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e^*-Almost Compactness via α-shading

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Abstract

This paper deals with a new type of compactness viz., α-e^*-almost compactness for crisp subsets of a space X where the underlying structure on X is a fuzzy topology. Several characterizations are made by using ordinary net and power set filterbases as basic tools. It is shown that α-e^*-almost compactness implies α-almost compactness but the converse is true only in α-e^*-regular space.

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1. Introduction

The generalized version of fuzzy cover is α-shading introduced by Gantner, Steinlage and Warren [9] in 1978. Using the concept of α-shading they introduced and studied α-compactness which is a generalization of compactness in a fuzzy topological space [6] (henceforth abbreviated as an fts, for short). Afterwards, α-almost compactness [10], α-δ_p-almost compactness [2], α-β-almost compactness [3] have been introduced and studied. In this paper using the definition of α-shading, α-e^*-almost compactness for crisp subsets in an fts is introduced and studied. This idea is also characterized by ordinary nets and power-set filterbases as basic appliances with the notion of adherence suitably defined via the fuzzy topology of the space. Afterwards, taking $A = X$, we get the characterizations of α-e^*-almost compact space.

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2. Preliminaries

In what follows, by (X,τ) or simply by X, we mean an fts in the sense of Chang [6]. A crisp set A in an fts X means an ordinary subset of the set X where the underlying structure of the set X being a fuzzy topology τ. A fuzzy set [13] A is a mapping from a nonempty set X into the closed interval $I=[0,1]$ of the real line, i.e., $A\in I^X$. For a fuzzy set A, the fuzzy closure [6] and fuzzy interior [6] of A in X are denoted by clA and $intA$ respectively. The support [13] of a fuzzy set A in X will be denoted by $suppA$ and is defined by $suppA=\{x\in X : A(x)\neq 0\}$. A fuzzy point [12] in X with the singleton support $\{x\}\subseteq X$ and the value α $(0<\alpha\leq 1)$ at x will be denoted by x_α. For a fuzzy set A , the complement [13] of A in X will be denoted by $1_X \setminus A$ and is defined by $(1_X\setminus A)(x)=1-A(x)$, for each $x\in X$. For any two fuzzy sets A and B in X, we write $A\leq B$ if $A(x)\leq B(x)$, for each $x\in X$ [13] while we write AqB if A is quasi-coincident (q-coincident, for short) with B [12], i.e., if there exists $x\in X$ such that $A(x)+B(x)>1$; the negation of these statements are written as $A\not\leq B$ and $A\not{q}B$ respectively. A fuzzy set A is called fuzzy regular open [1] (resp., fuzzy preopen [11], fuzzy β-open [7]) if $A=int(clA)$ (resp., $A\leq int(clA), A\leq cl(int(clA))$). The complement of a fuzzy preopen (resp., fuzzy β-open) set is called fuzzy preclosed [11] (resp., fuzzy β-closed [7]). A fuzzy set B is called a quasi-neighbourhood (q-nbd, for short) [12] of a fuzzy point x_t if there is a fuzzy open set U in X such that $x_t qU\leq B$. If, in addition, B is fuzzy open (resp., fuzzy regular open), then B is called a fuzzy open [12] (resp., fuzzy regular open [1]) q-nbd of x_t. A fuzzy point x_α is said to be a fuzzy δ-cluster point of a fuzzy set A in an fts X if every fuzzy regular open q-nbd U of x_α is q-coincident with A [8]. The union of all fuzzy δ-cluster points of A is called the fuzzy δ-closure of A and is denoted by δclA [8]. Afuzzy set A in an fts X is said to be fuzzy δ-preopen [4] if $A\leq int(\delta clA)$. The complement of a fuzzy δ-preopen set is called fuzzy δ-preclosed [4]. A fuzzy set A in an fts X is called a fuzzy δ-pre-q-nbd of a fuzzy point x_α in X if there exists a fuzzy δ-preopen set V in X such that $x_\alpha qV \le A$ [4]. A fuzzy point x_α in an fts X is called a fuzzy δ-precluster point of a fuzzy set A in X if every fuzzy δ-pre-q-nbd of x_α is q-coincident with A [4]. The union of all fuzzy δ-precluster points of A is called the fuzzy δ-preclosure of A and will be denoted by $\delta\text{-}pclA$ [4].

