$m$-$\gamma$-Continuous Multifunction in Fuzzy Setting

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Abstract

In this paper a new type of fuzzy multifunction is introduced between a set having minimal structure and a fuzzy topological space by introducing $m$-$\gamma$-open set in $m$-space. Several characterizations and properties of this fuzzy multifunction are studied here. Also the mutual relationships of this newly defined fuzzy multifunction with the fuzzy multifunctions defined in [8] are established here.

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1. Introduction

Fuzzy multifunction, a function between a topological space and a fuzzy topological space, is introduced by Papageorgiou [18]. He also defined fuzzy upper and lower inverses in [18] though fuzzy lower inverse was redefined by Mukherjee and Malakar [14] suitably. Throughout this paper the definition of fuzzy upper inverse given by Papageorgiou and the definition of fuzzy lower inverse given by Mukherjee and Malakar are used. Noiri and Popa [17] introduced minimal structure ($m$-structure, for short) on a non-empty set $X$ whereas
fuzzy minimal structure was introduced by Alimohammady and Roohi [1]. In [8], Bhat-
tacharyya introduced fuzzy upper (lower) M-continuous multifunctions between a set having
m-structure and a set having fuzzy minimal structure. In this paper we introduce a fuzzy
multifunction between a set having m-structure and a fuzzy topological space.

2. Preliminaries

Let \( Y \) be a non-empty set and \( I = [0, 1] \). Then a fuzzy set \([23]\) \( A \) in \( Y \) is a mapping from
\( Y \) into \( I \). The set of all fuzzy sets in \( Y \) is denoted by \( I^Y \). For a fuzzy set \( A \) in \( Y \), the
support of \( A \), denoted by \( \text{supp} A \) [23] and is defined by \( \text{supp} A = \{ y \in Y : A(y) \neq 0 \} \). A
t fuzzy point [21] with the singleton support \( y \in Y \) and the value \( t \) \( (0 < t \leq 1) \) at \( y \) will be
denoted by \( y_t \). \( 0_Y \) and \( 1_Y \) are the constant fuzzy sets taking respectively the constant values
0 and 1 on \( Y \). The complement of a fuzzy set \( A \) in \( Y \) will be denoted by \( 1_Y \setminus A \) [23] and is
defined by \( (1_Y \setminus A)(y) = 1 - A(y) \), for all \( y \in Y \). For two fuzzy sets \( A \) and \( B \) in \( Y \), we write
\( A \leq B \) iff \( A(y) \leq B(y) \), for each \( y \in Y \), while we write \( AqB \) to mean \( A \) is quasi-coincident
(q-coincident, for short) with \( B \) [21] if there exists \( y \in Y \) such that \( A(y) + B(y) > 1 \); the
negation of \( AqB \) is written as \( A \nq B \). \( \text{cl} A \) and \( \text{int} A \) of a set \( A \) in \( X \) (respectively, a fuzzy
set \( A \) [23] in \( Y \)) respectively stand for the closure and interior of \( A \) in \( X \) (respectively, fuzzy
closure and fuzzy interior of \( A \) in \( Y \)). A fuzzy set \( A \) in \( Y \) is called fuzzy regular open [3]
(resp., fuzzy semiopen [3], fuzzy \( \beta \)-open [4], fuzzy \( \alpha \)-open [10], fuzzy \( \gamma \)-open [9]) if
\( \text{int} \text{cl} A = A \) (resp., \( A \leq \text{cl} \text{int} A, A \leq \text{cl} \text{int} \text{cl} A, A \leq \text{int} \text{cl} A, A \leq (\text{cl}(\text{int} A)) \setminus (\text{int}(\text{cl} A))) \). The complement of a fuzzy semiopen (resp., fuzzy \( \beta \)-open,
fuzzy \( \alpha \)-open, fuzzy \( \gamma \)-open) set is called fuzzy semiclosed [3] (resp., fuzzy \( \beta \)-closed,
fuzzy \( \alpha \)-closed [10], fuzzy preclosed [16], fuzzy \( \gamma \)-closed [9]). The intersection of all fuzzy semiclosed (resp., fuzzy \( \beta \)-closed, fuzzy \( \alpha \)-closed, fuzzy preclosed, fuzzy \( \gamma \)-closed) sets containing a fuzzy set \( A \) in \( Y \) is called fuzzy semiclosure [3] (resp., fuzzy \( \beta \)-closure [4],
fuzzy \( \alpha \)-closure [10], fuzzy preclosure [16], fuzzy \( \gamma \)-closure [9]) of \( A \) and is denoted by \( \text{ scl} A \)
(resp., \( \beta \text{cl} A, \alpha \text{cl} A, \text{pcl} A, \gamma \text{cl} A \)). A fuzzy set \( A \) in \( Y \) is called a fuzzy neighbourhood (nbd, for
short) [21] of a fuzzy set \( B \) in \( Y \) if there is a fuzzy open set \( U \) in \( Y \) such that \( B \leq U \leq A \). A
fuzzy set \( B \) is called a quasi neighbourhood (q-nbd, for short) [21] of a fuzzy set \( A \) if there is a
fuzzy open set \( U \) in \( Y \) such that \( AqU \leq B \). If, in addition, \( B \) is fuzzy regular open, then \( B \) is
called a fuzzy regular open q-nbd of \( A \). A fuzzy point \( x_\alpha \) is said to be a fuzzy \( \delta \)-cluster point of
a fuzzy set \( A \) in an fts \( Y \) if every fuzzy regular open q-nbd \( U \) of \( x_\alpha \) is q-coincident with
The union of all fuzzy \( \delta \)-cluster points of a fuzzy set \( A \) is called the fuzzy \( \delta \)-closure of \( A \) and is denoted by \( \delta cl A \) \cite{12}. A fuzzy set \( A \) in an fts \( Y \) is called fuzzy \( \delta \)-preopen \cite{5} if \( A \subseteq \text{int}(\delta cl A) \). The complement of a fuzzy \( \delta \)-preopen set is called fuzzy \( \delta \)-preclosed \cite{5}. The intersection of all fuzzy \( \delta \)-preclosed sets containing a fuzzy set \( A \) in an fts \( Y \) is called fuzzy \( \delta \)-preclosure of \( A \) and is denoted by \( \delta pcl A \) \cite{5}. A subset \( A \) of an ordinary topological space \( X \) is called \( \gamma \)-open \cite{9,20} (formerly known as \( b \)-open \cite{2}) if \( A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}A) \).

3. Some Well Known Definitions, Lemmas and Theorems

In this section, we first recall some definitions, lemmas and theorems for ready references.

**Definition 3.1** [19, 20]. A subfamily \( m \) of the power set \( P(X) \) of a non empty set \( X \) is called a minimal structure (\( m \)-structure, for short) on \( X \) if \( \emptyset \in m \) and \( X \in m \). \((X, m)\) is called an \( m \)-space. The members of \( m \) are called \( m \)-open and the complement of an \( m \)-open set is called \( m \)-closed.

**Definition 3.2** [13]. Let \((X, m)\) be an \( m \)-space. For a subset \( A \) of \( X \), the \( m \)-closure and \( m \)-interior of \( A \) are defined as follows :

\[
\text{mCl} A = \bigcap \{ F : F \supseteq A, X \setminus F \in m \}
\]

\[
\text{mInt} A = \bigcup \{ U : U \subseteq A, U \in m \}
\]

**Remark 3.3.** From Definition 3.1 and Definition 3.2, it is to be noted that \( \text{mInt} A \) (resp., \( \text{mCl} A \)) may not be \( m \)-open (resp., \( m \)-closed) in an \( m \)-space \((X, m)\). But if we assume that \( m \) is closed under arbitrary union (this condition is known as Maki condition \cite{13}), then immediately, we have that \( \text{mInt} A \) is an element of \( m \) and hence \( A \subseteq X \) is \( m \)-open if and only if \( \text{mInt} A = A \) and \( m \)-closed if and only if \( \text{mCl} A = A \).

**Lemma 3.4** [13]. Let \((X, m)\) be an \( m \)-space. For two subsets \( A, B \) of \( X \), the following properties hold :

(i) \( \text{mCl}(X \setminus A) = X \setminus \text{mInt} A, \text{mInt}(X \setminus A) = X \setminus \text{mCl} A \),

(ii) If \( X \setminus A \in m \), then \( \text{mCl} A = A \) and if \( A \in m \), then \( \text{mInt} A = A \).
(iii) $m\text{Cl}(\emptyset) = \emptyset, m\text{Int}(\emptyset) = \emptyset, m\text{Cl}(X) = X, m\text{Int}(X) = X$,
(iv) If $A \subset B$, then $m\text{Cl}(A) \subset m\text{Cl}(B)$ and $m\text{Int}(A) \subset m\text{Int}(B)$,
(v) $A \subset m\text{Cl}(A)$ and $m\text{Int}(A) \subset A$
(vi) $m\text{Cl}(m\text{Cl}A) = m\text{Cl}A$ and $m\text{Int}(m\text{Int}A) = m\text{Int}A$.

**Lemma 3.5** [19]. Let $(X, m)$ be an $m$-space and $A$, a subset of $X$. Then $x \in m\text{Cl}A$ if and only if $U \cap A \neq \emptyset$ for every $U \in m$ containing $x$.

**Definition 3.6** [22]. Let $(X, m)$ be an $m$-space. A subset $A$ of $X$ is said to be
(i) $m$-regular if $A = m\text{Int}(m\text{Cl}A)$,
(ii) $m$-semiopen if $A \subseteq m\text{Cl}(m\text{Int}A)$,
(iii) $m$-$\alpha$-open if $A \subseteq m\text{Int}(m\text{Cl}(m\text{Int}A))$,
(iv) $m$-preopen if $A \subseteq m\text{Int}(m\text{Cl}A)$.

