\(\alpha e^*-\text{Closed Set}\)

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Abstract

This paper deals with a new type of crisp set viz., \(\alpha e^*\)-closed set which inherits \(\alpha\)-\(e^*\)-almost compactness of a space \(X\) where the underlying structure on \(X\) is a fuzzy topology. Also a new type of continuous-like function between two fuzzy topological spaces is introduced under which \(\alpha\)-\(e^*\)-almost compactness for crisp subsets remains invariant.

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1. Introduction

It is clear from definition that taking the idea of fuzzy cover given by Chang [4], many mathematicians have engaged themselves to introduce different types of compactness in fuzzy set theory. Gantner et al. [6] generalized the concept of fuzzy cover in 1978 by introducing \(\alpha\)-shading where \(0<\alpha<1\). Using this idea, in [3], \(\alpha\)-\(e^*\)-almost compactness for crisp set is introduced and studied. In [2], fuzzy \(e^*\)-open set is introduced. Here we introduce a crisp set, viz., \(\alpha e^*\)-closed set where the underlying topology is a fuzzy topology.

2. Preliminaries

In what follows, by \((X, \tau)\) or simply by \(X\), we mean an fts in the sense of Chang [4]. A crisp set \(A\) in an fts \(X\) means an ordinary subset of the set \(X\) where the underlying structure of the set \(X\) being a fuzzy topology \(\tau\). A fuzzy set [8] \(A\) is a mapping from a nonempty set \(X\) into the closed interval \([0,1]\) of the real line, i.e., \(A \in I^X\). For a fuzzy set \(A\), the fuzzy closure [4] and fuzzy interior [4] of \(A\) in \(X\) are denoted by \(clA\) and \(intA\) respectively. The support [8] of a fuzzy set \(A\) in \(X\) will be denoted by \(suppA\) and is defined by \(suppA = \{x \in X : A(x) \neq 0\}\). A fuzzy point [7] in \(X\) with the singleton support \(\{x\} \subseteq X\) and the value \(\alpha\) \((0 < \alpha \leq 1)\) at \(x\) will be denoted by \(x_\alpha\). For a fuzzy set \(A\), the complement [8] of \(A\) in \(X\) will be denoted by \(1 - A\) and is defined by \((1 - A)(x) = 1 - A(x)\), for each \(x \in X\). For any two fuzzy sets \(A\) and \(B\) in \(X\), we write \(A \leq B\)

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if $A(x) \leq B(x)$, for each $x \in X$ [8] while we write $AqB$ if $A$ is quasi-coincident ($q$-coincident, for short) with $B$ [7], i.e., if there exists $x \in X$ such that $A(x) + B(x) > 1$; the negation of these statements are written as $A \leq B$ and $AqB$ respectively. A fuzzy set $A$ is called fuzzy regular open [1] if $A = \text{int}(\text{cl}A)$. A fuzzy set $B$ is called a quasi-neighbourhood ($q$-nbd, for short) [7] of a fuzzy point $x$, if there exists $x \in X$ such that $x, qU \leq B$. If, in addition, $B$ is fuzzy open (resp., fuzzy regular open), then $B$ is called a fuzzy open [7] (resp., fuzzy regular open [1]) $q$-nbd of $x$. A fuzzy point $x_a$ is said to be a fuzzy $\delta$-cluster point of a fuzzy set $A$ in an fts $X$ if every fuzzy regular open q-nbd $U$ of $x_a$ is q-coincident with $A$ [5]. The union of all fuzzy $\delta$-cluster points of $A$ is called the fuzzy $\delta$-closure of $A$ and is denoted by $\delta \text{cl} A$ [5].

3. Fuzzy $e^*$-open Set: Some Properties

In this section we recall first the definition of fuzzy $e^*$-open set and some of its results from [2].

**Definition 3.1** [2]. A fuzzy set $A$ in an fts $X$ is said to be fuzzy $e^*$-open if $A \leq cl(int(\delta cl A))$. The complement of a fuzzy $e^*$-open set is called fuzzy $e^*$-closed.

**Definition 3.2** [2]. The intersection of all fuzzy $e^*$-closed sets containing a fuzzy set $A$ in $X$ is called fuzzy $e^*$-closure of $A$, to be denoted by $e \text{cl} A$.

**Result 3.3** [2]. A fuzzy set $A$ in $X$ is fuzzy $e^*$-closed iff $A = e \text{cl} A$.

**Definition 3.4** [2]. The union of all fuzzy $e^*$-open sets contained in a fuzzy set $A$ in $X$ is called fuzzy $e^*$-interior of $A$, to be denoted by $e \text{int} A$.

