$e^*$-Almost Compactness via $\alpha$-shading

Anjana Bhattacharyya

Department of Mathematics, Victoria Institution (College), 78 B, A.P.C. Road, Kolkata - 700009, India.

Abstract

This paper deals with a new type of compactness viz., $\alpha$-$e^*$-almost compactness for crisp subsets of a space $X$ where the underlying structure on $X$ is a fuzzy topology. Several characterizations are made by using ordinary net and power set filterbases as basic tools. It is shown that $\alpha$-$e^*$-almost compactness implies $\alpha$-almost compactness but the converse is true only in $\alpha$-$e^*$-regular space.

AMS Subject Classifications: 54A40, 54D20.

Keywords: $\alpha$-$e^*$-almost compactness, $\alpha$-$e^*$-almost compact set, $\alpha$-$e^*$-regularity, $\alpha^*$-adherent point of net and filterbase, $\alpha$-$e^*$-interiorly finite intersection property.

1. Introduction

The generalized version of fuzzy cover is $\alpha$-shading introduced by Gantner, Steinlage and Warren [9] in 1978. Using the concept of $\alpha$-shading they introduced and studied $\alpha$-compactness which is a generalization of compactness in a fuzzy topological space [6] (henceforth abbreviated as an fts, for short). Afterwards, $\alpha$-almost compactness [10], $\alpha$-$\delta_p$-almost compactness [2], $\alpha$-$\beta$-almost compactness [3] have been introduced and studied. In this paper using the definition of $\alpha$-shading, $\alpha$-$e^*$-almost compactness for crisp subsets in an fts is introduced and studied. This idea is also characterized by ordinary nets and power-set filterbases as basic appliances with the notion of adherence suitably defined via the fuzzy topology of the space. Afterwards, taking $A = X$, we get the characterizations of $\alpha$-$e^*$-almost compact space.

Corresponding author: Anjana Bhattacharyya, Department of Mathematics, Victoria Institution (College), 78 B, A.P.C. Road, Kolkata - 700009, India. E-mail: anjanabhattacharyya@hotmail.com.
2. Preliminaries

In what follows, by \((X, \tau)\) or simply by \(X\), we mean an fts in the sense of Chang [6]. A crisp set \(A\) in an fts \(X\) means an ordinary subset of the set \(X\) where the underlying structure of the set \(X\) being a fuzzy topology \(\tau\). A fuzzy set [13] \(A\) is a mapping from a nonempty set \(X\) into the closed interval \(I = [0,1]\) of the real line, i.e., \(A \in I^X\). For a fuzzy set \(A\), the fuzzy closure [6] and fuzzy interior [6] of \(A\) in \(X\) are denoted by \(clA\) and \(intA\) respectively. The support [13] of a fuzzy set \(A\) in \(X\) will be denoted by \(suppA\) and is defined by \(suppA = \{x \in X : A(x) \neq 0\}\). A fuzzy point [12] in \(X\) with the singleton support \(\{x\} \subseteq X\) and the value \(\alpha\) \((0 < \alpha \leq 1)\) at \(x\) will be denoted by \(x_\alpha\). For a fuzzy set \(A\), the complement [13] of \(A\) in \(X\) will be denoted by \(1_X \setminus A\) and is defined by \((1_X \setminus A)(x) = 1 - A(x)\), for each \(x \in X\). For any two fuzzy sets \(A\) and \(B\) in \(X\), we write \(A \leq B\) if \(A(x) \leq B(x)\), for each \(x \in X\) [13] while we write \(AqB\) if \(A\) is quasi-coincident \((q\)-coincident, for short) with \(B\) [12], i.e., if there exists \(x \in X\) such that \(A(x) + B(x) > 1\); the negation of these statements are written as \(A \leq/ B\) and \(AqB/\) respectively. A fuzzy set \(A\) is called fuzzy regular open [1] (resp., fuzzy preopen [11], fuzzy \(\beta\)-open [7]) if \(A = int(clA)\) (resp., \(A \leq int(clA), A \leq cl(int(clA))\)). The complement of a fuzzy preopen (resp., fuzzy \(\beta\)-open) set is called fuzzy preclosed [11] (resp., fuzzy \(\beta\)-closed [7]). A fuzzy set \(B\) is called a quasi-neighbourhood \((q\)-nbd, for short) [12] of a fuzzy point \(x_\xi\) if there is a fuzzy open set \(U\) in \(X\) such that \(x_\xi qU \leq B\). If, in addition, \(B\) is fuzzy open (resp., fuzzy regular open), then \(B\) is called a fuzzy open [12] (resp., fuzzy regular open [1]) \(q\)-nbd of \(x_\xi\). A fuzzy point \(x_\alpha\) is said to be a fuzzy \(\delta\)-cluster point of a fuzzy set \(A\) in an fts \(X\) if every fuzzy regular open \(q\)-nbd \(U\) of \(x_\alpha\) is q-coincident with \(A\) [8]. The union of all fuzzy \(\delta\)-cluster points of \(A\) is called the fuzzy \(\delta\)-closure of \(A\) and is denoted by \(\delta clA\) [8]. A fuzzy set \(A\) in an fts \(X\) is said to be fuzzy \(\delta\)-preopen [4] if \(A \leq int(\delta clA)\). The complement of a
fuzzy $\delta$-preopen set is called fuzzy $\delta$-preclosed [4]. A fuzzy set $A$ in an fts $X$ is called a fuzzy $\delta$-pre-$q$-nbd of a fuzzy point $x_\alpha$ in $X$ if there exists a fuzzy $\delta$-preopen set $V$ in $X$ such that $x_\alpha q V \leq A$ [4]. A fuzzy point $x_\alpha$ in an fts $X$ is called a fuzzy $\delta$-precluster point of a fuzzy set $A$ in $X$ if every fuzzy $\delta$-pre-$q$-nbd of $x_\alpha$ is q-coincident with $A$ [4]. The union of all fuzzy $\delta$-precluster points of $A$ is called the fuzzy $\delta$-preclosure of $A$ and will be denoted by $\delta \text{-pcl} A$ [4].