The complement of the above mentioned sets are called their respective closed sets.

**Definition 3.7** [22]. Let $(X, m)$ be an $m$-space and $A \subseteq X$. The $m$-$\delta$-closure and the $m$-$\delta$-interior of the set $A$, are defined, respectively as :

$m\delta\text{cl}A = \{ x \in X : A \cap m\text{Int}(m\text{Cl}U) \neq \emptyset, \text{ for all } U \in m, x \in U \}$

$m\delta\text{int}A = \bigcup \{ W : W \subseteq A, W \text{ is } m - \text{regular open set in } X \}$

**Definition 3.8** [22]. A subset $A$ of an $m$-space $(X, m)$ is called
(i) $m$-$\delta$-open if $A = m\delta\text{Int}A$,
(ii) $m$-$\delta$-preopen if $A \subseteq m\text{Int}(m\delta\text{Cl}A)$.

The complement of the above mentioned sets are called their respective closed sets.

**Definition 3.9** [22]. An $m$-space $(X, m)$ is said to be $m$-extremally disconnected if the $m$-closure of all $m$-open sets of $X$ is $m$-open.

**Definition 3.10** [11]. Let $A$ be a fuzzy set in an fts $Y$. A collection $\mathcal{U}$ of fuzzy sets in $Y$ is called a fuzzy cover of $A$ if $\sup \{ U(x) : U \in \mathcal{U} \} = 1$ for each $x \in \text{supp}A$. If, in addition, the members of $\mathcal{U}$ are fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of $A$. In particular, if
\( A = 1_Y \), we get the definition of fuzzy cover (resp., fuzzy open cover) of the fts \( Y \).

**Definition 3.11** [11]. A fuzzy cover \( \mathcal{U} \) of a fuzzy set \( A \) in an fts \( Y \) is said to have a finite subcover \( \mathcal{U}_0 \) if \( \mathcal{U}_0 \) is a finite subcollection of \( \mathcal{U} \) such that \( \bigcup \mathcal{U}_0 \geq A \). Clearly, if \( A = 1_Y \), in particular, then the requirements on \( \mathcal{U}_0 \) is \( \bigcup \mathcal{U}_0 = 1_Y \).

**Definition 3.12** [11]. An fts \( Y \) is said to be fuzzy compact if every fuzzy open cover of \( Y \) has a finite subcover.

**Definition 3.13** [18]. Let \((X, \tau)\) and \((Y, \tau_Y)\) be respectively an ordinary topological space and an fts. We say that \( F : X \to Y \) is a fuzzy multifunction if corresponding to each \( x \in X \), \( F(x) \) is a unique fuzzy set in \( Y \).

Henceforth by \( F : X \to Y \) we shall mean a fuzzy multifunction in the above sense.

**Definition 3.14** [18, 14]. For a fuzzy multifunction \( F : X \to Y \), the upper inverse \( F^+ \) and lower inverse \( F^- \) are defined as follows:
For any fuzzy set \( A \) in \( Y \), \( F^+(A) = \{ x \in X : F(x) \leq A \} \) and \( F^-(A) = \{ x \in X : F(x)qA \} \).

There is a following relationship between the upper and the lower inverses of a fuzzy multifunction.

**Theorem 3.15** [14]. For a fuzzy multifunction \( F : X \to Y \), we have \( F^- (1_Y \setminus A) = X \setminus F^+(A) \), for any fuzzy set \( A \) in \( Y \).

**Definition 3.16** [9]. A fuzzy multifunction \( F : X \to Y \) is called fuzzy
(i) upper \( \gamma \)-continuous at a point \( x \in X \) if for each fuzzy open set \( V \) in \( Y \) with \( F(x) \leq V \), there exists a \( \gamma \)-open set \( U \) in \( X \) containing \( x \) such that \( F(U) \leq V \),
(ii) lower \( \gamma \)-continuous at a point \( x \in X \) if for each fuzzy open set \( V \) in \( Y \) with \( F(x)qV \), there exists a \( \gamma \)-open set \( U \) in \( X \) containing \( x \) such that \( F(u)qV \), for all \( u \in U \),
(iii) upper (lower) \( \gamma \)-continuous if \( F \) has this property at each point of \( X \).

**Definition 3.17** [8]. A fuzzy multifunction \( F : (X, m) \to (Y, \tau_Y) \) is said to be fuzzy
(i) upper \( m \)-continuous (resp., upper \( m \)-quasi continuous, upper \( m \)-precontinuous, upper...
m-δ-precontinuous, upper m-α-continuous) if for each \( x \in X \) and each fuzzy open set \( V \) of \( Y \) with \( F(x) \leq V \), there exists an m-open (resp., m-semiopen, m-preopen, m-δ-preopen, m-α-open) set \( U \) of \( X \) containing \( x \) such that \( F(U) \leq V \),

(ii) lower m-continuous (resp., lower m-quasi continuous, lower m-precontinuous, lower m-δ-precontinuous, lower m-α-continuous) if for each \( x \in X \) and each fuzzy open set \( V \) of \( Y \) with \( F(x)qV \), there exists an m-open (resp., m-semiopen, m-preopen, m-δ-preopen, m-α-open) set \( U \) of \( X \) containing \( x \) such that \( F(u)qV \), for all \( u \in U \).

**Definition 3.18** [8]. A fuzzy multifunction \( F : (X, m) \to (Y, \tau_Y) \) is said to be fuzzy
(i) upper m-irresolute (resp., upper m-preirresolute, upper m-δ-preirresolute, upper m-α-irresolute) if for each \( x \in X \) and each fuzzy semiopen (resp., fuzzy preopen, fuzzy δ-preopen, fuzzy α-open) set \( V \) of \( Y \) with \( F(x) \leq V \), there exists an m-semiopen (resp., m-preopen, m-δ-preopen, m-α-open) set \( U \) of \( X \) containing \( x \) such that \( F(U) \leq V \),
(ii) lower m-irresolute (resp., lower m-preirresolute, lower m-δ-preirresolute, lower m-α-irresolute) if for each \( x \in X \) and each fuzzy semiopen (resp., fuzzy preopen, fuzzy δ-preopen, fuzzy α-open) set \( V \) of \( Y \) with \( F(x)qV \), there exists an m-semiopen (resp., m-preopen, m-δ-preopen, m-α-open) set \( U \) of \( X \) containing \( x \) such that \( F(u)qV \), for all \( u \in U \).

**Theorem 3.19** [8]. A fuzzy multifunction \( F : (X, m) \to (Y, \tau_Y) \) is said to be fuzzy
(i) upper m-continuous (resp., upper m-quasi continuous, upper m-precontinuous, upper m-δ-precontinuous, upper m-α-continuous) iff \( F^+(G) \) is m-open (resp., m-semiopen, m-preopen, m-δ-preopen, m-α-open) set in \( X \) for every fuzzy open set \( G \) of \( Y \).

**Theorem 3.20** [8]. A fuzzy multifunction \( F : (X, m) \to (Y, \tau_Y) \) is said to be fuzzy
(i) lower m-continuous (resp., lower m-quasi continuous, lower m-precontinuous, lower m-δ-precontinuous, lower m-α-continuous) iff \( F^-(G) \) is m-open (resp., m-semiopen, m-preopen, m-δ-preopen, m-α-open) set in \( X \) for every fuzzy open set \( G \) of \( Y \).

**Theorem 3.21** [8]. A fuzzy multifunction \( F : (X, m) \to (Y, \tau_Y) \) is said to be fuzzy
(i) upper m-irresolute (resp., upper m-preirresolute, upper m-δ-preirresolute, upper m-α-irresolute) iff \( F^+(G) \) is m-semiopen (resp., m-preopen, m-δ-preopen, m-α-open) set in \( X \) for every fuzzy semiopen (resp., fuzzy preopen, fuzzy δ-preopen, fuzzy α-open) set \( G \) of \( Y \).
Theorem 3.22 [8]. A fuzzy multifunction $F : (X, m) \rightarrow (Y, \tau_Y)$ is said to be fuzzy (i) lower $m$-irresolute (resp., lower $m$-preirresolute, lower $m$-$\delta$-preirresolute, lower $m$-$\alpha$-irresolute) iff $F^-(G)$ is $m$-semiopen (resp., $m$-preopen, $m$-$\delta$-preopen, $m$-$\alpha$-open) set in $X$ for every fuzzy semiopen (resp., fuzzy preopen, fuzzy $\delta$-preopen, fuzzy $\alpha$-open) set $G$ of $Y$.

4. Fuzzy Upper (Lower) $m$-$\gamma$-Continuous Multifunction: Characterizations

In this section we first define $m$-$\gamma$-open set in an $m$-space. Afterwards, fuzzy upper and fuzzy lower $m$-$\gamma$-continuous multifunctions are introduced and studied.

Definition 4.1. A subset $A$ in an $m$-space $(X, m)$ is said to be $m$-$\gamma$-open if $A \subseteq (m\text{Cl}(m\text{Int}A)) \cup (m\text{Int}(m\text{Cl}A))$.

The complement of an $m$-$\gamma$-open set in an $m$-space is called $m$-$\gamma$-closed. The union (intersection) of all $m$-$\gamma$-open (resp., $m$-$\gamma$-closed) sets contained in (resp., containing) a subset $A$ in an $m$-space $(X, m)$ is called $m$-$\gamma$-interior ($m$-$\gamma$-closure) of $A$, denoted by $m\gamma\text{int}A$ (resp., $m\gamma\text{cl}A$). $m\gamma\text{int}A$ (resp., $m\gamma\text{cl}A$) is not $m$-$\gamma$-open (resp., $m$-$\gamma$-closed), in general, but if $m$ satisfies Maki condition, then $m\gamma\text{int}A = A$ (resp., $m\gamma\text{cl}A = A$) if $A$ is $m$-$\gamma$-open (resp., $m$-$\gamma$-closed).