**Result 3.5** [2]. A fuzzy set $A$ is fuzzy $e^*$-open iff $A = e \text{int} A$.

**Result 3.6** [2]. (i) For any fuzzy set $A$ in $X$, $x, \in e \text{cl} A \Leftrightarrow U qA$ for any fuzzy $e^*$-open set $U$ in $X$ with $x, qU$.

(ii) For any two fuzzy sets $U, V$ in $X$ where $V$ is fuzzy $e^*$-open set, $U qV \Rightarrow e^* cl U qV$.

**Definition 3.7.** Let $X$ be an fts and $A$, a crisp subset of $X$. A collection $U$ of fuzzy sets in $X$ is called an $\alpha$-shading (where $0 < \alpha < 1$) of $A$ if for each $x \in A$, there is some $U, \in U$ such that $U, (x) > \alpha$ [6]. If, in addition, the members are fuzzy open (resp., fuzzy $e^*$-open) sets, then $U$ is called a fuzzy open [6] (resp., fuzzy $e^*$-open [3]) $\alpha$-shading of $A$.

**Definition 3.8** [3]. Let $X$ be an fts and $A$, a crisp subset of $X$. $A$ is said to be $\alpha$-$e^*$-almost compact if each fuzzy $e^*$-open $\alpha$-shading $U$ of $A$ has a finite $e^*$-proximate $\alpha$-subshading, i.e., there exists a finite subcollection $U, \in U$ of $U$ such that $\{e^* cl U : U \in U, \}$ is again an $\alpha$-shading of $A$. If, in particular $A = X$, we get the definition of $\alpha$-$e^*$-almost compact space $X$. 123
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4. \[\alpha^e\]-Closed Set: Some Properties

Let us now introduce a class of crisp sets in an fts \(X\), as follows.

**Definition 4.1.** Let \((X, \tau)\) be an fts and \(A \subseteq X\). A point \(x \in X\) is said to be an \[\alpha^e\]-limit point of \(A\) if for every fuzzy \(e^*\)-open set \(U\) in \(X\) with \(U(x) > \alpha\), there exists \(y \in A \setminus \{x\}\) such that \((e^*\text{cl}U)(y) > \alpha\). The set of all \[\alpha^e\]-limit points of \(A\) will be denoted by \([A]^e_{\alpha}\).

The \[\alpha^e\]-closure of \(A\), to be denoted by \(\alpha^e\text{-cl}A\), is defined by \(\alpha^e\text{-cl}A = \bigcup\{A\}^e_{\alpha}\).

**Definition 4.2.** A crisp subset \(A\) of an fts \(X\) is said to be \[\alpha^e\]-closed if it contains all its \[\alpha^e\]-limit points. Any subset \(B\) of \(X\) is called \[\alpha^e\]-open if \(X \setminus B\) is \[\alpha^e\]-closed.

**Remark 6.3.** It is clear from Definition 4.1 that for any set \(A \subseteq X\), \([A]^e_{\alpha}\) \(\subseteq \alpha^e\text{-cl}A\) and \(\alpha^e\text{-cl}A = A\) if and only if \([A]^e_{\alpha}\) \(\subseteq \alpha^e\text{-cl}A\).

**Theorem 4.4.** An \[\alpha^e\]-closed subset \(A\) of an \(\alpha\)-\(e^*\)-almost compact space \(X\) is \(\alpha\)-\(e^*\)-almost compact.

**Proof.** Let \(A(\subseteq X)\) be an \[\alpha^e\]-closed set in an \(\alpha\)-\(e^*\)-almost compact space \(X\). Then for any \(x \notin A\), there is a fuzzy \(e^*\)-open set \(U_x\) in \(X\) such that \(U_x(x) > \alpha\), and \((e^*\text{cl}U_x)(y) \leq \alpha\), for every \(y \in A\). Consider the collection \(U = \{U_x : x \notin A\}\). For proving \(A\) to be \(\alpha\)-\(e^*\)-almost compact, consider a fuzzy \(e^*\)-open \(\alpha\)-shading \(V\) of \(A\). Clearly \(U \cup V\) is a fuzzy \(e^*\)-open \(\alpha\)-shading of \(X\). Since \(X\) is \(\alpha\)-\(e^*\)-almost compact, there exists a finite subcollection \(\{V_1, V_2, \ldots, V_n\}\) of \(U \cup V\) such that for every \(t \in X\), there exists \(V_t(1 \leq i \leq n)\) such that \((e^*\text{cl}V_t)(t) > \alpha\). For every member \(U_x\) of \(U\), \((e^*\text{cl}U_x)(y) \leq \alpha\), for every \(y \in A\). So if this subcollection contains any member of \(U\), we omit it and hence we get the result.