### 3. Fuzzy $e^*$-open Set: Some Properties

In this section we recall first the definition of fuzzy $e^*$-open set and some of its results from [5].

**Definition 3.1** [5]. A fuzzy set $A$ in an fts $X$ is said to be fuzzy $e^*$-open if $A \leq cl(int(\delta cl A))$.

The complement of a fuzzy $e^*$-open set is called fuzzy $e^*$-closed.

**Definition 3.2** [5]. The intersection of all fuzzy $e^*$-closed sets containing a fuzzy set $A$ in $X$ is called fuzzy $e^*$-closure of $A$, to be denoted by $e^* cl A$.

**Result 3.3** [5]. A fuzzy set $A$ in $X$ is fuzzy $e^*$-closed iff $A = e^* cl A$.

**Definition 3.4** [5]. The union of all fuzzy $e^*$-open sets contained in a fuzzy set $A$ in $X$ is called fuzzy $e^*$-interior of $A$, to be denoted by $e^* int A$.

**Result 3.5** [5]. A fuzzy set $A$ is fuzzy $e^*$-open iff $A = e^* int A$.

**Result 3.6** [5]. (i) For any fuzzy set $A$ in $X$, $x_\epsilon \in e^* cl A \Leftrightarrow U q A$ for any fuzzy $e^*$-open set $U$ in $X$ with $x_\epsilon q U$.

(ii) for any two fuzzy sets $U, V$ in $X$ where $V$ is fuzzy $e^*$-open set, $U q V \Rightarrow e^* cl U q V$.

**Result 3.7.** For a fuzzy set $A$ in an fts $(X, \tau)$, $e^* cl(e^* cl A) = e^* cl A$.

**Proof.** It is clear that $e^* cl A \leq e^* cl(e^* cl A)$.
Conversely, let \( x_a \in e^*\text{cl}(e^*\text{cl}A) \). Then for every fuzzy \( e^* \)-open set \( U \) of \( X \) with \( x_a q U \), \( Uq e^*\text{cl}A \). We have to show that \( UqA \). If possible, let \( UqA \). Then \( A \leq 1_x \setminus U \Rightarrow e^*\text{cl}A \leq e^*\text{cl}(1_x \setminus U) = 1_x \setminus U \Rightarrow Uq e^*\text{cl}A \), a contradiction. Consequently, \( e^*\text{cl}(e^*\text{cl}A) \leq e^*\text{cl}A \) and hence \( e^*\text{cl}(e^*\text{cl}A) = e^*\text{cl}A \), for any fuzzy set \( A \) in \( X \).

**Result 3.8.** For a fuzzy \( e^* \)-open set \( U \) in \( X \), \( e^*\text{cl}(e^*\text{int}(e^*\text{cl}U)) = e^*\text{cl}U \).

**Proof.** Let \( U \) be fuzzy \( e^* \)-open in \( X \). Then clearly \( U \leq e^*\text{cl}(e^*\text{int}(e^*\text{cl}U)) \Rightarrow e^*\text{cl}U \leq e^*\text{cl}\left(e^*\text{cl}\left(e^*\text{int}(e^*\text{cl}U)\right)\right) = e^*\text{cl}(e^*\text{int}(e^*\text{cl}U)) \) (by **Result 3.7**). Again \( e^*\text{cl}(e^*\text{int}(e^*\text{cl}U)) \leq e^*\text{cl}(e^*\text{cl}U) = e^*\text{cl}U \) (by **Result 3.7**). Hence \( e^*\text{cl}(e^*\text{int}(e^*\text{cl}U)) = e^*\text{cl}U \), for any fuzzy \( e^* \)-open set \( U \) in \( X \).

**Result 3.9.** For any two fuzzy sets \( A, B \) in an fts \( X \), \( e^*\text{cl}(A \lor B) = e^*\text{cl}A \lor e^*\text{cl}B \).