The collection of all $m$-$\gamma$-open (resp., $m$-$\gamma$-closed) sets in an $m$-space $(X, m)$ is denoted by $m\gamma O(X)$ (resp., $m\gamma C(X)$).

If we put $m = \tau$, we get the definition of $\gamma$-open set [9].

Definition 4.2. A subset $A$ of an $m$-space $(X, m)$ is called an $m$-$\gamma$-nbd of a point $x \in X$ if there exists an $m$-$\gamma$-open set $U$ in $X$ such that $x \in U \subseteq A$.

Result 4.3. Let $(X, m)$ be an $m$-space and $A \subseteq X$. Then $x \in m\gamma\text{cl}A$ iff $U \cap A \neq \phi$ for every $m$-$\gamma$-open set $U$ containing $x$.

Proof. Let $x \in m\gamma\text{cl}A$ and $U$ be any $m$-$\gamma$-open set of $X$ containing $x$. If possible, let $U \cap A = \phi$. Then $A \subseteq X \setminus U$ where $X \setminus U$ is $m$-$\gamma$-closed set of $X$ and $x \notin X \setminus U$ and so by definition, $x \notin m\gamma\text{cl}A$, a contradiction.

Conversely, let $U \cap A \neq \phi$, for every $m$-$\gamma$-open set $U$ containing $x$. Let $V$ be an $m$-$\gamma$-open
closed set of $X$ containing $A$. We have to show that $x \in V$. If possible, let $x \notin V$. Then $x \in X \setminus V$ which is $m$-$\gamma$-open set of $X$. By assumption, $(X \setminus V) \cap A \neq \emptyset \Rightarrow A \nsubseteq V$, a contradiction.

**Remark 4.4.** It is clear from definition that $m$-open, $m$-semiopen, $m$-preopen, $m$-$\alpha$-open sets are $m$-$\gamma$-open, but not conversely follow from next examples. Also $m$-$\gamma$-open set and $m$-$\delta$-preopen set are independent concepts follow from next examples.

**Example 4.5.** $m$-$\gamma$-open set $\not= m$-open, $m$-semiopen, $m$-$\alpha$-open set

Let $X = \{a,b,c\}$, $m = \{\phi,X\}$. Then $(X,m)$ is an $m$-space. Now $\{a\}$ is clearly $m$-$\gamma$-open in $X$, but $\{a\} \notin m \Rightarrow \{a\}$ is not $m$-open in $X$. Again, $mCl(mInt(\{a\})) = \emptyset \Rightarrow \{a\}$ is not $m$-semiopen. Also, $mInt(mCl(mInt(\{a\}))) = \emptyset \Rightarrow \{a\}$ is not $m$-$\alpha$-open in $X$.

**Example 4.6.** $m$-$\gamma$-open set $\not= m$-preopen, $m$-$\delta$-preopen

Let $X = \{a,b,c\}$, $m = \{\phi,X,\{a\},\{b\}\}$. Then $(X,m)$ is an $m$-space. Then $\{b,c\}$ is $m$-$\gamma$-open as $mCl(mInt(\{b,c\})) = mCl(\{b\}) = \{b\}$. But $\{b,c\}$ is not $m$-preopen as $\{b,c\} \nsubseteq mInt(mCl(\{b,c\})) = mInt(\{b\}) = \{b\}$.

Again $\{b,c\} \nsubseteq mInt(m\delta cl(\{b,c\})) = mInt(\{b\}) = \{b\} \Rightarrow \{b,c\}$ is not $m$-$\delta$-preopen in $X$.

**Example 4.7.** $m$-$\delta$-preopen set $\not= m$-$\gamma$-open set

Let $X = \{a,b,c\}$, $m = \{\phi,X,\{b\}\}$. Then $(X,m)$ is an $m$-space. Consider the set $\{a,c\}$. Now $mCl(mInt(\{a,c\})) = \phi$ and $mInt(mCl(\{a,c\})) = \phi \Rightarrow (mCl(mInt(\{a,c\})) \cup (mInt(mCl(\{a,c\})))) = \phi \Rightarrow \{a,c\}$ is not $m$-$\gamma$-open in $X$. But $\{a,c\} \subset X = mIntX = mInt(m\delta cl(\{a,c\})) \Rightarrow \{a,c\}$ is $m$-$\delta$-preopen in $X$.

**Note 4.8.** Let $(X,m)$ be an $m$-space where $m$ satisfies Maki condition. If $X$ is $m$-extremally disconnected, then $m$-$\gamma$-open set is $m$-preopen and $m$-$\delta$-preopen.

**Definition 4.9.** A fuzzy multifunction $F : (X,m) \rightarrow (Y,\tau_Y)$ is called fuzzy

(i) upper $m$-$\gamma$-continuous at a point $x \in X$ if for each fuzzy open set $V$ in $Y$ with $F(x) \leq V$, there exists an $m$-$\gamma$-open set $U$ in $X$ containing $x$ such that $F(U) \leq V$,

(ii) lower $m$-$\gamma$-continuous at a point $x \in X$ if for each fuzzy open set $V$ in $Y$ with $F(x)qV$, there exists a $m$-$\gamma$-open set $U$ in $X$ containing $x$ such that $F(u)qV$, for all $u \in U$,
(iii) upper (lower) \( m_{-\gamma}\)-continuous if \( F \) has this property at each point of \( X \).

**Theorem 4.10.** For a fuzzy multifunction \( F : (X, m) \rightarrow (Y, \tau_Y) \) where \( m \) satisfies Maki condition, the following statements are equivalent:

(a) \( F \) is fuzzy upper \( m_{-\gamma}\)-continuous,
(b) \( F^+(V) \in m_{\gamma}O(X) \), for any fuzzy open set \( V \) of \( Y \),
(c) \( F^-(V) \in m_{\gamma}C(X) \), for any fuzzy closed set \( V \) of \( Y \),
(d) \( m_{\gamma}cl(F^-(B)) \subseteq F^-(clB) \), for any \( B \in I^Y \),
(e) for each point \( x \in X \) and each fuzzy nbd \( V \) of \( F(x) \), \( F^+(V) \) is an \( m_{-\gamma}\)-nbd of \( x \),
(f) for each point \( x \in X \) and each fuzzy nbd \( V \) of \( F(x) \), there exists an \( m_{-\gamma}\)-nbd \( U \) of \( x \) such that \( F(U) \leq V \),

\[ mCl(mInt(F^-(B))) \cap mInt(mCl(F^-(B))) \subseteq F^-(clB), \text{ for any } B \in I^Y. \]

\[ F^+(intB) \subseteq mInt(mCl(F^+(B))) \cup mCl(mInt(F^+(B))), \text{ for any } B \in I^Y. \]

**Proof.** (a) \( \Rightarrow \) (b) Let \( V \) be a fuzzy open set of \( Y \) and \( x \in F^+(V) \). Then \( F(x) \leq V \). By (a), there exists an \( m_{-\gamma}\)-open set \( U \) containing \( x \) such that \( F(U) \leq V \). Therefore, we obtain, \( x \in U \subseteq (mCl(mIntU)) \cup (mInt(mClU)) \subseteq (mCl(mInt(F^+(V)))) \cup (mInt(mCl(F^+(V)))) \) and so we have \( F^+(V) \subseteq (mCl(mInt(F^+(V)))) \cup (mInt(mCl(F^+(V)))) \Rightarrow F^+(V) \in m_{\gamma}O(X) \).

(b) \( \Leftrightarrow \) (c) Follows from Theorem 3.14.

(c) \( \Rightarrow \) (d) Let \( B \in I^Y \). Then \( clB \) is fuzzy closed set in \( Y \) and so by (c), \( F^-(clB) \in m_{\gamma}C(X) \) and so \( m_{\gamma}cl(F^-(clB)) \subseteq F^-(clB) \Rightarrow m_{\gamma}cl(F^-(B)) \subseteq m_{\gamma}cl(F^-(clB)) \subseteq F^-(clB) \).

(d) \( \Rightarrow \) (c) Let \( V \) be a fuzzy closed set of \( Y \). Then \( clV = V \) and so by (d), \( m_{\gamma}cl(F^-(V)) \subseteq F^-(clV) = F^-(V) \Rightarrow F^-(V) \in m_{\gamma}C(X) \).

(b) \( \Rightarrow \) (e) Let \( x \in X \) and \( V \) be a fuzzy nbd of \( F(x) \). Then there exists a fuzzy open set \( G \) of \( Y \) such that \( F(x) \leq G \leq V \Rightarrow x \in F^+(G) \subseteq F^+(V) \). Since by (b), \( F^+(G) \) is \( m_{-\gamma}\)-open in \( X \); \( F^+(V) \) is an \( m_{-\gamma}\)-nbd of \( x \).

(e) \( \Rightarrow \) (f) Let \( x \in X \) and \( V \) be a fuzzy nbd of \( F(x) \). Put \( U = F^+(V) \). By (e), \( U \) is an \( m_{-\gamma}\)-nbd of \( x \) and \( F(U) \leq V \).