To achieve the converse of Theorem 4.4, we define the following.

**Definition 4.5.** An fts \((X, \tau)\) is said to be \(\alpha\)-\(e^*\)-Urysohn if for any two distinct points \(x, y\) of \(X\), there exist a fuzzy open set \(U\) and a fuzzy \(e^*\)-open set \(V\) in \(X\) with \(U(x) > \alpha\), \(V(y) > \alpha\) and \(\min((e^*\text{cl}U)(z), (e^*\text{cl}V(z))) \leq \alpha\), for each \(z \in X\).

**Theorem 4.6.** An \(\alpha\)-\(e^*\)-almost compact set in an \(\alpha\)-\(e^*\)-Urysohn space \(X\) is \(\alpha^e\)-closed.

**Proof.** Let \(A\) be an \(\alpha\)-\(e^*\)-almost compact set and \(x \in X \setminus A\). Then for each \(y \in A\), \(x \neq y\). As \(X\) is \(\alpha\)-\(e^*\)-Urysohn, there exist a fuzzy open set \(U_x\) and a fuzzy \(e^*\)-open set \(V_y\) in \(X\) such that \(U_x(x) > \alpha, V_y(y) > \alpha\) and \(\min((e^*\text{cl}U_x)(z), (e^*\text{cl}V_y(z))) \leq \alpha\), for all \(z \in X\) ... (1).
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Then \(U = \{V_y : y \in A\}\) is a fuzzy \(e^*\)-open \(\alpha\)-shading of \(A\) and so by \(\alpha - e^*\)-almost compactness of \(A\), there exist finitely many points \(y_1, y_2, \ldots, y_n\) of \(A\) such that 
\(U_0 = \{e^*clV_{y_1}, e^*clV_{y_2}, \ldots, e^*clV_{y_n}\}\) is again an \(\alpha\)-shading of \(A\). Now \(U = U_{y_1} \cap \ldots \cap U_{y_n}\) being a fuzzy open set is a fuzzy \(e^*\)-open \(\alpha\)-shading of \(A\) and so by \(\alpha\)-\(e^*\)-almost compactness of \(A\), there exist finitely many points \(1, 2, \ldots, n\) of \(A\) such that 
\(U_{y_1} \cap \ldots \cap U_{y_n}\) is again an \(\alpha\)-shading of \(A\). Now \(U = U_{y_1} \cup \ldots \cup U_{y_n}\) being a fuzzy open set is a fuzzy \(e^*\)-open set in \(X\) such that 
\((e^*cl U_{y_1})(z) > \alpha\), for some \(k\) \((1 \leq k \leq n)\). Also 
\((e^*cl U_{y_k})(z) > \alpha\). Hence 
\(\min((e^*cl U_{y_1})(z), (e^*cl V_{y_k})(z)) > \alpha\), contradicting (1).

**Corollary 4.7.** In an \(\alpha - e^*\)-almost compact, \(\alpha - e^*\)-Urysohn space \(X\), a subset \(A\) of \(X\) is \(\alpha - e^*\)-almost compact if and only if it is \(\alpha - e^*\)-closed.

**Theorem 4.8.** In an \(\alpha - e^*\)-almost compact space \(X\), every cover of \(X\) by \(\alpha - e^*\)-open sets has a finite subcover.

**Proof.** Let \(U = \{U_i : i \in \Lambda\}\) be a cover of \(X\) by \(\alpha - e^*\)-open sets. Then for each \(x \in X\), there exists \(U_x \in U\) such that \(x \in U_x\). Since \(X \setminus U_x\) is \(\alpha - e^*\)-closed, there exists a fuzzy \(e^*\)-open set \(V_x\) in \(X\) such that \(V_x(x) > \alpha\) and \((e^*cl V_x)(y) \leq \alpha\), for each \(y \in X \setminus U_x\) ... (1).

Then \(\{V_x : x \in X\}\) forms a fuzzy \(e^*\)-open \(\alpha\)-shading of the \(\alpha - e^*\)-almost compact space \(X\). Thus there exists a finite subset \(\{x_1, x_2, \ldots, x_n\}\) of \(X\) such that \(\{e^*cl V_{x_i} : i = 1, 2, \ldots, n\}\) is an \(\alpha\)-shading of \(X\) ... (2).