**Proof.** It is clear that \( e^*\text{cl}A \lor e^*\text{cl}B \leq e^*\text{cl}(A \lor B) \).

Conversely, let \( x_a \in e^*\text{cl}(A \lor B) \). Then for any fuzzy \( e^* \)-open set \( U \) in \( X \) with \( x_a q U \), \( Uq(A \lor B) \Rightarrow \) there exists \( y \in X \) such that \( U(y) + (A \lor B)(y) > 1 \Rightarrow U(y) + \max\{A(y), B(y)\} > 1 \Rightarrow \) either \( UqA \) or \( UqB \) \Rightarrow \) either \( x_a \in e^*\text{cl}A \) or \( x_a \in e^*\text{cl}B \) \Rightarrow \( x_a \in e^*\text{cl}A \lor e^*\text{cl}B \). Hence the proof.

**Remark 3.10.** The union (intersection) of any collection of fuzzy \( e^* \)-open (resp., fuzzy \( e^* \)-closed) sets in an fts \( X \) is fuzzy \( e^* \)-open (resp., fuzzy \( e^* \)-closed) in \( X \).

**Remark 3.11.** It is clear from **Remark 3.10** that \( e^*\text{cl}(\bigvee_{i=1}^n A_i) = \bigvee_{i=1}^n e^*\text{cl}A_i \) for fuzzy \( e^* \)-open sets \( A_1, A_2, \ldots, A_n \) in \( X \).
4. $\alpha$-$e^*$-Almost Compactness: Some Characterizations

The concept of $\alpha$-shading given by Gantner et al. [9] when applied to arbitrary crisp subsets of $X$ we get the following definition.

**Definition 4.1.** Let $X$ be an fts and $A$, a crisp subset of $X$. A collection $\mathcal{U}$ of fuzzy sets in $X$ is called an $\alpha$-shading (where $0 < \alpha < 1$) of $A$ if for each $x \in A$, there is some $U_x \in \mathcal{U}$ such that $U_x(x) > \alpha$ [9]. If, in addition, the members of $\mathcal{U}$ are fuzzy open (resp., fuzzy $e^*$-open) sets, then $\mathcal{U}$ is called a fuzzy open [9] (resp., fuzzy $e^*$-open) $\alpha$-shading of $A$.

**Definition 4.2.** Let $X$ be an fts and $A$, a crisp subset of $X$. $A$ is said to be $\alpha$-compact [9] (resp. $\alpha$-almost compact [10], $\alpha$-$\beta$-almost compact [3], $\alpha$-$p_\delta$-almost compact [2]) if each fuzzy open $\alpha$-shading $(0 < \alpha < 1) \mathcal{U}$ of $A$, there is a finite (resp. finite proximate, finite $\beta$-proximate, finite $p_\delta$-proximate) $\alpha$-subshading of $A$, i.e., there is a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that

$\{U : U \in \mathcal{U}_0\}$ (resp., $\{\text{cl}U : U \in \mathcal{U}_0\}$, $\{\beta\text{cl}U : U \in \mathcal{U}_0\}$, $\{\delta - p\text{cl}U : U \in \mathcal{U}_0\}$) is again an $\alpha$-shading of $A$. If $A = X$ in addition, then $X$ is called an $\alpha$-compact (resp., $\alpha$-almost compact, $\alpha$-$\beta$-almost compact, $\alpha$-$p_\delta$-almost compact) space.

Now we set the following definition.

**Definition 4.3.** Let $X$ be an fts and $A$, a crisp subset of $X$. $A$ is said to be $\alpha$-$e^*$-almost compact if each fuzzy $e^*$-open $\alpha$-shading $\mathcal{U}$ of $A$ has a finite $e^*$-proximate $\alpha$-subshading, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $U$ such that $\{e^*\text{cl}U : U \in \mathcal{U}_0\}$ is again an $\alpha$-shading of $A$.

If in particular $A = X$, we get the definition of $\alpha$-$e^*$-almost compact space $X$.

It then follows that

**Theorem 4.4.** (a) Every finite subset of an fts $X$ is $\alpha$-$e^*$-almost compact.

(b) If $A_1$ and $A_2$ are $\alpha$-$e^*$-almost compact subsets of an fts $X$, then so is $A_1 \cup A_2$. 

(c) $X$ is $\alpha - e^*$-almost compact if $X$ can be written as the union of finite number of $\alpha - e^*$-almost compact sets in $X$.

Since $e^* cl A \leq \beta cl A \leq pcl A \leq \delta - pcl A$ for any fuzzy set $A$ in an fts $X$, it is clear from Definition 4.2 and Definition 4.3 that $\alpha - e^*$-almost compactness implies $\alpha - \beta$-almost compactness implies $\alpha - p$-almost compactness implies $\alpha - \delta_p$-almost compactness.