(f) \( \Rightarrow \) (a) Let \( x \in X \) and \( V \) be a fuzzy open set of \( Y \) with \( F(x) \leq V \). Then \( V \) is a fuzzy nbd of \( F(x) \). By (f), there exists an \( m_{-\gamma}\)-nbd \( U \) of \( x \) such that \( F(U) \leq V \). Therefore, there exists \( W \in m_{\gamma}O(X) \) such that \( x \in W \subseteq U \) and so \( F(W) \leq F(U) \leq V \Rightarrow F(W) \leq V \).

(c) \( \Rightarrow \) (g) Let \( B \in I^Y \). Then \( clB \) is fuzzy closed in \( Y \) and so by (c), \( F^-(clB) \in m_{\gamma}C(X) \Rightarrow (mInt(mCl(F^-(B)))) \cap (mCl(mInt(F^-(B)))) \subseteq (mInt(mCl(F^-(clB)))) \cap (mCl(mInt(F^-(clB)))) \subseteq F^-(clB) \).

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(g) ⇒ (h) Let \( B \in I^Y \). Then \( 1_Y \setminus B \in I^Y \). By (g), \((mCl(mInt(F^-(1_Y \setminus B)))) \cap (mInt(mCl(F^-(1_Y \setminus B)))) \subseteq F^-(cl(1_Y \setminus B)) \Rightarrow (mCl(mInt(X \setminus F^+(B)))) \cap (mInt(mCl(X \setminus F^+(B)))) \subseteq F^-(1_Y \setminus intB) \Rightarrow X \setminus (mInt(mCl(F^+(B)))) \cup (mCl(mInt(F^+(B)))) = (X \setminus (mInt(mCl(F^+(B)))) \cap (X \setminus (mCl(mInt(F^+(B))))))) \subseteq X \setminus F^+(intB) \Rightarrow F^+(intB) \subseteq (mInt(mCl(F^+(B)))) \cup (mCl(mInt(F^+(B))))(mCl(mInt(F^+(B))))). (h) ⇒ (b) Let \( V \) be a fuzzy open set of \( Y \). By (h), \( F^+(intV) = F^+(V) \subseteq (mInt(mCl(F^+(V)))) \cup (mCl(mInt(F^+(V)))) \Rightarrow F^+(V) \in m\gamma O(X) \).

**Theorem 4.11.** For a fuzzy multifunction \( F : (X, m) \to (Y, \tau_Y) \) where \( m \) satisfies Maki condition, the following statements are equivalent:

(a) \( F \) is fuzzy lower \( m-\gamma \)-continuous,
(b) \( F^+(V) \in m\gamma O(X) \), for any fuzzy open set \( V \) of \( Y \),
(c) \( F^+(V) \in m\gamma C(X) \), for any fuzzy closed set \( V \) of \( Y \),
(d) \( m\gamma cl(F^+(B)) \subseteq F^+(clB) \), for any \( B \in I^Y \),
(e) \( F(m\gamma clA) \leq cl(F(A)) \), for any subset \( A \) of \( X \),
(f) \( mCl(mInt(F^+(B))) \cap mInt(mCl(F^+(B))) \subseteq F^+(clB) \), for any \( B \in I^Y \),
(g) \( F^-(intB) \subseteq mInt(mCl(F^-(B))) \cup mCl(mInt(F^-(B))) \), for any \( B \in I^Y \),
(h) for each point \( x \in X \) and each fuzzy \( q \)-nbd \( V \) of \( F(x) \), \( F^-(V) \) is an \( m-\gamma \)-nbd of \( x \),
(i) for each point \( x \in X \) and each fuzzy \( q \)-nbd \( V \) of \( F(x) \), there exists an \( m-\gamma \)-nbd \( U \) of \( x \) such that \( F(u)qV \), for all \( u \in U \).

**Proof** (a) ⇒ (b) Let \( x \in X \) and \( V \) be a fuzzy open set of \( Y \) such that \( x \in F^-(V) \). Then \( F(x)qV \). By (a), there exists \( U \in m\gamma O(X) \) containing \( x \) such that \( F(u)qV \), for all \( u \in U \) ⇒ \( U \subseteq F^-(V) \). Thus we have \( x \in U \subseteq (mCl(mIntU)) \cup (mInt(mClU)) \subseteq (mCl(mInt(F^-(V)))) \cup (mInt(mCl(F^-(V)))) \Rightarrow F^-(V) \subseteq (mCl(mInt(F^-(V)))) \cup (mInt(mCl(F^-(V)))) \Rightarrow F^-(V) \in m\gamma O(X) \).

(b) ⇔ (c) Follows from Theorem 3.14.

(c) ⇒ (d) Let \( B \in I^Y \). Then \( clB \) is fuzzy closed set of \( Y \). By (c), \( F^+(clB) \in m\gamma C(X) \Rightarrow m\gamma cl(F^+(B)) \subseteq m\gamma cl(F^+(clB)) \subseteq F^+(clB) \).

(d) ⇒ (c) Let \( V \) be a fuzzy closed set of \( Y \). Then \( clV = V \). By (d), \( m\gamma cl(F^+(V)) = m\gamma cl(F^+(clV)) \subseteq F^+(clV) \subseteq F^+(V) \Rightarrow F^+(V) \subseteq m\gamma C(X) \).

(e) ⇒ (a) Let \( A \) be a subset of \( X \). Then \( cl(F(A)) \) is fuzzy closed set of \( Y \). By (c), \( F^+(cl(F(A))) \in m\gamma C(X) \Rightarrow m\gamma cl(F^+(cl(F(A)))) \subseteq F^+(cl(F(A))) \Rightarrow F(m\gamma cl(F^+(cl(F(A)))))) \leq F(F^+(cl(F(A)))) \leq cl(F(A)) \Rightarrow cl(F(A)) \subseteq F(m\gamma cl(F^+(cl(F(A)))))) \geq F(m\gamma clA) \).

(e) ⇒ (d) Let \( B \in I^Y \). Then \( F^+(B) \subseteq X \). By (e), \( F(m\gamma cl(F^+(B))) \leq cl(F(F^+(B))) \leq cl(F(F^+(V))) \subseteq F^+(cl(F(A)))) \Rightarrow cl(F(A)) \subseteq F(m\gamma cl(F^+(cl(F(A)))))) \geq F(m\gamma clA) \).

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\[ clB \Rightarrow m\gamma cl(F^+(B)) \subseteq F^+(clB). \]

(c) \Rightarrow (f) Let \( B \in \mathcal{I}^Y \). Then \( clB \) is fuzzy closed set of \( Y \). By (c), \( F^+(clB) \in m\gamma C(X) \Rightarrow F^+(clB) \supseteq (m\text{Int}(mCl(F^+(clB)))) \cap (m\text{Cl}(m\text{Int}(F^+(clB)))) \supseteq (m\text{Int}(mCl(F^+(B)))) \cap (mCl(m\text{Int}(F^+(B)))) \).

(f) \Rightarrow (g) Let \( B \in \mathcal{I}^Y \). Then \( 1_Y \setminus B \in \mathcal{I}^Y \). By (f), \( F^+(cl(1_Y \setminus B)) \supseteq (m\text{Cl}(m\text{Int}(F^+(1_Y \setminus B)))) \cap (m\text{Int}(m\text{Cl}(F^+(1_Y \setminus B)))) \Rightarrow F^+(1_Y \setminus \text{int}B) \supseteq (m\text{Cl}(m\text{Int}(X \setminus F^-(B)))) \cap (m\text{Int}(m\text{Cl}(X \setminus F^-(B)))) \Rightarrow X \setminus F^-(\text{int}B) \supseteq (X \setminus (m\text{Int}(m\text{Cl}(F^-(B)))) \cap (X \setminus (m\text{Int}(m\text{Cl}(F^-(B))))) = X \setminus ((m\text{Int}(m\text{Cl}(F^-(B)))) \cup (m\text{Cl}(m\text{Int}(F^-(B)))) \cup (m\text{Cl}(m\text{Int}(F^-(B))))).

(g) \Rightarrow (b) Let \( V \) be a fuzzy open set of \( Y \). Then \( F^-(V) = F^-(\text{int}V) \subseteq m\text{Int}(m\text{Cl}(F^-(V))) \cup m\text{Cl}(m\text{Int}(F^-(V))) \) by (g) \( \Rightarrow F^-(V) \in m\gamma O(X) \).

(b) \Rightarrow (h) Let \( x \in X \) and \( V \) be a fuzzy q-nbd of \( F(x) \). Then there exists a fuzzy open set \( G \) of \( Y \) such that \( F(x)qG \subseteq V \). Then \( x \in F^-(G) \subseteq F^-(V) \). By (b), \( F^-(G) \in m\gamma O(X) \) and so \( F^-(V) \) is an \( m\gamma \)-nbd of \( x \).

(h) \Rightarrow (i) Let \( x \in X \) and \( V \) be a fuzzy q-nbd of \( F(x) \). Put \( U = F^-(V) \). By (h), \( U \) is an \( m\gamma \)-nbd of \( x \) and \( F(u)qV \), for all \( u \in U \).

(i) \Rightarrow (a) Let \( x \in X \) and \( V \) be a fuzzy open set of \( Y \) such that \( F(x)qV \). Then \( V \) is a fuzzy q-nbd of \( F(x) \). By (i), there exists an \( m\gamma \)-nbd \( U \) of \( x \) such that \( F(u)qV \), for all \( u \in U \Rightarrow U \subseteq F^-(V) \). Therefore, there exists \( W \in m\gamma O(X) \) containing \( x \) such that \( x \in W \subseteq U \) and so \( W \subseteq F^-(V) \Rightarrow F(w)qV \), for all \( w \in W \).

If we take \( m = \tau \), we get fuzzy upper (lower) \( \gamma \)-continuous multifunction.