We claim that \(\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}\) is a finite subcover of \(U\). If not, then there exists \(y \in X \setminus \bigcup_{i=1}^{n} U_{x_i} = \bigcap_{i=1}^{n} (X \setminus U_{x_i})\). Then by (1), \((e^*cl V_{x_i})(y) \leq \alpha\), for \(i = 1, 2, \ldots, n\) and so 
\((\bigcup_{i=1}^{n} e^*cl V_{x_i})(y) \leq \alpha\), contradicting (2).

**Theorem 4.9.** Let \((X, \tau)\) be an fts. If \(X\) is \(\alpha - e^*\)-almost compact, then every collection of \(\alpha - e^*\)-closed sets in \(X\) with finite intersection property has non-empty intersection.

**Proof.** Let \(\mathcal{F} = \{F_i : i \in \Lambda\}\) be a collection of \(\alpha - e^*\)-closed sets in an \(\alpha - e^*\)-almost compact space \(X\) having finite intersection property. If possible, let \(\bigcap_{i \in \Lambda} F_i = \emptyset\). Then \(X \setminus \bigcap_{i \in \Lambda} F_i = \bigcup_{i \in \Lambda} (X \setminus F_i) = X \Rightarrow U = \{X \setminus F_i : i \in \Lambda\}\) is an \(\alpha - e^*\)-open cover of \(X\). Then by Theorem 4.8, there is a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(\bigcup_{i \in \Lambda_0} (X \setminus F_i) = X \Rightarrow \bigcap_{i \in \Lambda_0} F_i = \emptyset\), a contradiction.
In this section, we now introduce a class of functions under which \( \alpha - e^* \)-almost compactness remains invariant.

**Definition 5.1.** Let \( X, Y \) be fts’s. A function \( f : X \rightarrow Y \) is said to be \( \alpha - e^* \)-continuous if for each point \( x \in X \) and each fuzzy \( e^* \)-open set \( V \) in \( Y \) with \( V(f(x)) > \alpha \), there exists a fuzzy \( e^* \)-open set \( U \) in \( X \) with \( U(x) > \alpha \) such that \( e^* cl U \subseteq f^{-1}(e^* cl V) \).

**Theorem 5.2.** If \( f : X \rightarrow Y \) is \( \alpha - e^* \)-continuous (where \( X, Y \) are, as usual, fts’s), then the following are true:

(a) \( f([A]_\alpha^\alpha) \subseteq [f(A)]_\alpha^\alpha \), for every \( A \subseteq X \).

(b) \( [f^{-1}(A)]_\alpha^\alpha \subseteq f^{-1}([A]_\alpha^\alpha) \), for every \( A \subseteq Y \).

(c) For each \( \alpha - \)open set \( A \) in \( Y \), \( f^{-1}(A) \) is \( \alpha - \)closed in \( X \).

(d) For each \( \alpha - \)closed set \( A \) in \( Y \), \( f(A) \) is \( \alpha - \)open in \( X \).

**Proof.**

(a). Let \( x \in [A]_\alpha^\alpha \) and \( U \) be any fuzzy \( e^* \)-open set in \( Y \) with \( U(x) > \alpha \). Then there is a fuzzy \( e^* \)-open set \( V \) in \( X \) such that \( V(x) > \alpha \) and \( \alpha < e^* cl V(x_0) \leq (f^{-1}(e^* cl U))(x_0) = (e^* cl U)(f(x_0)) \) where \( f(x_0) \in f(A) \setminus \{f(x)\} \Rightarrow f(x) \in [f(A)]_\alpha^\alpha \). Thus (a) follows.

(b) By (a), \( f([f^{-1}(A)]_\alpha^\alpha) \subseteq [f^{-1}(A)]_\alpha^\alpha \subseteq [A]_\alpha^\alpha \Rightarrow [f^{-1}(A)]_\alpha^\alpha \subseteq f^{-1}([A]_\alpha^\alpha) \).

(c) We have \( [A]_\alpha^\alpha = A \). By (b), \( [f^{-1}(A)]_\alpha^\alpha \subseteq f^{-1}([A]_\alpha^\alpha) = f^{-1}(A) \Rightarrow f^{-1}(A) \Rightarrow f^{-1}(A) \) is \( \alpha - \)closed set in \( X \).

(d) Follows from (c).

**Theorem 5.3.** Let \( X, Y \) be fts’s and \( f : X \rightarrow Y \) be fuzzy \( \alpha - e^* \)-continuous function. If \( A(\subseteq X) \) is \( \alpha - e^* \)-almost compact set in \( X \), then so is \( f(A) \) in \( Y \).