Again as $e^* cl A \leq cl A$, for any fuzzy set $A$ in an fts $X$, it is clear from literature that $\alpha - e^*$-almost compactness implies $\alpha$-almost compactness, but not conversely. To achieve the converse we need to define some sort of regularity condition in our setting. The following definition serves our purpose.

**Definition 4.5.** An fts $X$ is said to be $\alpha - e^*$-regular, if for each point $x \in X$ and each fuzzy $e^*$-open set $U_x$ in $X$ with $U_x(x) > \alpha$, there exists a fuzzy open set $V_x$ in $X$ with $V_x(x) > \alpha$ such that $cl V_x \leq U_x$.

Two other equivalent ways of defining $\alpha - e^*$-regularity are given by the following theorem.

**Theorem 4.6.** For an fts $X$, the following are equivalent:

(a) $X$ is $\alpha - e^*$-regular.

(b) For each point $x \in X$ and each fuzzy $e^*$-closed set $F$ with $F(x) < 1 - \alpha$, there is a fuzzy open set $U$ such that $(cl U)(x) < 1 - \alpha$ and $F \leq U$.

(c) For each crisp point $x \in X$ and each fuzzy $e^*$-closed set $F$ with $F(x) < 1 - \alpha$, there exist fuzzy open sets $U$ and $V$ in $X$ such that $V(x) > \alpha$, $F \leq U$ and $U \cap V$.

**Proof.** (a) $\Rightarrow$ (b): Let $x \in X$ and $F$ be a fuzzy $e^*$-closed set with $F(x) < 1 - \alpha$. Put $V = 1_x \setminus F$. Then $V(x) > \alpha$ where $V$ is fuzzy $e^*$-open in $X$. By (a), there exists a fuzzy open set $W$ in $X$ with $W(x) > \alpha$ and $cl W \leq V = 1_x \setminus F$. Then $F \leq 1_x \setminus cl W = int (1_x \setminus W) = U$ (say).
Then $U$ is fuzzy open in $X$. Also, $clU = cl(int(1_x \setminus W)) = cl(1_x \setminus clW) = 1_x \setminus int(clW) \leq 1_x \setminus W$. Thus $(clU)(x) \leq (1_x \setminus W)(x) < 1 - \alpha$.

(b) $\Rightarrow$ (a): Let $x \in X$ and $U$ be fuzzy $e^*$-open set in $X$ with $U(x) > \alpha$. Let $F = 1_x \setminus U$. Then $F$ is fuzzy $e^*$-closed set in $X$ with $F(x) < 1 - \alpha$. By (b), there is a fuzzy open set $V$ in $X$ such that $(clV)(x) < 1 - \alpha$ and $F \leq V$. So $(1_x \setminus clV)(x) > \alpha$, i.e., $W(x) > \alpha$ where $W = 1_x \setminus clV = int(1_x \setminus V)$ is a fuzzy open set in $X$. Now $clW = cl(1_x \setminus clV) = 1_x \setminus int(clV) \leq 1_x \setminus V \leq 1_x \setminus F = U$. Hence (a) follows.

(b) $\Rightarrow$ (c): Let $x \in X$ and $F$, a fuzzy $e^*$-closed set in $X$ with $F(x) < 1 - \alpha$. By (b), there exists a fuzzy open set $U$ in $X$ such that $(clU)(x) < 1 - \alpha$ and $F \leq U$. Then $x_{1-\alpha} \notin clU$. Consequently, there is a fuzzy open set $V$ in $X$ such that $x_{1-\alpha} q V$ and $V q U$, i.e., $V(x) + 1 - \alpha > 1 \Rightarrow V(x) > \alpha$.

(c) $\Rightarrow$ (b): Let $x \in X$, and $F$, a fuzzy $e^*$-closed set in $X$ with $F(x) < 1 - \alpha$. By (c), there exist fuzzy open sets $U$ and $V$ in $X$ such that $V(x) > \alpha, F \leq U$ and $U q V$. Now $V(x) > \alpha \Rightarrow x_{1-\alpha} q V$. Then $U q V$, $clU q V \Rightarrow (clU)(x) \leq 1 - V(x) < 1 - \alpha$.

**Theorem 4.7.** In an $\alpha$-$e^*$-regular fts $X$, the $\alpha$-almost compactness of a crisp subset $A$ of $X$ implies its $\alpha$-$e^*$-almost compactness.

**Proof.** Let $\mathcal{U}$ be a fuzzy $e^*$-open $\alpha$-shading of an $\alpha$-almost compact set $A$ in an $\alpha$-$e^*$-regular fts $X$. Then for each $a \in A$, there exists $U_a \in \mathcal{U}$ such that $U_a(a) > \alpha$. By hypothesis, there is a fuzzy open set $V_a$ in $X$ with $V_a(a) > \alpha$ such that $clV_a \leq U_a$ ... (1).