**Definition 4.12.** For a fuzzy multifunction \( F : X \rightarrow Y \), fuzzy multifunction \( \gamma clF : X \rightarrow Y [9] \), \( aclF : X \rightarrow Y [9] \), \( \beta clF : X \rightarrow Y [9] \), \( clF : X \rightarrow Y [6] \), \( sclF : X \rightarrow Y [6] \), \( pclF : X \rightarrow Y [9] \), \( \delta pclF : X \rightarrow Y [7] \) are defined by \( (\gamma clF)(x) = \gamma clF(x), (\alpha clF)(x) = \alpha clF(x), (\beta clF)(x) = \beta clF(x), (clF)(x) = clF(x), (sclF)(x) = sclF(x), (pclF)(x) = pclF(x), (\delta pclF)(x) = \delta pclF(x) \), for all \( x \in X \).

**Lemma 4.13** [9]. Let \( F : X \rightarrow Y \) be a fuzzy multifunction. Then we have \( (\gamma clF)^-(G) = F^-(G), (\alpha clF)^-(G) = F^-(G), (\beta clF)^-(G) = F^-(G), (clF)^-(G) = F^-(G), (sclF)^-(G) = F^-(G), (pclF)^-(G) = F^-(G), (\delta pclF)^-(G) = F^-(G) \), for each fuzzy open set \( G \) of \( Y \).

Using Lemma 4.13, we can easily state the following theorem.
Theorem 4.14. For a fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$, the following statements are equivalent:

(i) $F$ is fuzzy lower $m$-$\gamma$-continuous,
(ii) $\gamma cl F$ is fuzzy lower $m$-$\gamma$-continuous,
(iii) $\alpha cl F$ is fuzzy lower $m$-$\gamma$-continuous,
(iv) $\beta cl F$ is fuzzy lower $m$-$\gamma$-continuous,
(v) $scl F$ is fuzzy lower $m$-$\gamma$-continuous,
(vi) $cl F$ is fuzzy lower $m$-$\gamma$-continuous,
(vii) $pcl F$ is fuzzy lower $m$-$\gamma$-continuous,
(viii) $\delta pcl F$ is fuzzy lower $m$-$\gamma$-continuous.

5. Mutual Relationship

In this section, the mutual relationship between fuzzy upper (lower) $m$-$\gamma$-continuous multifunction and fuzzy multifunctions in Section 3 are established.

Remark 5.1. Using Remark 4.4, we have from Theorem 3.19 and Theorem 3.20 that fuzzy upper (lower) $m$-continuous, fuzzy upper (lower) $m$-quasi continuous, fuzzy upper (lower) $m$-precontinuous, fuzzy upper (lower) $m$-$\alpha$-continuous multifunctions are fuzzy upper (lower) $m$-$\gamma$-continuous multifunction. But the converses are not true, in general, as shown from the following examples.

Example 5.2. Fuzzy upper $m$-$\gamma$-continuity $\not\Rightarrow$ fuzzy upper $m$-continuity

Let $X = \{a, b, c\}$, $m = \{\phi, X\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.35, B(y) = 0.4$, for all $y \in Y$. Then $(X, m)$ and $(Y, \tau_Y)$ are $m$-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy multifunction defined by $F(a) = A, F(b) = B, F(c) = C$ where $C(y) = 0.6$ for all $y \in Y$. Now $F^+(A) = \{x \in X : F(x) \leq A\} = \{a\} \notin m$ and so $F$ is not fuzzy upper $m$-continuous multifunction. But $F^+(A) = \{a\} \Rightarrow \text{int(cl}\{a\}) = X \Rightarrow F^+(A)$ is $m$-$\gamma$-open in $X$. Again $F^+(B) = \{a, b\} \Rightarrow \text{int(cl}\{a, b\}) = X \Rightarrow F^+(B)$ is $m$-$\gamma$-open in $X \Rightarrow F$ is fuzzy upper $m$-$\gamma$-continuous multifunction.

Example 5.3. Fuzzy lower $m$-$\gamma$-continuity $\not\Rightarrow$ fuzzy lower $m$-continuity

Let $X = \{a, b, c\}$, $m = \{\phi, X\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.35, B(y) =$
0.5, for all \( y \in Y \). Then \((X, m)\) and \((Y, \tau_Y)\) are \( m \)-space and an fts respectively. Let \( F : (X, m) \to (Y, \tau_Y) \) be a fuzzy multifunction defined by \( F(a) = A, F(b) = B, F(c) = C \) where \( C(y) = 0.6 \) for all \( y \in Y \). Now \( F^{-}(A) = \{ x \in X : F(x)qA \} = \phi \in m \Rightarrow F^{-}(A) \in m\gamma O(X) \), \( F^{-}(B) = \{ x \in X : F(x)qB \} = \{ c \} \). Now \( int(cl(\{c\})) = X \Rightarrow F^{-}(B) \) is \( m\gamma \)-open in \( X \Rightarrow F \) is fuzzy lower \( m\gamma \)-continuous multifunction. But \( F^{-}(B) \notin m \Rightarrow F \) is not fuzzy lower \( m \)-continuous multifunction.

**Example 5.4.** Fuzzy upper \( m\gamma \)-continuity \( \not\Rightarrow \) fuzzy upper \( m \)-quasi continuity
Consider Example 5.2. Here \( F^{+}(A) = \{ a \} \not\subseteq mCl(mInt(\{a\})) = \phi \Rightarrow F^{+}(A) \) is not fuzzy upper \( m \)-quasi continuous multifunction though it is fuzzy upper \( m\gamma \)-continuous multifunction.

**Example 5.5.** Fuzzy lower \( m\gamma \)-continuity \( \not\Rightarrow \) fuzzy lower \( m \)-quasi continuity
Consider Example 5.3. Here \( F^{-}(B) = \{ c \} \not\subseteq mCl(mInt(\{c\})) = \phi \Rightarrow F^{-}(B) \) is not \( m \)-semiopen in \( X \Rightarrow F \) is not fuzzy lower \( m \)-quasi continuous multifunction though it is fuzzy lower \( m\gamma \)-continuous multifunction.

**Example 5.6.** Fuzzy upper \( m\gamma \)-continuity \( \not\Rightarrow \) fuzzy upper \( m\alpha \)-continuity
Consider Example 5.2. Here \( F^{+}(A) = \{ a \} \not\subseteq mInt(mCl(mInt(\{a\})) = \phi \Rightarrow F^{+}(A) \) is not \( m\alpha \)-open in \( X \Rightarrow F \) is not fuzzy upper \( m\alpha \)-continuous multifunction though it is fuzzy upper \( m\gamma \)-continuous multifunction.

**Example 5.7.** Fuzzy lower \( m\gamma \)-continuity \( \not\Rightarrow \) fuzzy lower \( m\alpha \)-continuity
Consider Example 5.3. Here \( F^{-}(B) = \{ c \} \not\subseteq mInt(mCl(mInt(\{c\})) = \phi \Rightarrow F^{-}(B) \) is not \( m\alpha \)-open \( X \Rightarrow F \) is not fuzzy lower \( m\alpha \)-continuous multifunction though it is fuzzy lower \( m\gamma \)-continuous multifunction.

**Example 5.8.** Fuzzy upper \( m\gamma \)-continuity \( \not\Rightarrow \) fuzzy upper \( m \)-precontinuity
Let \( X = \{ a, b, c \}, m = \{ \phi, X, \{ b \}, \{ c \} \}, Y = [0, 1], \tau_Y = \{ 0_Y, 1_Y, A, B \} \) where \( A(y) = 0.35, B(y) = 0.4 \), for all \( y \in Y \). Then \((X, m)\) and \((Y, \tau_Y)\) are \( m \)-space and an fts respectively.
Let \( F : (X, m) \to (Y, \tau_Y) \) be a fuzzy multifunction defined by \( F(a) = F(c) = A, F(b) = B \). Now \( F^{+}(A) = \{ a, c \} \). Now \( mInt(mCl(\{a, c\})) = mInt(\{a, c\}) = \{ c \} \not\supseteq \{ a, c \} \Rightarrow F^{+}(A) \) is not \( m \)-preopen in \( X \Rightarrow F \) is not fuzzy upper \( m \)-precontinuous multifunction though it is fuzzy upper \( m\gamma \)-continuous multifunction.
Example 5.9. Fuzzy lower $m$-$\gamma$-continuity $\not\Rightarrow$ fuzzy lower $m$-precontinuity
Let $X = \{a, b, c\}$, $m = \{\phi, X, \{a\}, \{c\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A\}$ where $A(y) = 0.7$ for all $y \in Y$. Then $(X, m)$ and $(Y, \tau_Y)$ are $m$-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy multifunction defined by $F(a) = A, F(b) = B, F(c) = C$ where $B(y) = 0.4, C(y) = 0.01$ for all $y \in Y$. Now $F^-(A) = \{a, b\}$. Then $m\text{Cl}(m\text{Int}(\{a, b\})) = m\text{Cl}(\{a\}) = \{a, b\} \Rightarrow F^-(A) \in m\gamma O(X) \Rightarrow F$ is lower $m$-$\gamma$-continuous multifunction. But $m\text{Int}(m\text{Cl}(\{a, b\})) = m\text{Int}(\{a, b\}) = \{a\} \not\supset \{a, b\} \Rightarrow F^-(A)$ is not $m$-preopen in $X \Rightarrow F$ is not fuzzy lower $m$-precontinuous multifunction.

Remark 5.10. Fuzzy upper (lower) $m$-$\gamma$-continuity and fuzzy upper (lower) $m$-$\delta$-precontinuity are independent concepts follow from next examples.

