $x_0 \in X$ and each fuzzy lower nbd M of $F(x_0)$, $F^-(M)$ is an m-γ-nbd of x_0.

Proof. Let F be fuzzy lower m-γ-continuous multifunction on X. Then for any $x_0 \in X$ and for any fuzzy lower nbd M of $F(x_0)$, there exists a fuzzy open set V of Y such that $F(x_0)qV$ and $V \not q (1_Y \setminus M) \Rightarrow V \leq M$. Since F is fuzzy lower m-γ-continuous multifunction, there exists $U \in m\gamma O(X)$ containing x_0 such that $U \subseteq F^-(V) \subseteq F^-(M)$. Therefore, $x_0 \in U \subseteq F^-(M) \Rightarrow F^-(M)$ is an m-γ-nbd of x_0.

Conversely, let for any $x_0 \in X$ and any fuzzy open set V of Y with $F(x_0)qV$. Since $V \not q (1_Y \setminus V)$, V is a fuzzy lower nbd of $F(x_0)$. Then by hypothesis, $F^-(V)$ is an m-γ-nbd of x_0. Then there exists $U \in m\gamma O(X)$ containing x_0 such that $x_0 \in U \subseteq F^-(V) \Rightarrow F(u)qV$, for all $u \in U \Rightarrow F$ is fuzzy lower m-γ-continuous multifunction.

Definition 6.4. An m-space (X, m) is said to be m-γ-compact if for every covering of X by m-γ-open sets of X has a finite subcover.

Theorem 6.5. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy upper m-γ-continuous surjective multifunction and $F(x)$ be a fuzzy compact set of Y for each $x \in X$. If X is m-γ-compact space, then Y is fuzzy compact space.

Proof. Let $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of Y. Now for each $x \in X$, $F(x)$ is fuzzy compact in Y and so there is a finite subset Λ_x of Λ such that $F(x) \leq \bigcup\{A_\alpha : \alpha \in \Lambda_x\}$. Let $A_x = \bigcup\{A_\alpha : \alpha \in \Lambda_x\}$. Then $F(x) \leq A(x)$ where A_x is a fuzzy open set of Y. Since F is fuzzy upper m-γ-continuous multifunction, there exists $U_x \in m\gamma O(X)$ containing x such that $U_x \subseteq F^+(A_x)$. Then $\mathcal{U} = \{U_x : x \in X\}$ is a cover of X by m-γ-open sets of X. Since X is m-γ-compact, there exists finitely many points $x_1, x_2, ..., x_n$ of X such that $X = \bigcup_{i=1}^{n} U_{x_i}$. As F is surjrctive, $1_Y = F(X) = F(\bigcup_{i=1}^{n} U_{x_i}) = \bigcup_{i=1}^{n} F(U_{x_i}) \leq \bigcup_{i=1}^{n} A_{x_i} = \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_{x_i}} A_\alpha \Rightarrow Y$ is fuzzy compact space.

Definition 6.6 [15]. An fts (Y, τ_Y) is said to be FNC-space if every fuzzy regular open cover of Y has a finite subcover.

Remark 6.7. As every fuzzy regular open set is fuzzy open, we can set the following theorem easily.

Theorem 6.8. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy upper m-γ-continuous surjective multifunction and $F(x)$ be a fuzzy compact set of Y for each $x \in X$. If X is m-γ-compact space, then Y is FNC-space.

Theorem 6.9. Every m-γ-closed subset of an m-γ-compact space is m-γ-compact.

Proof. Let A be an m-γ-closed subset of an m-γ-compact space (X, m). Let $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ be a covering of A by m-γ-open sets of X. Then $(X \setminus A) \bigcup (\bigcup_{\alpha \in \Lambda} A_\alpha)$ is a covering of X by m-γ-open sets of X. As X is m-γ-compact, there exists a finite subset Λ_0 of Λ such that $(X \setminus A) \bigcup (\bigcup_{\alpha \in \Lambda_0} A_\alpha)$ covers X. Now discarding the set $X \setminus A$, we get the finite subcover $\{A_\alpha : \alpha \in \Lambda_0\}$ of A by m-γ-open sets of X. Hence A is m-γ-compact.

Definition 6.10 [14]. For a fuzzy multifunction $F : X \to Y$, the fuzzy graph multifunction $G_F : X \to X \times Y$ of F is defined as $G_F(x) =$ the fuzzy set $x_1 \times F(x)$ of $X \times Y$, where x_1 is the fuzzy set in X, whose value is 1 at $x \in X$ and 0 at other points of X. We shall write $\{x\} \times F(x)$ for $x_1 \times F(x)$.