Let $\mathcal{V} = \{V_a : a \in A\}$. Then $\mathcal{V}$ is a fuzzy open $\alpha$-shading of $A$. By $\alpha$-almost compactness of
Almost Compactness via $\alpha$-shading

For a crisp subset $A$ of an fts $X$, there is a finite subset $A_0$ of $A$ such that $V_0 = \{clV_a : a \in A_0\}$ is an $\alpha$-shading of $A$. Using (1), $U_0 = \{U_a : a \in A_0\}$ and hence $\{clU_a : a \in A_0\}$ is then a finite $\alpha$-subshading of $U$. Hence $A$ is $\alpha$-$e^*$-almost compact space.

**Theorem 4.8.** A crisp subset $A$ of an fts $X$ is $\alpha$-$e^*$-almost compact if and only if every family of fuzzy $e^*$-open sets, the $e^*$-interiors of whose $e^*$-closures form an $\alpha$-shading of $A$, contains a finite subfamily, the $e^*$-closures of whose members form an $\alpha$-shading of $A$.

**Proof.** It is sufficient to prove that for a fuzzy $e^*$-open set $U$, $U \leq e^{int}(e^{cl}U) \leq e^{cl}(e^{int}(e^{cl}U)) = e^{cl}U$ (By Result 3.8).

**Theorem 4.9.** A crisp subset $A$ of an fts $X$ is $\alpha$-$e^*$-almost compact if and only if for every collection $\{F_i : i \in \Lambda\}$ of fuzzy $e^*$-open sets in $X$ with the property that for each finite subset $\Lambda_0$ of $\Lambda$, there is $x \in A$ such that $\inf_{i \in \Lambda_0} F_i(x) \geq 1 - \alpha$, one has $\inf_{i \in \Lambda} (e^{cl}F_i)(y) \geq 1 - \alpha$, for some $y \in A$.

**Proof.** Let $A$ be $\alpha$-$e^*$-almost compact. If possible, let for a collection $\{F_i : i \in \Lambda\}$ of fuzzy $e^*$-open sets in $X$ with the property that for each finite subset $\Lambda_0$ of $\Lambda$, there is $x \in A$ such that $\inf_{i \in \Lambda_0} F_i(x) \geq 1 - \alpha$, but $\inf_{i \in \Lambda} (e^{cl}F_i)(x) < 1 - \alpha$, for all $x \in A$. Then $\alpha < (1_X \cap \bigcap_{i \in \Lambda} e^{cl}F_i)(x) = \bigcup_{i \in \Lambda} (1_X \setminus e^{cl}F_i)(x)$, for each $x \in A$ which shows that $\{1_X \setminus e^{cl}F_i : i \in \Lambda\}$ is a fuzzy $e^*$-open $\alpha$-shading of $A$. By assumption, there is a finite subset $\Delta_0$ of $\Lambda$ such that $\{e^{cl}(1_X \setminus e^{cl}F_i) : i \in \Delta_0\} = \{1_X \setminus e^{int}(e^{cl}F_i) : i \in \Delta_0\}$ is an $\alpha$-shading of $A$. Hence $\alpha < \bigcup_{i \in \Delta_0} (1_X \setminus e^{int}e^{cl}F_i))(x) = [1_X \setminus (\bigcap_{i \in \Lambda_0} e^{int}(e^{cl}F_i))](x)$, for each $x \in A$. Then $(\bigcap F_i)(x) \leq [\bigcap_{i \in \Delta_0} e^{int}(e^{cl}F_i)](x) < 1 - \alpha$, for each $x \in A$, which contradicts our assumption.

Conversely, let under the given hypothesis, $A$ be not $\alpha$-$e^*$-almost compact. Then there exists a
fuzzy $e^*$-open $\alpha$-shading $U = \{U_i : i \in \Lambda\}$ of $A$ having no finite $e^*$-proximate $\alpha$-subshading, i.e., for every finite subset $\Lambda_0$ of $\Lambda$, $\{e^*clU_i : i \in \Lambda_0\}$ is not an $\alpha$-shading of $A$, i.e., there exists $x \in A$ such that $\sup_{i \in \Lambda_0}(e^*clU_i)(x) \leq \alpha$, i.e., $1_x \setminus \sup_{i \in \Lambda_0}(e^*clU_i)(x) = \inf_{i \in \Lambda_0}(1_x \setminus e^*clU_i)(x) \geq 1 - \alpha$.

Hence $\{1_x \setminus e^*clU_i : i \in \Lambda\}$ is a family of fuzzy $e^*$-open sets with the stated property. Consequently, there is some $y \in A$ such that $\inf_{i \in \Lambda}(e^*cl(1_x \setminus e^*clU_i))(y) \geq 1 - \alpha$. Then $\sup_{i \in \Lambda}(y) \leq \sup_{i \in \Lambda}(e^*int(e^*clU_i))(y) = 1 - \inf_{i \in \Lambda}(e^*int(e^*clU_i))(y) = 1 - \inf_{i \in \Lambda}(e^*cl(1_x \setminus e^*clU_i))(y) \leq \alpha$. This shows that $\{U_i : i \in \Lambda\}$ fails to be an $\alpha$-shading of $A$, a contradiction.