3. Fuzzy e^*-open Set: Some Properties

In this section we recall first the definition of fuzzy e^*-open set and some of its results from [5].

Definition 3.1 [5]. A fuzzy set A in an fts X is said to be fuzzy e^*-open if $A \le cl(int(\delta clA))$. The complement of a fuzzy e^*-open set is called fuzzy e^*-closed.

Definition 3.2 [5]. The intersection of all fuzzy e^*-closed sets containing a fuzzy set A in X is called fuzzy e^*-closure of A, to be denoted by e^*clA.

Result 3.3 [5]. A fuzzy set A in X is fuzzy e^*-closed iff $A = e^*clA$.

Definition 3.4 [5]. The union of all fuzzy e^*-open sets contained in a fuzzy set A in X is called fuzzy e^*-interior of A, to be denoted by e^*intA.

Result 3.5 [5]. A fuzzy set A is fuzzy e^*-open iff $A = e^*intA$.

Result 3.6 [5]. (i) For any fuzzy set A in X, $x_t \in e^*clA \Leftrightarrow UqA$ for any fuzzy e^*-open set U in X with $x_t qU$.

(ii) for any two fuzzy sets U, V in X where V is fuzzy e^*-open set, $U \not{q} V \Rightarrow e^*clU \not{q} V$.

Result 3.7. For a fuzzy set A in an fts (X, τ), $e^*cl(e^*clA) = e^*clA$.

Proof. It is clear that $e^*clA \le e^*cl(e^*clA)$.

Conversely, let $x_\alpha \in e^*cl(e^*clA)$. Then for every fuzzy e^*-open set U of X with $x_\alpha qU$, Uqe^*clA. We have to show that UqA. If possible, let $U\not{q}A$. Then $A \le 1_X \setminus U \Rightarrow e^*clA \le e^*cl(1_X \setminus U) = 1_X \setminus U \Rightarrow U\not{q}e^*clA$, a contradiction. Consequently, $e^*cl(e^*clA) \le e^*clA$ and hence $e^*cl(e^*clA) = e^*clA$, for any fuzzy set A in X.

Result 3.8. For a fuzzy e^*-open set U in X, $e^*cl(e^*int(e^*clU)) = e^*clU$.

Proof. Let U be fuzzy e^*-open in X. Then clearly $U \le e^*cl(e^*int(e^*clU)) \Rightarrow e^*clU \le e^*cl\left(e^*cl\left(e^*\operatorname{int}\left(e^*clU\right)\right)\right) = e^*cl(e^*int(e^*clU))$ (by Result 3.7). Again $e^*cl(e^*int(e^*clU)) \le e^*cl(e^*clU) = e^*clU$ (by Result 3.7). Hence $e^*cl(e^*int(e^*clU)) = e^*clU$, for any fuzzy e^*-open set U in X.

Result 3.9. For any two fuzzy sets A, B in an fts X, $e^*cl(A \vee B) = e^*clA \vee e^*clB$.

Proof. It is clear that $e^*clA \vee e^*clB \le e^*cl(A \vee B)$.

Conversely, let $x_\alpha \in e^*cl(A \vee B)$. Then for any fuzzy e^*-open set U in X with $x_\alpha qU$, $Uq(A \vee B) \Rightarrow$ there exists $y \in X$ such that $U(y) + (A \vee B)(y) > 1 \Rightarrow U(y) + max\{A(y), B(y)\} > 1 \Rightarrow$ either UqA or $UqB \Rightarrow$ either $x_\alpha \in e^*clA$ or $x_\alpha \in e^*clB \Rightarrow x_\alpha \in e^*clA \vee e^*clB$. Hence the proof.

Remark 3.10. The union (intersection) of any collection of fuzzy e^*-open (resp., fuzzy e^*-closed) sets in an fts X is fuzzy e^*-open (resp., fuzzy e^*-closed) in X.