Example 5.11. Fuzzy upper $m$-$\gamma$-continuity $\not\Rightarrow$ fuzzy upper $m$-$\delta$-precontinuity
Consider Example 5.2. Here $F^+(A) = \{a\}$. Now $m\delta\text{cl}(\{a\}) = \{x \in X : \{a\} \cap m\text{Int}(m\text{Cl}U) \neq \phi, U \in m, x \in U\} = \{a\}$, $m\text{Int}(m\delta\text{cl}(\{a\})) = \phi \not\supset \{a\} \Rightarrow \{a\}$ is not $m$-$\delta$-preopen in $X \Rightarrow F$ is not fuzzy upper $m$-$\delta$-precontinuous multifunction though it is fuzzy upper $m$-$\gamma$-continuous multifunction.

Example 5.12. Fuzzy lower $m$-$\gamma$-continuity $\not\Rightarrow$ fuzzy lower $m$-$\delta$-precontinuity
Consider Example 5.3. Here $F^-(B) = \{c\}$. Now $m\delta\text{cl}(\{c\}) = \{c\} \Rightarrow m\text{Int}(m\delta\text{cl}(\{c\})) = \phi \not\supset \{c\} \Rightarrow F$ is not fuzzy lower $m$-$\delta$-precontinuous multifunction though it is fuzzy lower $m$-$\gamma$-continuous multifunction.

Example 5.13. Fuzzy upper $m$-$\delta$-precontinuity $\not\Rightarrow$ fuzzy upper $m$-$\gamma$-continuity
Let $X = \{a, b, c\}$, $m = \{\phi, X, \{b\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A\}$ where $A(y) = 0.5$ for all $y \in Y$. Then $(X, m)$ and $(Y, \tau_Y)$ are $m$-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = F(c) = A, F(b) = B$, where $B(y) = 0.6$ for all $y \in Y$. Now $F^+(A) = \{a, c\}$. Then $m\text{Int}(m\delta\text{cl}(\{a, c\})) = X \supset \{a, c\} \Rightarrow \{a, c\}$ is $m$-$\delta$-preopen in $X \Rightarrow F$ is fuzzy upper $m$-$\delta$-precontinuous multifunction. But $m\text{Int}(m\text{Cl}(\{a, c\})) = m\text{Int}(\{a, c\}) = \phi \not\supset \{a, c\} \Rightarrow F^+(A)$ is not $m$-$\gamma$-open in $X \Rightarrow F$ is not fuzzy upper $m$-$\gamma$-continuous multifunction.
Example 5.14. Fuzzy lower $m$-$\delta$-precontinuity $\not\Rightarrow$ fuzzy lower $m$-$\gamma$-continuity

Let $X = \{a, b, c\}$, $m = \{\phi, X, \{b\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A\}$ where $A(y) = 0.6$ for all $y \in Y$. Then $(X, m)$ and $(Y, \tau_Y)$ are $m$-space and an fts respectively. Let $F : (X, m) \rightarrow (Y, \tau_Y)$ be defined by $F(a) = F(c) = A, F(b) = B$, where $B(y) = 0.3$ for all $y \in Y$. Now $F^{-}(A) = \{a, c\}$. Then $mInt(m\delta cl(\{a, c\})) = X \supset \{a, c\} \Rightarrow \{a, c\}$ is $m$-$\delta$-preopen in $X \Rightarrow F$ is fuzzy lower $m$-$\delta$-precontinuous multifunction. Now $mInt(mCl(mInt(\{a, c\})) = \phi$, $mCl(mInt(\{a, c\})) = \phi \Rightarrow mInt(mCl(mInt(\{a, c\})) \cup mCl(mInt(\{a, c\})) = \phi \Rightarrow \{a, c\}$ is not $m$-$\gamma$-open in $X \Rightarrow F$ is not fuzzy lower $m$-$\gamma$-continuous multifunction.

Remark 5.15. It is clear from Theorem 3.21 and Theorem 3.22 that fuzzy upper (lower) $m$-irresolute, fuzzy upper (lower) $m$-preirresolute, fuzzy upper (lower) $m$-$\alpha$-irresolute multifunctions are fuzzy upper (lower) $m$-$\gamma$-continuous multifunction. But the converses are not true, in general, follow from next examples. Also fuzzy upper (lower) $m$-$\gamma$-continuous multifunction and fuzzy upper (lower) $m$-$\delta$-preirresolute multifunction are independent concepts follow from next examples.

Example 5.16. Fuzzy upper $m$-$\gamma$-continuous multifunction $\not\Rightarrow$ fuzzy upper $m$-irresolute multifunction

Consider Example 5.2. Here the fuzzy set $A$ being fuzzy open in $Y$ is fuzzy semiopen in $Y$. Now $F^{+}(A) = \{a\} \not\subseteq mCl(mInt(\{a\})) = \phi \Rightarrow F^{+}(A)$ is not $m$-semiopen in $X \Rightarrow F$ is not fuzzy upper $m$-irresolute multifunction though it is fuzzy upper $m$-$\gamma$-continuous multifunction.

Example 5.17. Fuzzy lower $m$-$\gamma$-continuous multifunction $\not\Rightarrow$ fuzzy lower $m$-irresolute multifunction

Consider Example 5.3. Here the fuzzy set $B$ is fuzzy semiopen in $Y$. Now $F^{-}(B) = \{c\} \not\subseteq mCl(mInt(\{c\})) = \phi \Rightarrow F^{-}(B)$ is not $m$-semiopen in $X \Rightarrow F$ is not fuzzy lower $m$-irresolute multifunction though it is fuzzy lower $m$-$\gamma$-continuous multifunction.

Example 5.18. Fuzzy upper $m$-$\gamma$-continuous multifunction $\not\Rightarrow$ fuzzy upper $m$-preirresolute multifunction

Let $X = \{a, b, c\}$, $m = \{\phi, X, \{b\}, \{c\}\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.35, B(y) = 0.4$, for all $y \in Y$. Then $(X, m)$ and $(Y, \tau_Y)$ are $m$-space and an fts respectively.
Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = F(c) = A, F(b) = B$. Now $F^+(A) = \{a, c\}$. Then $mCl(mInt(\{a, c\})) = mInt(\{c\}) = \{a, c\} \Rightarrow F^+(A) \subseteq (mCl(mInt(\{a, c\})) \cup (mInt(mCl(\{a, c\}))) \Rightarrow F^+(A) \in m\gamma O(X), F^+(B) = \{b\} \in m$ and so $F^+(B) \in m\gamma O(X) \Rightarrow F$ is fuzzy upper $m$-$\gamma$-continuous multifunction. Consider the fuzzy set $D$ defined by $D(y) = 0.37$ for all $y \in Y$. Then $D$ is fuzzy preopen in $Y$. Now $F^+(D) = \{a, c\}$. Now $mInt(mCl(\{a, c\})) = mInt(\{a, c\}) = \{c\} \not\subseteq \{a, c\} \Rightarrow F^+(D)$ is not $m$-preopen in $X \Rightarrow F$ is not fuzzy upper $m$-preirresolute.

**Example 5.19.** Fuzzy lower $m$-$\gamma$-continuous multifunction $\not\Rightarrow$ fuzzy lower $m$-preirresolute multifunction
Let $X = \{a, b, c\}, m = \{\phi, X, \{c\}\}, Y = [0, 1], \tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.4, B(y) = 0.44$ for all $y \in Y$. Then $(X, m)$ and $(Y, \tau_Y)$ are $m$-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = A, F(b) = B, F(c) = C$ where $C(y) = 0.39$ for all $y \in Y$. Here $F^-(A) = F^-(B) = \phi \in m \Rightarrow F$ is fuzzy lower $m$-$\gamma$-continuous multifunction. Consider the fuzzy set $D$ defined by $D(y) = 0.61$ for all $y \in Y$. Then $int(clD) = 1_Y \geq D \Rightarrow D$ is fuzzy preopen in $Y$. Now $F^-(D) = \{a, b\}$. Then $mInt(mCl(\{a, b\})) = mInt(\{a, b\}) = \phi \not\subseteq \{a, b\} \Rightarrow F^-(D)$ is not $m$-preopen in $X \Rightarrow F$ is not fuzzy lower $m$-preirresolute multifunction.

**Example 5.20.** Fuzzy upper $m$-$\gamma$-continuous multifunction $\not\Rightarrow$ fuzzy upper $m$-$\alpha$-irresolute multifunction
Consider Example 5.18. Here $D$ is fuzzy $\alpha$-open in $Y$. Now $F^+(D) = \{a, c\}$. Then $mInt(mCl(mInt(\{a, c\})) = mInt(mCl(\{c\})) = mInt(\{a, c\}) = \{c\} \not\subseteq \{a, c\} \Rightarrow F^+(D)$ is not $m$-$\alpha$-open in $X \Rightarrow F$ is not fuzzy upper $m$-$\alpha$-irresolute multifunction though it is fuzzy upper $m$-$\gamma$-continuous multifunction.