**Definition 4.10.** A family $\{F_i : i \in \Lambda\}$ of fuzzy sets in an fts $X$ is said to have $\alpha$-$e^*$-interiorly finite intersection property ($\alpha$-$e^*$-IFIP, for short) in a subset $A$ of $X$, if for each finite subset $\Lambda_0$ of $\Lambda$, there exists $x \in A$ such that $\bigcap_{i \in \Lambda_0} e^*intF_i(x) \geq 1 - \alpha$.

**Theorem 4.11.** A crisp subset $A$ of an fts $X$ is $\alpha$-$e^*$-almost compact if and only if for every family $\mathcal{F} = \{F_i : i \in \Lambda\}$ of fuzzy $e^*$-closed sets in $X$ with $\alpha$-$e^*$-IFIP in $A$, there exists $x \in A$ such that $\inf_{i \in \Lambda} F_i(x) \geq 1 - \alpha$.

**Proof.** Let $\mathcal{F} = \{F_i : i \in \Lambda\}$ be a family of fuzzy $e^*$-closed sets in $X$ with $\alpha$-$e^*$-IFIP in $A$ where $A$ is $\alpha$-$e^*$-almost compact subset of $X$. If possible, let for each $x \in A$, $\inf_{i \in \Lambda} F_i(x) < 1 - \alpha$, i.e., $\bigcap_{i \in \Lambda} F_i(x) < 1 - \alpha$ and hence $\bigcup_{i \in \Lambda}(1_x \setminus F_i)(x) > \alpha$. Therefore $\mathcal{U} = \{1_x \setminus F_i : i \in \Lambda\}$ is a fuzzy $e^*$-open $\alpha$-shading of $A$. By hypothesis, there exists a finite subfamily $\Lambda_0$ of $\Lambda$ such that $\bigcup_{i \in \Lambda_0}(e^*cl(1_x \setminus F_i))(x) > \alpha$, i.e., $1 - (\bigcap_{i \in \Lambda_0} e^*intF_i)(x) > \alpha$, i.e., $\bigcap_{i \in \Lambda_0} e^*intF_i(x) < 1 - \alpha$, for each $x \in A$, which shows that $\mathcal{F}$ does not have $\alpha$-$e^*$-IFIP in $A$, a contradiction.

Conversely, let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be a fuzzy $e^*$-open $\alpha$-shading of $A$. Then...
$\mathcal{F} = \{1_X \setminus U_i : i \in \Lambda\}$ is a family of fuzzy $\epsilon^*$-closed sets in $X$ with $\inf_{i \in \Lambda} (1_X \setminus U_i)(x) < 1 - \alpha$, for each $x \in A$, so that $\mathcal{F}$ does not have $\alpha$-$\epsilon^*$-IFIP in $A$. Then there exists some finite subset $\Lambda_0$ of $\Lambda$ such that for each $x \in A$, $\left( \bigcap_{i \in \Lambda_0} e^* \text{int} (1_X \setminus U_i) \right)(x) < 1 - \alpha$. Then $\mathcal{F}$ is a family of fuzzy $\epsilon^*$-closed sets in $X$ with $\inf_{i \in \Lambda} (1_X \setminus U_i)(x) < 1 - \alpha$, for each $x \in A$.

### 5. Characterizations of $\alpha$-$\epsilon^*$-Almost Compactness Via Ordinary Nets and Filterbases

In this section, $\alpha$-$\epsilon^*$-almost compactness of a crisp subset $A$ of an fts $X$ is characterized via $\alpha^*$-adherent points of ordinary nets and power-set filterbases.

Let us now introduce the following definition:

**Definition 5.1.** Let $\{S_n : n \in (D, \succeq)\}$ (where $(D, \succeq)$ is a directed set) be an ordinary net in $A$ and $\mathcal{F}$ be a power-set filterbase on $A$, and $x \in X$ be any crisp point in $X$. Then $x$ is called an $\alpha^*$-adherent point of

(a) the net $\{S_n\}$ if for each fuzzy $\epsilon^*$-open set $U$ in $X$ with $U(x) > \alpha$ and for each $m \in D$, there exists $k \in D$ such that $k \succeq m$ in $D$ and $(\epsilon^* \text{cl} U)(S_k) > \alpha$,

(b) the filterbase $\mathcal{F}$ if for each fuzzy $\epsilon^*$-open set $U$ with $U(x) > \alpha$ and for each $F \in \mathcal{F}$, there exists a crisp point $x_F$ in $F$ such that $(\epsilon^* \text{cl} U)(x_F) > \alpha$.

**Theorem 5.2.** A crisp subset $A$ of an fts $X$ is $\alpha$-$\epsilon^*$-almost compact if and only if every net in $A$ has an $\alpha^*$-adherent point in $A$.