Remark 3.11. It is clear from Remark 3.10 that $e^*cl(\bigvee_{i=1}^{n} A_i) = \bigvee_{i=1}^{n} e^*clA_i$ for fuzzy e^*-open sets $A_1, A_2, ..., A_n$ in X.

4. α-e^*-Almost Compactness: Some Characterizations

The concept of α-shading given by Gantner et al. [9] when applied to arbitrary crisp subsets of X we get the following definition.

Definition 4.1. Let X be an fts and A, a crisp subset of X. A collection $\mathcal{U}$ of fuzzy sets in X is called an α-shading (where $0<\alpha<1$) of A if for each $x \in A$, there is some $U_x \in \mathcal{U}$ such that $U_x(x) > \alpha$ [9]. If, in addition, the members of $\mathcal{U}$ are fuzzy open (resp., fuzzy e^*-open) sets, then $\mathcal{U}$ is called a fuzzy open [9] (resp., fuzzy e^*-open) α-shading of A.

Definition 4.2. Let X be an fts and A, a crisp subset of X. A is said to be α-compact [9] (resp. α-almost compact [10], α-β-almost compact [3], α-δ_p-almost compact [2]) if each fuzzy open α-shading $(0<\alpha<1)\,\mathcal{U}$ of A, there is a finite (resp. finite proximate, finite β-proximate, finite δ_p-proximate) α-subshading of A, i.e., there is a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{U : U \in \mathcal{U}_0\}$ (resp., $\{clU : U \in \mathcal{U}_0\}$, $\{\beta clU : U \in \mathcal{U}_0\}$, $\{\delta\text{-}pclU : U \in \mathcal{U}_0\}$) is again an α-shading of A. If $A = X$ in addition, then X is called an α-compact (resp., α-almost compact, α-β-almost compact, α-δ_p-almost compact) space.

Now we set the following definition.

Definition 4.3. Let X be an fts and A, a crisp subset of X. A is said to be α-e^*-almost compact if each fuzzy e^*-open α-shading $\mathcal{U}$ of A has a finite e^*-proximate α-subshading, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{e^*clU : U \in \mathcal{U}_0\}$ is again an α-shading of A. If in particular $A = X$, we get the definition of α-e^*-almost compact space X.

It then follows that

Theorem 4.4. (a) Every finite subset of an fts X is α-e^*-almost compact.

(b) If A_1 and A_2 are α-e^*-almost compact subsets of an fts X, then so is $A_1 \vee A_2$.

(c) X is α - e^*-almost compact if X can be written as the union of finite number of α - e^*-almost compact sets in X.

Since $e^*clA \leq \beta clA \leq pclA \leq \delta\text{ - }pclA$ for any fuzzy set A in an fts X, it is clear from Definition 4.2 and Definition 4.3 that α - e^*-almost compactness implies α - β-almost compactness implies α - p-almost compactness implies α - δ_p-almost compactness.

Again as $e^*clA \leq clA$, for any fuzzy set A in an fts X, it is clear from literature that α - e^*-almost compactness implies α-almost compactness, but not conversely. To achieve the converse we need to define some sort of regularity condition in our setting. The following definition serves our purpose.

Definition 4.5. An fts X is said to be α - e^*-regular, if for each point $x \in X$ and each fuzzy e^*-open set U_x in X with $U_x(x) > \alpha$, there exists a fuzzy open set V_x in X with $V_x(x) > \alpha$ such that $clV_x \leq U_x$.

Two other equivalent ways of defining α - e^*-regularity are given by the following theorem.

Theorem 4.6. For an fts X, the following are equivalent :

(a) X is α - e^*-regular.

(b) For each point $x \in X$ and each fuzzy e^*-closed set F with $F(x) < 1-\alpha$, there is a fuzzy open set U such that $(clU)(x) < 1-\alpha$ and $F \leq U$.

(c) For each crisp point $x \in X$ and each fuzzy e^*-closed set F with $F(x) < 1-\alpha$, there exist fuzzy open sets U and V in X such that $V(x) > \alpha$, $F \leq U$ and $U \not{q} V$.