**Example 5.21.** Fuzzy lower $m$-$\gamma$-continuous multifunction $\not\Rightarrow$ fuzzy lower $m$-$\alpha$-irresolute multifunction
Let $X = \{a, b, c\}, m = \{\phi, X, \{a\}, \{c\}\}, Y = [0, 1], \tau_Y = \{0_Y, 1_Y, A\}$ where $A(y) = 0.7$ for all $y \in Y$. Then $(X, m)$ and $(Y, \tau_Y)$ are $m$-space and an fts respectively. Let $F : (X, m) \to (Y, \tau_Y)$ be defined by $F(a) = A, F(b) = B, F(c) = C$ where $B(y) = 0.2, C(y) = 0.01$ for all $y \in Y$. Now $F^-(A) = \{a\} \in m \Rightarrow F^-(A) \in m\gamma O(X) \Rightarrow F$ is fuzzy lower $m$-$\gamma$-continuous multifunction. Consider the fuzzy set $D$ defined by $D(y) = 0.81$, for all $y \in Y$. 272
Then $D$ is fuzzy $\alpha$-open in $Y$. Now $F^{-}(D) = \{a,b\}$. Then $mInt(mCl(mInt(\{a,b\}))) = mInt(mCl(\{a\})) = mInt(\{a,b\}) = \{a\} \not\supseteq \{a,b\} \Rightarrow F^{-}(D)$ is not $m$-$\alpha$-open in $X \Rightarrow F$ is not fuzzy lower $m$-$\alpha$-irresolute multifunction.

**Example 5.22.** Fuzzy upper $m$-$\gamma$-continuous multifunction $\nRightarrow$ fuzzy upper $m$-$\delta$-preirresolute multifunction

Consider Example 5.18. Here $D$ is fuzzy $\delta$-preopen in $Y$. Now $F^{+}(D) = \{a,c\}$. Then $mInt(m\delta cl(\{a,c\})) = mInt(\{a,c\}) = \{c\} \not\supseteq \{a,c\} \Rightarrow F^{+}(D)$ is not $m$-$\delta$-preopen in $X \Rightarrow F$ is not fuzzy upper $m$-$\delta$-preirresolute multifunction though it is fuzzy upper $m$-$\gamma$-continuous multifunction.

**Example 5.23.** Fuzzy lower $m$-$\gamma$-continuous multifunction $\nRightarrow$ fuzzy lower $m$-$\delta$-preirresolute multifunction

Let $X = \{a,b,c\}$, $m = \{\phi,X,\{b\},\{c\}\}$, $Y = [0,1]$, $\tau_{Y} = \{0_{Y},1_{Y},A,B\}$ where $A(y) = 0.4, B(y) = 0.44$, for all $y \in Y$. Then $(X,m)$ and $(Y,\tau_{Y})$ are $m$-space and an fts respectively. Let $F : (X,m) \rightarrow (Y,\tau_{Y})$ be defined by $F(a) = A, F(b) = B, F(c) = C$ where $C(y) = 0.29$ for all $y \in Y$. Then $F^{-}(A) = F^{-}(B) = \phi \in m \Rightarrow F$ is fuzzy lower $m$-$\gamma$-continuous multifunction. Now consider the fuzzy set $D$ defined by $D(y) = 0.61$ for all $y \in Y$. Then $D$ is fuzzy $\delta$-preopen in $Y$. Now $F^{-}(D) = \{a,b\}$. Then $mInt(m\delta cl(\{a,b\})) = mInt(\{a,b\}) = \{b\} \not\supseteq \{a,b\} \Rightarrow F^{-}(D)$ is not $m$-$\delta$-preopen in $X \Rightarrow F$ is not fuzzy lower $m$-$\delta$-preirresolute multifunction.

**Example 5.24.** Fuzzy upper $m$-$\delta$-preirresolute multifunction $\nRightarrow$ fuzzy upper $m$-$\gamma$-continuous multifunction

Let $X = \{a,b,c\}$, $m = \{\phi,X,\{c\}\}, Y = [0,1], \tau_{Y} = \{0_{Y},1_{Y},A\}$ where $A(y) = 0.4$ for all $y \in Y$. Then $(X,m)$ and $(Y,\tau_{Y})$ are $m$-space and an fts respectively. Let $F : (X,m) \rightarrow (Y,\tau_{Y})$ be defined by $F(a) = F(b) = B, F(c) = D$ where $B(y) = 0.3, D(y) = 0.7$ for all $y \in Y$. Now the collection of all fuzzy $\delta$-preopen sets in $Y$ is $\{0_{Y},1_{Y},U,V\}$ where $U \leq A,V > 1_{Y} \setminus A$. Then $F^{+}(U) = \phi$, if $U < B$, $F^{+}(U) = \{a,b\}$, if $B \leq U < D$, $F^{+}(U) = X$, if $U \geq D$. Then $\phi,X$ are obviously $m$-$\delta$-preopen in $X$. Now $mInt(m\delta cl(\{a,b\})) = mIntX = X \supset \{a,b\} \Rightarrow \{a,b\}$ is $m$-$\delta$-preopen in $X \Rightarrow F^{+}(U)$ is $m$-$\delta$-preopen in $X$ for every fuzzy $\delta$-preopen set $U$ of $Y$. But $mInt(mCl(mInt(\{a,b\})) = mInt(\{a,b\}) = \phi, mCl(mInt(\{a,b\})) = \phi \Rightarrow mInt(mCl(mInt(\{a,b\})) \cup mCl(mInt(\{a,b\})) = \phi \not\supseteq \{a,b\} \Rightarrow \{a,b\}$ is not $m$-$\gamma$-open in.
Therefore, \( x \in U \) and \( \gamma \). and in for any fuzzy upper nbd \( F \). Proof In this section fuzzy upper (lower) \( m \)-\( \gamma \)-continuous multifunction: \( F \). Let \( X = \{ a, b, c \}, m = \{ \phi, X, \{ c \} \}, Y = [0, 1], \tau_Y = \{ 0, 1, A \} \) where \( A(y) = 0.5 \) for all \( y \in Y \). Then \( (X, m) \) and \( (Y, \tau_Y) \) are \( m \)-space and an fts respectively. Let \( F : (X, m) \to (Y, \tau_Y) \) be defined by \( F(a) = F(b) = B, F(c) = C \) where \( B(y) = 0.51, C(y) = 0.3 \) for all \( y \in Y \). Any fuzzy set in \( Y \) is fuzzy \( \delta \)-preopen in \( Y \). Now \( F^{-1}(U) = \phi \), if \( U \subseteq 1 \setminus B, F^{-1}(U) = \{ a, b \} \), if \( 1 \setminus B < U \subseteq 1 \setminus C, F^{-1}(U) = X \), if \( U > 1 \setminus C \). Then as in Example 5.24, \( F^{-1}(U) \) is \( m \)-\( \delta \)-preopen in \( X \) \( \Rightarrow \) \( F \) is fuzzy lower \( m \)-\( \delta \)-preirresolute multifunction. But \( \{ a, b \} \) is not \( m \)-\( \gamma \)-open in \( X \) as shown in Example 5.24. So \( F \) is not fuzzy lower \( m \)-\( \gamma \)-continuous multifunction.

6. Fuzzy Upper (Lower) \( m \)-\( \gamma \)-Continuous Multifunction:

More Characterizations and Applications

In this section fuzzy upper (lower) \( m \)-\( \gamma \)-continuous multifunction is characterized by fuzzy upper (lower) nbd \( [9] \) of a fuzzy set and also some applications of these fuzzy multifunctions have been shown.

Definition 6.1 [9]. A fuzzy set \( A \) in an fts \( Y \) is said to be a fuzzy lower (upper) nbd of a fuzzy set \( B \) of \( Y \) if there exists a fuzzy open set \( V \) of \( Y \) such that \( BqV \) (resp., \( B \leq V \)) and \( V \notq (1_Y \setminus A) \).

Theorem 6.2. A fuzzy multifunction \( F : (X, m) \to (Y, \tau_Y) \) is fuzzy upper \( m \)-\( \gamma \)-continuous on \( X \) iff for each point \( x_0 \in X \) and each fuzzy upper nbd \( M \) of \( F(x_0) \), \( F^+(M) \) is an \( m \)-\( \gamma \)-nbd of \( x_0 \).

Proof. Let \( F \) be fuzzy upper \( m \)-\( \gamma \)-continuous multifunction on \( X \). Then for any \( x_0 \in X \) and for any fuzzy upper nbd \( M \) of \( F(x_0) \), there exists a fuzzy open set \( V \) of \( Y \) such that \( F(x_0) \leq V \) and \( V \notq (1_Y \setminus M) \Rightarrow V \leq M \). Since \( F \) is fuzzy upper \( m \)-\( \gamma \)-continuous multifunction, there exists \( U \in m\gamma O(X) \) containing \( x_0 \) such that \( U \subseteq F^+(V) \Rightarrow F(U) \leq V \leq M \Rightarrow U \subseteq F^+(M) \). Therefore, \( x_0 \in U \subseteq F^+(M) \Rightarrow F^+(M) \) is an \( m \)-\( \gamma \)-nbd of \( x_0 \).

Conversely, let for any \( x_0 \in X \) and any fuzzy open set \( V \) of \( Y \) with \( F(x_0) \leq V \), we
have $V/\varnothing(1_Y \setminus V)$. Therefore, $V$ is a fuzzy upper nbd of $F(x_0)$. Then by hypothesis, $F^+(V)$ is an $m$-$\gamma$-nbd of $x_0$. Then there exists $U \in m\gamma O(X)$ containing $x_0$ such that $x_0 \in U \subseteq F^+(V) \Rightarrow F(U) \subseteq V \Rightarrow F$ is fuzzy upper $m$-$\gamma$-continuous multifunction.

**Theorem 6.3.** A fuzzy multifunction $F : (X, m) \to (Y, \tau_Y)$ is fuzzy lower $m$-$\gamma$-continuous on $X$ iff for each point $x_0 \in X$ and each fuzzy lower nbd $M$ of $F(x_0)$, $F^{-}(M)$ is an $m$-$\gamma$-nbd of $x_0$.