Theorem 6.11. When X is product related to Y, a fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is fuzzy upper m-γ-continuous if its fuzzy graph multifunction $G_F : X \to X \times Y$ is fuzzy upper m-γ-continuous multifunction.

Proof. Let G_F be a fuzzy upper m-γ-continuous multifunction. Let $x \in X$ and V be a fuzzy open set of Y such that $F(x) \leq V$. Then $G_F(x) \leq X \times V$ and $X \times V$ is easily seen to be open in $X \times Y$. By hypothesis, there exists $U \in m\gamma O(X)$ containing x such that $G_F(U) \leq X \times V$. Now for any $z \in U$ and any $y \in Y$, $[F(z)](y) = [G_F(z)](z, y) \leq (X \times V)(z, y) = V(y)$, i.e., $[F(z)](y) \leq V(y)$, for all $y \in Y \Rightarrow F(z) \leq V$, for any $z \in U \Rightarrow F(U) \leq V \Rightarrow F$ is fuzzy upper m-γ-continuous multifunction.

Definition 6.12. The m-γ-frontier of a subset A of an m-space (X, m), denoted by $m\gamma Fr(A)$, is defined by $m\gamma Fr(A) = m\gamma clA \bigcap m\gamma cl(X \setminus A) = m\gamma clA \setminus m\gamma intA$.

Theorem 6.13. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy multifunction where m satisfies Maki condition. Let $A = \{x \in X : F$ is not fuzzy upper m-γ-continuous at $x\}$, $B = \bigcup\{m\gamma Fr(F^+(V)) : F(x) \leq V$ and V is fuzzy open in $Y\}$. Then $A = B$.

Proof. Let $x \in X$ be such that F is not fuzzy upper m-γ-continuous at x. Then there exists a fuzzy open set V of Y with $F(x) \leq V$ such that $U \nsubseteq F^+(V)$, for all $U \in m\gamma O(X)$ containing $x \Rightarrow U \bigcap (X \setminus F^+(V)) \neq \phi \Rightarrow x \in m\gamma cl(X \setminus F^+(V)) = X \setminus m\gamma int(F^+(V)) \Rightarrow x \notin m\gamma int(F^+(V))$. But $x \in F^+(V) \subseteq m\gamma cl(F^+(V))$. Therefore, $x \in m\gamma Fr(F^+(V))$.

Conversely, let $x \in X$ and V be a fuzzy open set of Y with $F(x) \leq V$ such that $x \in m\gamma Fr(F^+(V))$. If possible, let F be fuzzy upper m-γ-continuous at x. Then there exists $U \in m\gamma O(X)$ containing x such that $U \subseteq F^+(V)$. Then $x \in U = m\gamma intU \subseteq m\gamma int(F^+(V)) \Rightarrow x \in m\gamma int(F^+(V)) \Rightarrow x \notin m\gamma Fr(F^+(V))$, a contradiction and hence F is not fuzzy upper m-γ-continuous at x.

Theorem 6.14. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy multifunction where m satisfies Maki condition. Let $A = \{x \in X : F$ is not fuzzy lower m-γ-continuous at $x\}$, $B = \bigcup\{m\gamma Fr(F^-(V)) : F(x)qV$ and V is fuzzy open in $Y\}$. Then $A = B$.

Proof. Let $x \in X$ be such that F is not fuzzy lower m-γ-continuous at x. Then there exists a fuzzy open set V of Y with $F(x)qV$ such that $U \nsubseteq F^-(V)$, for all $U \in m\gamma O(X)$ containing $x \Rightarrow U \bigcap (X \setminus F^-(V)) \neq \phi \Rightarrow x \in m\gamma cl(X \setminus F^-(V)) = X \setminus m\gamma int(F^-(V)) \Rightarrow x \notin m\gamma int(F^-(V))$. But $x \in F^-(V) \subseteq m\gamma cl(F^-(V))$. Therefore, $x \in m\gamma Fr(F^-(V))$.

Conversely, let $x \in X$ and V be a fuzzy open set of Y with $F(x)qV$ such that $x \in m\gamma Fr(F^-(V))$. If possible, let F be fuzzy lower m-γ-continuous at x. Then there exists $U \in m\gamma O(X)$ containing x such that $U \subseteq F^-(V)$. Then $x \in U = m\gamma intU \subseteq m\gamma int(F^-(V)) \Rightarrow x \in m\gamma int(F^-(V)) \Rightarrow x \notin m\gamma Fr(F^-(V))$, a contradiction and hence F is not fuzzy lower m-γ-continuous at x.

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