Proof. (a) $\Rightarrow$ (b): Let $x \in X$ and F be a fuzzy e^*-closed set with $F(x) < 1-\alpha$. Put $V = 1_X \setminus F$. Then $V(x) > \alpha$ where V is fuzzy e^*-open in X. By (a), there exists a fuzzy open set W in X with $W(x) > \alpha$ and $clW \leq V = 1_X \setminus F$. Then $F \leq 1_X \setminus clW = int(1_X \setminus W) = U$ (say).

Then U is fuzzy open in X. Also, $clU = cl(int(1_X \setminus W)) = cl(1_X \setminus clW) = 1_X \setminus int(clW) \leq 1_X \setminus W$. Thus $(clU)(x) \leq (1_X \setminus W)(x) < 1-\alpha$.

(b) $\Rightarrow$ (a): Let $x \in X$ and U be fuzzy e^*-open set in X with $U(x) > \alpha$. Let $F = 1_X \setminus U$. Then F is fuzzy e^*-closed set in X with $F(x) < 1-\alpha$. By (b), there is a fuzzy open set V in X such that $(clV)(x) < 1-\alpha$ and $F \leq V$. So $(1_X \setminus clV)(x) > \alpha$, i.e., $W(x) > \alpha$ where $W = 1_X \setminus clV = int(1_X \setminus V)$ is a fuzzy open set in X. Now $clW = cl(1_X \setminus clV) = 1_X \setminus int(clV)$ $\leq 1_X \setminus V \leq 1_X \setminus F = U$. Hence (a) follows.

(b) $\Rightarrow$ (c): Let $x \in X$ and F, a fuzzy e^*-closed set in X with $F(x) < 1-\alpha$. By (b), there exists a fuzzy open set U in X such that $(clU)(x) < 1-\alpha$ and $F \leq U$. Then $x_{1-\alpha} \notin clU$. Consequently, there is a fuzzy open set V in X such that $x_{1-\alpha} q V$ and $V \not{q} U$, i.e., $V(x) + 1 - \alpha > 1 \Rightarrow V(x) > \alpha$.

(c) $\Rightarrow$ (b): Let $x \in X$, and F, a fuzzy e^*-closed set in X with $F(x) < 1-\alpha$. By (c), there exist fuzzy open sets U and V in X such that $V(x) > \alpha, F \leq U$ and $U \not{q} V$. Now $V(x) > \alpha \Rightarrow x_{1-\alpha} q V$. Then as $U \not{q} V$, $clU \not{q} V \Rightarrow (clU)(x) \leq 1 - V(x) < 1-\alpha$.

Theorem 4.7. In an α-e^*-regular fts X, the α-almost compactness of a crisp subset A of X implies its α-e^*-almost compactness.

Proof. Let $\mathcal{U}$ be a fuzzy e^*-open α-shading of an α-almost compact set A in an α-e^*-regular fts X. Then for each $a \in A$, there exists $U_a \in \mathcal{U}$ such that $U_a(a) > \alpha$. By hypothesis, there is a fuzzy open set V_a in X with $V_a(a) > \alpha$ such that $clV_a \leq U_a$ ... (1).

Let $\mathcal{V} = \{V_a : a \in A\}$. Then $\mathcal{V}$ is a fuzzy open α-shading of A. By α-almost compactness of

A, there is a finite subset A_0 of A such that $\mathcal{V}_0 = \{clV_a : a \in A_0\}$ is an α-shading of A. Using (1), $\mathcal{U}_0 = \{U_a : a \in A_0\}$ and hence $\{clU_a : a \in A_0\}$ is then a finite α-subshading of $\mathcal{U}$. Hence A is α-e^*-almost compact space.

Theorem 4.8. A crisp subset A of an fts X is α-e^*-almost compact if and only if every family of fuzzy e^*-open sets, the e^*-interiors of whose e^*-closures form an α-shading of A, contains a finite subfamily, the e^*-closures of whose members form an α-shading of A.