**Proof.** Let a crisp subset $A$ of an fts $X$ be $\alpha$-$\epsilon^*$-almost compact. If possible, let there be a net $\{S_n : n \in (D, \succeq)\}$ in $A$ ($(D, \succeq)$ being a directed set, as usual) having no $\alpha^*$-adherent point in $A$. 


Then for each \( x \in A \), there is a fuzzy \( e^* \)-open set \( U_x \) in \( X \) with \( U_x(x) > \alpha \) and an \( m_x \in D \) such that \( (e^*\text{cl} U_x)(S_n) \leq \alpha \), for all \( n \geq m_x \) (\( n \in D \)). Now, \( \mathcal{U} = \{1_X \setminus e^*\text{cl} U_x : x \in A \} \) is a collection of fuzzy \( e^* \)-open sets in \( X \) such that for any finite subcollection \( \{1_X \setminus e^*\text{cl} U_{x_i} : i = 1, 2, \ldots, k \} \) (say) of \( \mathcal{U} \), there exists \( m \in D \) with \( m \geq m_x, i = 1, 2, \ldots, k \) in \( D \) such that \( (\bigcup_{i=1}^k e^*\text{cl} U_{x_i})(S_n) \leq \alpha \), for all \( n \geq m \), i.e., \( \inf_{x \in A} (1_X \setminus e^*\text{cl} U_x) \geq 1 - \alpha \), for all \( n \geq m \). Hence by Theorem 4.9, there exists some \( y \in A \) such that \( \inf_{x \in A} (1_X \setminus e^*\text{cl} U_x)(y) \geq 1 - \alpha \), i.e.,

\[
(\bigcup_{x \in A} U_x)(y) \geq [\bigcup_{x \in A} e^*\text{int}(e^*\text{cl} U_x)](y)
= 1 - [1 - (\bigcup_{x \in A} e^*\text{int}(e^*\text{cl} U_x))(y)]
= 1 - \inf_{x \in A} (e^*\text{cl}(1-e^*\text{cl} U_x))(y) \leq 1 - 1 + \alpha = \alpha.
\]

We have, in particular, \( U_y(y) \leq \alpha \), contradicting the definition of \( U_y \). Hence the result is proved.

Conversely, suppose that every net in \( A \) has an \( \alpha^* \)-adherent point in \( A \). Let \( \{F_i : i \in \Lambda \} \) be an arbitrary collection of fuzzy \( e^* \)-open sets in \( X \). Let \( \Lambda_f \) denote the collection of all finite subsets of \( \Lambda \), then \( (\Lambda_f, \supseteq) \) is a directed set, where for \( \mu, \lambda \in \Lambda_f, \mu \supseteq \lambda \) iff \( \mu \supseteq \lambda \). For each \( \mu \in \Lambda_f \), put

\[
F_{\mu} = \bigcap \{F_i : i \in \mu \}.
\]

Let for each \( \mu \in \Lambda_f \), there be a point \( x_\mu \in A \) such that

\[
\inf_{x \in \mu} F_i(x_\mu) \geq 1 - \alpha \ldots(1).
\]

Then by Theorem 4.9 it is enough to show that \( \inf_{i \in \Lambda} (e^*\text{cl} F_i)(z) \geq 1 - \alpha \) for some \( z \in A \). If possible, let \( \inf_{i \in \Lambda} (e^*\text{cl} F_i)(z) < 1 - \alpha \), for each \( z \in A \) \ldots(2).

Now, \( S = \{x_\mu : \mu \in (\Lambda_f, \supseteq)\} \) is clearly a net of points in \( A \). By hypothesis, there is an \( \alpha^* \)-adherent point \( z \) in \( A \) of this net. By (2), \( \inf_{i \in \Lambda} (e^*\text{cl} F_i)(z) < 1 - \alpha \Rightarrow \) there exists \( i_0 \in \Lambda \) such that

\[
(e^*\text{cl} F_{i_0})(z) < 1 - \alpha \], i.e., \( 1 - e^*\text{cl} F_{i_0}(z) > \alpha \). Since \( z \) is an \( \alpha^* \)-adherent point of \( S \), for the index \( \{i_0\} \in \Lambda_f \), there is \( \mu_0 \in \Lambda_f \) with \( \mu_0 \supseteq \{i_0\} \) (i.e., \( i_0 \in \mu_0 \)) such that \( e^*\text{cl}(1_X \setminus e^*\text{cl} F_{i_0})(x_{\mu_0}) > \alpha \), i.e.,
(\epsilon^* \text{int} (\epsilon^* \text{cl} F_y))(x_{l_y}) < 1 - \alpha. \quad \text{Since } \inf_{t = 0}^{\mu_l} F_t(x_{l_y}) \leq F_{l_y}(x_{l_y}) \leq (\epsilon^* \text{int} (\epsilon^* \text{cl} F_y))(x_{l_y}) < 1 - \alpha,

which contradicts (1). This completes the proof.