**Proof.** Let $F$ be fuzzy lower $m$-$\gamma$-continuous multifunction on $X$. Then for any $x_0 \in X$ and for any fuzzy lower nbd $M$ of $F(x_0)$, there exists a fuzzy open set $V$ of $Y$ such that $F(x_0)qV$ and $V \not\in \varnothing(1_Y \setminus M) \Rightarrow V \subseteq M$. Since $F$ is fuzzy lower $m$-$\gamma$-continuous multifunction, there exists $U \in m\gamma O(X)$ containing $x_0$ such that $U \subseteq F^{-}(V) \subseteq F^{-}(M)$. Therefore, $x_0 \in U \subseteq F^{-}(M) \Rightarrow F^{-}(M)$ is an $m$-$\gamma$-nbd of $x_0$.

Conversely, let for any $x_0 \in X$ and any fuzzy open set $V$ of $Y$ with $F(x_0)qV$. Since $V \not\in \varnothing(1_Y \setminus V)$, $V$ is a fuzzy lower nbd of $F(x_0)$. Then by hypothesis, $F^{-}(V)$ is an $m$-$\gamma$-nbd of $x_0$. Then there exists $U \in m\gamma O(X)$ containing $x_0$ such that $x_0 \in U \subseteq F^{-}(V) \Rightarrow F(u)qV$, for all $u \in U \Rightarrow F$ is fuzzy lower $m$-$\gamma$-continuous multifunction.

**Definition 6.4.** An $m$-space $(X, m)$ is said to be $m$-$\gamma$-compact if for every covering of $X$ by $m$-$\gamma$-open sets of $X$ has a finite subcover.

**Theorem 6.5.** Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy upper $m$-$\gamma$-continuous surjective multifunction and $F(x)$ be a fuzzy compact set of $Y$ for each $x \in X$. If $X$ is $m$-$\gamma$-compact space, then $Y$ is fuzzy compact space.

**Proof.** Let $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of $Y$. Now for each $x \in X$, $F(x)$ is fuzzy compact in $Y$ and so there is a finite subset $\Lambda_x$ of $\Lambda$ such that $F(x) \subseteq \bigcup \{A_\alpha : \alpha \in \Lambda_x\}$. Let $A_x = \bigcup \{A_\alpha : \alpha \in \Lambda_x\}$. Then $F(x) \subseteq A(x)$ where $A_x$ is a fuzzy open set of $Y$. Since $F$ is fuzzy upper $m$-$\gamma$-continuous multifunction, there exists $U_x \in m\gamma O(X)$ containing $x$ such that $U_x \subseteq F^+(A_x)$. Then $\mathcal{U} = \{U_x : x \in X\}$ is a cover of $X$ by $m$-$\gamma$-open sets of $X$. Since $X$ is $m$-$\gamma$-compact, there exists finitely many points $x_1, x_2, ..., x_n$ of $X$ such that $X = \bigcup_{i=1}^{n} U_{x_i}$. As $F$ is surjective, $1_Y = F(X) = F\left(\bigcup_{i=1}^{n} U_{x_i}\right) = \bigcup_{i=1}^{n} F(U_{x_i}) \subseteq \bigcup_{i=1}^{n} A_{x_i} = \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_{x_i}} A_\alpha \Rightarrow Y$ is fuzzy compact space.
Definition 6.6 [15]. An fts (Y, τ_Y) is said to be FNC-space if every fuzzy regular open cover of Y has a finite subcover.

Remark 6.7. As every fuzzy regular open set is fuzzy open, we can set the following theorem easily.

Theorem 6.8. Let F : (X, m) → (Y, τ_Y) be a fuzzy upper m-γ-continuous surjective multifunction and F(x) be a fuzzy compact set of Y for each x ∈ X. If X is m-γ-compact space, then Y is FNC-space.

Theorem 6.9. Every m-γ-closed subset of an m-γ-compact space is m-γ-compact.

Proof. Let A be an m-γ-closed subset of an m-γ-compact space (X, m). Let A = {A_α : α ∈ Λ} be a covering of A by m-γ-open sets of X. Then (X \ A) ∪ (\bigcup_{α ∈ Λ} A_α) is a covering of X by m-γ-open sets of X. As X is m-γ-compact, there exists a finite subset Λ_0 of Λ such that (X \ A) ∪ (\bigcup_{α ∈ Λ_0} A_α) covers X. Now discarding the set X \ A, we get the finite subcover \{A_α : α ∈ Λ_0\} of A by m-γ-open sets of X. Hence A is m-γ-compact.

Definition 6.10 [14]. For a fuzzy multifunction F : X → Y, the fuzzy graph multifunction G_F : X → X × Y of F is defined as G_F(x) = the fuzzy set x_1 × F(x) of X × Y, where x_1 is the fuzzy set in X, whose value is 1 at x ∈ X and 0 at other points of X. We shall write \{x\} × F(x) for x_1 × F(x).

Theorem 6.11. When X is product related to Y, a fuzzy multifunction F : (X, m) → (Y, τ_Y) is fuzzy upper m-γ-continuous if its fuzzy graph multifunction G_F : X → X × Y is fuzzy upper m-γ-continuous multifunction.

Proof. Let G_F be a fuzzy upper m-γ-continuous multifunction. Let x ∈ X and V be a fuzzy open set of Y such that F(x) ≤ V. Then G_F(x) ≤ X × V and X × V is easily seen to be open in X × Y. By hypothesis, there exists U ∈ mγO(X) containing x such that G_F(U) ≤ X × V. Now for any z ∈ U and any y ∈ Y, [F(z)](y) = [G_F(z)](z, y) ≤ (X × V)(z, y) = V(y), i.e., [F(z)](y) ≤ V(y), for all y ∈ Y ⇒ F(z) ≤ V, for any z ∈ U ⇒ F(U) ≤ V ⇒ F is fuzzy upper m-γ-continuous multifunction.
Definition 6.12. The $m$-$\gamma$-frontier of a subset $A$ of an $m$-space $(X, m)$, denoted by $m\gamma Fr(A)$, is defined by $m\gamma Fr(A) = m\gamma cl A \cap m\gamma cl (X \setminus A) = m\gamma cl A \setminus m\gamma int A$.

Theorem 6.13. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy multifunction where $m$ satisfies Maki condition. Let $A = \{x \in X : F$ is not fuzzy upper $m$-$\gamma$-continuous at $x\}$, $B = \bigcup\{m\gamma Fr(F^+(V)) : F(x) \leq V \text{ and } V \text{ is fuzzy open in } Y\}$. Then $A = B$.

Proof. Let $x \in X$ be such that $F$ is not fuzzy upper $m$-$\gamma$-continuous at $x$. Then there exists a fuzzy open set $V$ of $Y$ with $F(x) \leq V$ such that $U \not\subseteq F^+(V)$, for all $U \in m\gamma O(X)$ containing $x \Rightarrow U \cap (X \setminus F^+(V)) \neq \emptyset \Rightarrow x \in m\gamma cl (X \setminus F^+(V)) = X \setminus m\gamma int(F^+(V)) \Rightarrow x \notin m\gamma int(F^+(V))$. But $x \in F^+(V) \subseteq m\gamma cl(F^+(V))$. Therefore, $x \in m\gamma Fr(F^+(V))$.

Conversely, let $x \in X$ and $V$ be a fuzzy open set of $Y$ with $F(x) \leq V$ such that $x \in m\gamma Fr(F^+(V))$. If possible, let $F$ be fuzzy upper $m$-$\gamma$-continuous at $x$. Then there exists $U \in m\gamma O(X)$ containing $x$ such that $U \subseteq F^+(V)$. Then $x \in U = m\gamma int U \subseteq m\gamma int(F^+(V)) \Rightarrow x \in m\gamma int(F^+(V)) \Rightarrow x \notin m\gamma Fr(F^+(V))$, a contradiction and hence $F$ is not fuzzy upper $m$-$\gamma$-continuous at $x$.

Theorem 6.14. Let $F : (X, m) \to (Y, \tau_Y)$ be a fuzzy multifunction where $m$ satisfies Maki condition. Let $A = \{x \in X : F$ is not fuzzy lower $m$-$\gamma$-continuous at $x\}$, $B = \bigcup\{m\gamma Fr(F^-(V)) : F(x)qV \text{ and } V \text{ is fuzzy open in } Y\}$. Then $A = B$.

Proof. Let $x \in X$ be such that $F$ is not fuzzy lower $m$-$\gamma$-continuous at $x$. Then there exists a fuzzy open set $V$ of $Y$ with $F(x)qV$ such that $U \not\subseteq F^-(V)$, for all $U \in m\gamma O(X)$ containing $x \Rightarrow U \cap (X \setminus F^-(V)) \neq \emptyset \Rightarrow x \in m\gamma cl (X \setminus F^-(V)) = X \setminus m\gamma int(F^-(V)) \Rightarrow x \notin m\gamma int(F^-(V))$. But $x \in F^-(V) \subseteq m\gamma cl(F^-(V))$. Therefore, $x \in m\gamma Fr(F^-(V))$.

Conversely, let $x \in X$ and $V$ be a fuzzy open set of $Y$ with $F(x)qV$ such that $x \in m\gamma Fr(F^-(V))$. If possible, let $F$ be fuzzy lower $m$-$\gamma$-continuous at $x$. Then there exists $U \in m\gamma O(X)$ containing $x$ such that $U \subseteq F^-(V)$. Then $x \in U = m\gamma int U \subseteq m\gamma int(F^-(V)) \Rightarrow x \in m\gamma int(F^-(V)) \Rightarrow x \notin m\gamma Fr(F^-(V))$, a contradiction and hence $F$ is not fuzzy lower $m$-$\gamma$-continuous at $x$.
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