Proof. It is sufficient to prove that for a fuzzy e^*-open set U, $U \le e^*int(e^*clU) \le e^*cl(e^*int(e^*clU)) = e^*clU$ (By Result 3.8).

Theorem 4.9. A crisp subset A of an fts X is α-e^*-almost compact if and only if for every collection $\{F_i : i \in \Lambda\}$ of fuzzy e^*-open sets with the property that for each finite subset Λ_0 of Λ, there is $x \in A$ such that $\inf_{i \in \Lambda_0} F_i(x) \ge 1-\alpha$, one has $\inf_{i \in \Lambda}(e^*clF_i)(y) \ge 1-\alpha$, for some $y \in A$.

Proof. Let A be α-e^*-almost compact. If possible, let for a collection $\{F_i : i \in \Lambda\}$ of fuzzy e^*-open sets in X with the property that for each finite subset Λ_0 of Λ, there is $x \in A$ such that $\inf_{i \in \Lambda_0} F_i(x) \ge 1-\alpha$, but $\inf_{i \in \Lambda}(e^*clF_i)(x) < 1-\alpha$, for all $x \in A$. Then $\alpha < (1_X \setminus \bigcap_{i \in \Lambda} e^*clF_i)(x)$ = $[\bigcup_{i \in \Lambda}(1_X \setminus e^*clF_i)](x)$, for each $x \in A$ which shows that $\{1_X \setminus e^*clF_i : i \in \Lambda\}$ is a fuzzy e^*-open α-shading of A. By assumption, there is a finite subset Λ_0 of Λ such that $\{e^*cl(1_X \setminus e^*clF_i) : i \in \Lambda_0\}$ = $\{1_X \setminus e^*int(e^*clF_i) : i \in \Lambda_0\}$ is an α-shading of A. Hence $\alpha < [\bigcup_{i \in \Lambda_0}(1_X \setminus e^*int(e^*clF_i))](x)$ = $[1_X \setminus (\bigcap_{i \in \Lambda_0} e^*int(e^*clF_i))](x)$, for each $x \in A$. Then $(\bigcap_{i \in \Lambda_0} F_i)(x) \le [\bigcap_{i \in \Lambda_0} e^*int(e^*clF_i)](x) < 1-\alpha$, for each $x \in A$, which contradicts our assumption.

Conversely, let under the given hypothesis, A be not α-e^*-almost compact. Then there exists a fuzzy e^*-open α-shading $\mathcal{U}=\{U_i : i\in\Lambda\}$ of A having no finite e^*-proximate α-subshading, i.e., for every finite subset Λ_0 of Λ, $\{e^*clU_i : i\in\Lambda_0\}$ is not an α-shading of A, i.e., there exists $x\in A$ such that $\sup_{i\in\Lambda_0}(e^*clU_i)(x)\le\alpha$, i.e., $1_X\setminus sup_{i\in\Lambda_0}(e^*clU_i)(x)=\inf_{i\in\Lambda_0}(1_X\setminus e^*clU_i)(x)\ge 1-\alpha$. Hence $\{1_X\setminus e^*clU_i : i\in\Lambda\}$ is a family of fuzzy e^*-open sets with the stated property. Consequently, there is some $y\in A$ such that $\inf_{i\in\Lambda}[e^*cl(1_X\setminus e^*clU_i)](y)\ge 1-\alpha$. Then $\sup_{i\in\Lambda}U_i(y)\le \sup_{i\in\Lambda}(e^*int(e^*clU_i))(y) = 1-\inf_{i\in\Lambda}(1_X\setminus e^*int(e^*clU_i))(y)=1-\inf_{i\in\Lambda}[e^*cl(1_X\setminus e^*clU_i)]((y)\le\alpha$. This shows that $\{U_i : i\in\Lambda\}$ fails to be an α-shading of A, a contradiction.

Definition 4.10. A family $\{F_i : i\in\Lambda\}$ of fuzzy sets in an fts X is said to have α-e^*-interiorly finite intersection property (α-e^*-IFIP, for short) in a subset A of X, if for each finite subset Λ_0 of Λ, there exists $x\in A$ such that $[\bigcap_{i\in\Lambda_0} e^*intF_i](x)\ge 1-\alpha$.