**Theorem 5.3.** A crisp subset \( A \) of an fts \( X \) is \( \alpha - \epsilon^* \)-almost compact if and only if every filterbase \( \mathcal{F} \) on \( A \) has an \( \alpha^\epsilon \)-adherent point in \( A \).

**Proof.** Let \( A \) be \( \alpha - \epsilon^* \)-almost compact and if possible, let there be a filterbase \( \mathcal{F} \) on \( A \) having no \( \alpha^\epsilon \)-adherent point in \( A \). Then for each \( x \in A \), there exist a fuzzy \( \epsilon^* \)-open set \( U_x \) with \( U_x(x) > \alpha \), and an \( F_x \in \mathcal{F} \) such that \( (\epsilon^* \text{cl} U_y)(y) \leq \alpha \), for each \( y \in F_x \). Then \( U = \{ U_x : x \in A \} \) is a fuzzy \( \epsilon^* \)-open \( \alpha \)-shading of \( A \). Thus there exist finitely many points \( x_1, x_2, \ldots, x_n \) in \( A \) such that \( U_i = \{ (\epsilon^* \text{cl} U_{x_i}) : i = 1, 2, \ldots, n \} \) is also an \( \alpha \)-shading of \( A \). Now let \( F \in \mathcal{F} \) be such that \( F \leq \bigcap_{i=1}^{n} F_{x_i} \).

Then \( (\epsilon^* \text{cl} U_{x_i})(y) \leq \alpha \), for all \( y \in F \) and for \( i = 1, 2, \ldots, n \). Thus \( U_0 \) fails to be an \( \alpha \)-shading of \( A \), a contradiction.

Conversely, let the condition hold and suppose, if possible, \( \{ y_n : n \in (D, \geq) \} \) be a net in \( A \) having no \( \alpha^\epsilon \)-adherent point in \( A \). Then for each \( x \in A \), there are a fuzzy \( \epsilon^* \)-open set \( U_x \) with \( U_x(x) > \alpha \) and an \( m_x \in D \) such that \( (\epsilon^* \text{cl} U_x)(y_n) \leq \alpha \), for all \( n \geq m_x \) \( (n \in D) \). Thus \( B = \{ F_x : x \in A \} \), where \( F_x = \{ y_n : n \geq m_x \} \) is a subbase for a filterbase \( \mathcal{F} \) on \( A \), where \( \mathcal{F} \) consists of all finite intersections of members of \( B \). By hypothesis, \( \mathcal{F} \) has an \( \alpha^\epsilon \)-adherent point \( z \) (say) in \( A \). But there are a fuzzy \( \epsilon^* \)-open set \( U_z \) with \( U_z(z) > \alpha \) and an \( m_z \in D \) such that \( (\epsilon^* \text{cl} U_z)(y_n) \leq \alpha \), for all \( n \geq m_z \), i.e., for all \( p \in F_z \in B (\subseteq \mathcal{F}) \), \( (\epsilon^* \text{cl} U_z)(p) \leq \alpha \) which implies that \( z \) cannot be an \( \alpha^\epsilon \)-adherent point of the filterbase \( \mathcal{F} \), a contradiction. Hence by Theorem 5.2, \( A \) is \( \alpha - \epsilon^* \)-almost compact.
Putting $A = X$ in the characterization theorems so far, of $\alpha - e^\ast$-almost compact crisp subset $A$, we obtain as follows:

**Theorem 5.4.** For an fts $(X, \tau)$, the following are equivalent:

(a) $X$ is $\alpha - e^\ast$-almost compact.

(b) For every family $\mathcal{U} = \{U_i : i \in \Lambda\}$ of fuzzy $e^\ast$-open sets in $X$ such that $\{e^* \text{int}(e^\ast \text{cl} U_i) : i \in \Lambda\}$ is an $\alpha$-shading of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $\{e^\ast \text{cl} U_i : i \in \Lambda_0\}$ is an $\alpha$-shading of $X$.

(c) For every collection $\{F_i : i \in \Lambda\}$ of fuzzy $e^\ast$-open sets in $X$ with the property that for each finite subset $\Lambda_0$ of $\Lambda$, there is $x \in X$ such that $\inf_{i \in \Lambda_0} F_i(x) \geq 1 - \alpha$, one has $\inf_{i \in \Lambda} (e^\ast \text{cl} F_i)(y) \geq 1 - \alpha$, for some $y \in X$.

(d) For every family $\{F_i : i \in \Lambda\}$ of fuzzy $e^\ast$-closed sets in $X$ with $\alpha - e^\ast$-IFIP in $X$, there exists $x \in X$ such that $\inf_{i \in \Lambda} F_i(x) \geq 1 - \alpha$.

(e) Every net in $X$ has an $\alpha^\ast$-adherent point in $X$.

(f) Every filterbase on $X$ has an $\alpha^\ast$-adherent point in $X$.

**References**


[5]. Bhattacharyya, Anjana; *Several concepts of continuity in fuzzy setting*, e-Proceedings, The 10th International


*Published: Volume 2018, Issue 4 / April 25, 2018*