**Proof.** Let \( V = \{ V_i : i \in \Lambda \} \) be a fuzzy \( e^* \)-open \( \alpha \)-shading of \( f(A) \), where \( A \) is \( \alpha - e^* \)-almost compact set in \( X \). For each \( x \in A \), \( f(x) \in f(A) \) and so there exists \( V_x \in \mathcal{V} \) such that \( V_x(f(x)) > \alpha \). As \( f \) is fuzzy \( \alpha - e^* \)-continuous, there exists a fuzzy \( e^* \)-open set \( U_x \) in \( X \) such that \( U_x(x) > \alpha \) and \( f(e^* cl U_x) \leq e^* cl V_x \). Then \( \{ U_x : x \in A \} \) is a fuzzy \( e^* \)-open \( \alpha \)-shading of \( A \). By \( \alpha - e^* \)-almost compactness of \( A \), there are finitely many points \( a_1, a_2, ..., a_n \) in \( A \) such that \( \{ e^* cl U_{a_i} : i = 1, 2, ..., n \} \) is again an \( \alpha \)-shading of \( A \).
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We claim that \( \{ e^{c}clV_{a_i} : i=1,2,...,n \} \) is an \( \alpha \)-shading of \( f(A) \). In fact, \( y \in f(A) \Rightarrow \) there exists \( x \in A \) such that \( y = f(x) \). Now there is a \( U_{a_j} \) (for some \( j, 1 \leq j \leq n \)) such that \( (e^{c}clU_{a_j})(x) > \alpha \) and hence \( (e^{c}clV_{a_j})(y) \geq f(e^{c}clU_{a_j})(y) \geq e^{c}clU_{a_j}(x) > \alpha \).

We now introduce a function under which \( \alpha^{e^*} \)-closedness of a set remains invariant.

**Definition 5.4.** Let \( X,Y \) be fts’s. A function \( f : X \rightarrow Y \) is said to be fuzzy \( e^{*} \)-open if \( f(A) \) is fuzzy \( e^{*} \)-open in \( Y \) whenever \( A \) is fuzzy \( e^{*} \)-open in \( X \).

**Remark 5.5.** For a fuzzy \( e^{*} \)-open function \( f : X \rightarrow Y \), for every fuzzy \( e^{*} \)-closed set \( A \) in \( X \), \( f(A) \) is fuzzy \( e^{*} \)-closed in \( Y \).

**Theorem 5.6.** If \( f : (X, \tau) \rightarrow (Y, \tau_{1}) \) is a bijective fuzzy \( e^{*} \)-open function, then the image of a \( \alpha^{e^*} \)-closed set in \( (X, \tau) \) is \( \alpha^{e^*} \)-closed in \( (Y, \tau_{1}) \).

**Proof.** Let \( A \) be a \( \alpha^{e^*} \)-closed set in \( (X, \tau) \) and let \( y \in Y \setminus f(A) \). Then there exists a unique \( z \in X \) such that \( f(z) = y \). As \( y \notin f(A) \), \( z \notin A \). Now, \( A \) being \( \alpha^{e^*} \)-closed in \( X \), there exists a fuzzy \( e^{*} \)-open set \( V \) in \( X \) such that \( V(z) > \alpha \) and \( e^{c}clV(p) \leq \alpha \), for each \( p \in A \) ... (1).

As \( f \) is fuzzy \( e^{*} \)-open, \( f(V) \) is a fuzzy \( e^{*} \)-open set in \( Y \), and also \( (f(V))(y) = V(z) > \alpha \).

Let \( t \in f(A) \). Then there is a unique \( t_0 \in A \) such that \( f(t_0) = t \). As \( f \) is bijective and fuzzy \( e^{*} \)-open, by Remark 5.5, \( e^{c}clf(V) \leq f(e^{c}clV) \). Then \( (e^{c}clf(V))(t) \leq f(e^{c}clV)(t) = e^{c}clV(t_0) \leq \alpha \), by (1). Thus \( y \) is not an \( \alpha^{e^*} \)-limit point of \( f(A) \). Hence the proof.

From Theorem 5.2 (c) and Theorem 5.6, it follows that

**Corollary 5.7.** Let \( f : X \rightarrow Y \) be a fuzzy \( \alpha^{e^*} \)-continuous, bijective and fuzzy \( e^{*} \)-open function. Then \( A \) is \( \alpha^{e^*} \)-closed in \( Y \) if and only if \( f^{-1}(A) \) is \( \alpha^{e^*} \)-closed in \( X \).

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