Theorem 4.11. A crisp subset A of an fts X is α-e^*-almost compact if and only if for every family $\mathcal{F}=\{F_i : i\in\Lambda\}$ of fuzzy e^*-closed sets in X with α-e^*-IFIP in A, there exists $x\in A$ such that $\inf_{i\in\Lambda} F_i(x)\ge 1-\alpha$.

Proof. Let $\mathcal{F}=\{F_i : i\in\Lambda\}$ be a family of fuzzy e^*-closed sets in X with α-e^*-IFIP in A where A is α-e^*-almost compact subset of X. If possible, let for each $x\in A$, $\inf_{i\in\Lambda} F_i(x)<1-\alpha$, i.e., $(\bigcap_{i\in\Lambda} F_i)(x)<1-\alpha$ and hence $[\bigcup_{i\in\Lambda}(1_X\setminus F_i)](x)>\alpha$. Therefore $\mathcal{U}=\{1_X\setminus F_i : i\in\Lambda\}$ is a fuzzy e^*-open α-shading of A. By hypothesis, there exists a finite subfamily Λ_0 of Λ such that $[\bigcup_{i\in\Lambda_0} e^*cl(1_X\setminus F_i)](x)>\alpha$, i.e., $1-(\bigcap_{i\in\Lambda_0} e^*intF_i)(x)>\alpha$, i.e., $(\bigcap_{i\in\Lambda_0} e^*intF_i)(x)<1-\alpha$, for each $x\in A$, which shows that $\mathcal{F}$ does not have α-e^*-IFIP in A, a contradiction.

Conversely, let $\mathcal{U}=\{U_i : i\in\Lambda\}$ be a fuzzy e^*-open α-shading of A. Then

$\mathcal{F} = \{1_X \setminus U_i : i \in \Lambda\}$ is a family of fuzzy e^*-closed sets in X with $\inf_{i\in\Lambda}(1_X \setminus U_i)(x) < 1-\alpha$, for each $x \in A$, so that $\mathcal{F}$ does not have α-e^*-IFIP in A. Then there exists some finite subset Λ_0 of Λ such that for each $x \in A$, $[\bigcap_{i\in\Lambda_0} e^*int(1_X \setminus U_i)](x) < 1-\alpha \Rightarrow 1-(\bigcup_{i\in\Lambda_0} e^*clU_i)(x) < 1-\alpha$, for each $x \in A \Rightarrow (\bigcup_{i\in\Lambda_0} e^*clU_i)(x) > \alpha$, for each $x \in A \Rightarrow A$ is α-e^*-almost compact space.

5. Characterizations of *α*-*e**-Almost Compactness Via Ordinary Nets and Filterbases

In this section, α-e^*-almost compactness of a crisp subset A of an fts X is characterized via α^{e^*}-adherent points of ordinary nets and power-set filterbases.

Let us now introduce the following definition:

Definition 5.1. Let $\{S_n : n \in (D,\geq)\}$ (where $(D,\geq)$ is a directed set) be an ordinary net in A and $\mathcal{F}$ be a power-set filterbase on A, and $x \in X$ be any crisp point in X. Then x is called an α^{e^*}-adherent point of

(a) the net $\{S_n\}$ if for each fuzzy e^*-open set U in X with $U(x) > \alpha$ and for each $m \in D$, there exists $k \in D$ such that $k \geq m$ in D and $(e^*clU)(S_k) > \alpha$,

(b) the filterbase $\mathcal{F}$ if for each fuzzy e^*-open set U with $U(x) > \alpha$ and for each $F \in \mathcal{F}$, there exists a crisp point x_F in F such that $(e^*clU)(x_F) > \alpha$.

Theorem 5.2. A crisp subset A of an fts X is α-e^*-almost compact if and only if every net in A has an α^{e^*}-adherent point in A.

Proof. Let a crisp subset A of an fts X be α-e^*-almost compact. If possible, let there be a net $\{S_n : n \in (D,\geq)\}$ in A ($(D,\geq)$ being a directed set, as usual) having no α^{e^*}-adherent point in A.

Then for each $x \in A$, there is a fuzzy e^*-open set U_x in X with $U_x(x) > \alpha$ and an $m_x \in D$ such that $(e^*clU_x)(S_n) \le \alpha$, for all $n \ge m_x$ $(n \in D)$. Now, $\mathcal{U} = \{1_X \setminus e^*clU_x : x \in A\}$ is a collection of fuzzy e^*-open sets in X such that for any finite subcollection $\{1_X \setminus e^*clU_{x_i} : i = 1, 2, ..., k\}$ (say) of $\mathcal{U}$, there exists $m \in D$ with $m \ge m_{x_i}, i = 1, 2, ..., k$ in D such that $(\bigcup_{i=1}^{k} e^*clU_{x_i})(S_n) \le \alpha$, for all $n \ge m$ $(n \in D)$, i.e., $\inf_{1 \le i \le k}(1_X \setminus e^*clU_{x_i})(S_n) \ge 1 - \alpha$, for all $n \ge m$. Hence by Theorem 4.9, there exists some $y \in A$ such that $\inf_{x \in A}[e^*cl(1_X \setminus e^*clU_x)(y)] \ge 1 - \alpha$, i.e., $(\bigcup_{x \in A} U_x)(y) \le [\bigcup_{x \in A} e^*int(e^*clU_x)](y) = 1 - [1 - (\bigcup_{x \in A} e^*int(e^*clU_x))(y)] = 1 - \inf_{x \in A}[e^*cl(1 - e^*clU_x)](y) \le 1 - 1 + \alpha = \alpha$. We have, in particular, $U_y(y) \le \alpha$, contradicting the definition of U_y. Hence the result is proved.

Conversely, suppose that every net in A has an α^{e^*}-adherent point in A. Let $\{F_i : i \in \Lambda\}$ be an arbitrary collection of fuzzy e^*-open sets in X. Let Λ_f denote the collection of all finite subsets of Λ, then $(\Lambda_f, \ge)$ is a directed set, where for $\mu, \lambda \in \Lambda_f$, $\mu \ge \lambda$ iff $\mu \supseteq \lambda$. For each $\mu \in \Lambda_f$, put $F_\mu = \bigcap\{F_i : i \in \mu\}$. Let for each $\mu \in \Lambda_f$, there be a point $x_\mu \in A$ such that $\inf_{x \in \mu} F_i(x_\mu) \ge 1 - \alpha$ ....(1).

Then by Theorem 4.9 it is enough to show that $\inf_{i \in \Lambda}(e^*clF_i)(z) \ge 1 - \alpha$ for some $z \in A$. If possible, let $\inf_{i \in \Lambda}(e^*clF_i)(z) < 1 - \alpha$, for each $z \in A$ ....(2).

Now, $S = \{x_\mu : \mu \in (\Lambda_f, \ge)\}$ is clearly a net of points in A. By hypothesis, there is an α^{e^*}-adherent point z in A of this net. By (2), $\inf_{i \in \Lambda}(e^*clF_i)(z) < 1 - \alpha \Rightarrow$ there exists $i_0 \in \Lambda$ such that $(e^*clF_{i_0})(z) < 1 - \alpha$, i.e., $1 - e^*clF_{i_0}(z) > \alpha$. Since z is an α^{e^*}-adherent point of S, for the index $\{i_0\} \in \Lambda_f$, there is $\mu_0 \in \Lambda_f$ with $\mu_0 \ge \{i_0\}$ (i.e., $i_0 \in \mu_0$) such that $e^*cl(1_X \setminus e^*clF_{i_0})(x_{\mu_0}) > \alpha$, i.e.,

$(e^*int(e^*clF_{i_0}))(x_{\mu_0}) < 1-\alpha$. Since $i_0 \in \mu_0$, $\inf_{i\in\mu_0} F_i(x_{\mu_0}) \le F_{i_0}(x_{\mu_0}) \le (e^*int(e^*clF_{i_0}))(x_{\mu_0}) < 1-\alpha$, which contradicts (1). This completes the proof.

Theorem 5.3. A crisp subset A of an fts X is α - e^*-almost compact if and only if every filterbase $\mathcal{F}$ on A has an α^{e^*}-adherent point in A.

Proof. Let A be α - e^*-almost compact and if possible, let there be a filterbase $\mathcal{F}$ on A having no α^{e^*}-adherent point in A. Then for each $x \in A$, there exist a fuzzy e^*-open set U_x with $U_x(x) > \alpha$, and an $F_x \in \mathcal{F}$ such that $(e^*clU_x)(y) \le \alpha$, for each $y \in F_x$. Then $\mathcal{U} = \{U_x : x \in A\}$ is a fuzzy e^*-open α-shading of A. Thus there exist finitely many points $x_1, x_2, ..., x_n$ in A such that $\mathcal{U}_0 = \{e^*clU_{x_i} : i = 1, 2, ..., n\}$ is also an α-shading of A. Now let $F \in \mathcal{F}$ be such that $F \le \bigcap_{i=1}^{n} F_{x_i}$. Then $(e^*clU_{x_i})(y) \le \alpha$, for all $y \in F$ and for $i = 1, 2, ..., n$. Thus $\mathcal{U}_0$ fails to be an α-shading of A, a contradiction.

Conversely, let the condition hold and suppose, if possible, $\{y_n : n \in (D, \ge)\}$ be a net in A having no α^{e^*}-adherent point in A. Then for each $x \in A$, there are a fuzzy e^*-open set U_x with $U_x(x) > \alpha$ and an $m_x \in D$ such that $(e^*clU_x)(y_n) \le \alpha$, for all $n \ge m_x$ $(n \in D)$. Thus $\mathcal{B} = \{F_x : x \in A\}$, where $F_x = \{y_n : n \ge m_x\}$ is a subbase for a filterbase $\mathcal{F}$ on A, where $\mathcal{F}$ consists of all finite intersections of members of $\mathcal{B}$. By hypothesis, $\mathcal{F}$ has an α^{e^*}-adherent point z (say) in A. But there are a fuzzy e^*-open set U_z with $U_z(z) > \alpha$ and an $m_z \in D$ such that $(e^*clU_z)(y_n) \le \alpha$, for all $n \ge m_z$, i.e., for all $p \in F_z \in \mathcal{B} (\subseteq \mathcal{F})$, $(e^*clU_z)(p) \le \alpha$ which implies that z cannot be an α^{e^*}-adherent point of the filterbase $\mathcal{F}$, a contradiction. Hence by Theorem 5.2, A is α - e^*-almost compact.

Putting $A = X$ in the characterization theorems so far, of α-e^*-almost compact crisp subset A, we obtain as follows :

Theorem 5.4. For an fts (X,τ), the following are equivalent:

(a) X is α-e^*-almost compact.

(b) For every family $\mathcal{U} = \{U_i : i \in \Lambda\}$ of fuzzy e^*-open sets in X such that $\{e^*int(e^*clU_i) : i \in \Lambda\}$ is an α-shading of X, there exists a finite subset Λ_0 of Λ such that $\{e^*clU_i : i \in \Lambda_0\}$ is an α-shading of X.

(c) For every collection $\{F_i : i \in \Lambda\}$ of fuzzy e^*-open sets in X with the property that for each finite subset Λ_0 of Λ, there is $x \in X$ such that $\inf_{i\in\Lambda_0} F_i(x) \geq 1-\alpha$, one has $\inf_{i\in\Lambda} (e^*clF_i)(y) \geq 1-\alpha$, for some $y \in X$.

(d) For every family $\{F_i : i \in \Lambda\}$ of fuzzy e^*-closed sets in X with α-e^*-IFIP in X, there exists $x \in X$ such that $\inf_{i\in\Lambda} F_i(x) \geq 1-\alpha$.

(e) Every net in X has an α^{e^*}-adherent point in X.

(f) Every filterbase on X has an α^{e^*}-adherent point in X.

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