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α^{e^*}-Closed Set

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Abstract

This paper deals with a new type of crisp set viz., α^{e^*}-closed set which inherits α-e^*-almost compactness of a space X where the underlying structure on X is a fuzzy topology. Also a new type of continuous-like function between two fuzzy topological spaces is introduced under which α-e^*-almost compactness for crisp subsets remains invariant.

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1. Introduction

It is clear from definition that taking the idea of fuzzy cover given by Chang [4], many mathematicians have engaged themselves to introduce different types of compactness in fuzzy set theory. Gantner et al. [6] generalized the concept of fuzzy cover in 1978 by introducing α-shading where $0<\alpha<1$. Using this idea, in [3], α-e^*-almost compactness for crisp set is introduced and studied. In [2], fuzzy e^*-open set is introduced. Here we introduce a crisp set, viz., α^{e^*}-closed set where the underlying topology is a fuzzy topology.

2. Preliminaries

In what follows, by (X,τ) or simply by X, we mean an fts in the sense of Chang [4]. A crisp set A in an fts X means an ordinary subset of the set X where the underlying structure of the set X being a fuzzy topology τ. A fuzzy set [8] A is a mapping from a nonempty set X into the closed interval $I=[0,1]$ of the real line, i.e., $A\in I^X$. For a fuzzy set A, the fuzzy closure [4] and fuzzy interior [4] of A in X are denoted by clA and $intA$ respectively. The support [8] of a fuzzy set A in X will be denoted by $suppA$ and is defined by $suppA=\{x\in X : A(x)\neq 0\}$. A fuzzy point [7] in X with the singleton support $\{x\}\subseteq X$ and the value α $(0<\alpha\leq 1)$ at x will be denoted by x_α. For a fuzzy set A , the complement [8] of A in X will be denoted by $1_X\setminus A$ and is defined by $(1_X\setminus A)(x)=1-A(x)$, for each $x\in X$. For any two fuzzy sets A and B in X, we write $A\leq B$

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if $A(x) \leq B(x)$, for each $x \in X$ [8] while we write AqB if A is quasi-coincident (q-coincident, for short) with B [7], i.e., if there exists $x \in X$ such that $A(x)+B(x)>1$; the negation of these statements are written as $A \not\leq B$ and $A\not{q}B$ respectively. A fuzzy set A is called fuzzy regular open [1] if $A = int(clA)$. A fuzzy set B is called a quasi-neighbourhood (q-nbd, for short) [7] of a fuzzy point x_t if there is a fuzzy open set U in X such that $x_t qU \leq B$. If, in addition, B is fuzzy open (resp., fuzzy regular open), then B is called a fuzzy open [7] (resp., fuzzy regular open [1]) q-nbd of x_t. A fuzzy point x_α is said to be a fuzzy δ-cluster point of a fuzzy set A in an fts X if every fuzzy regular open q-nbd U of x_α is q-coincident with A [5]. The union of all fuzzy δ-cluster points of A is called the fuzzy δ-closure of A and is denoted by δclA [5].

3. Fuzzy e^*-open Set: Some Properties

In this section we recall first the definition of fuzzy e^*-open set and some of its results from [2].

Definition 3.1 [2]. A fuzzy set A in an fts X is said to be fuzzy e^*-open if $A \leq cl(int(\delta clA))$. The complement of a fuzzy e^*-open set is called fuzzy e^*-closed.

Definition 3.2 [2]. The intersection of all fuzzy e^*-closed sets containing a fuzzy set A in X is called fuzzy e^*-closure of A, to be denoted by e^*clA.

Result 3.3 [2]. A fuzzy set A in X is fuzzy e^*-closed iff $A = e^*clA$.

Definition 3.4 [2]. The union of all fuzzy e^*-open sets contained in a fuzzy set A in X is called fuzzy e^*-interior of A, to be denoted by e^*intA.

Result 3.5 [2]. A fuzzy set A is fuzzy e^*-open iff $A = e^*intA$.

Result 3.6 [2]. (i) For any fuzzy set A in X, $x_t \in e^*clA \Leftrightarrow UqA$ for any fuzzy e^*-open set U in X with $x_t qU$.

(ii) for any two fuzzy sets U, V in X where V is fuzzy e^*-open set, $U\not{q}V \Rightarrow e^*clU\not{q}V$.

Definition 3.7. Let X be an fts and A, a crisp subset of X. A collection $\mathcal{U}$ of fuzzy sets in X is called an α-shading (where $0 < \alpha < 1$) of A if for each $x \in A$, there is some $U_x \in \mathcal{U}$ such that $U_x(x) > \alpha$ [6]. If, in addition, the members are fuzzy open (resp., fuzzy e^*-open) sets, then $\mathcal{U}$ is called a fuzzy open [6] (resp., fuzzy e^*-open [3]) α-shading of A.

Definition 3.8 [3]. Let X be an fts and A, a crisp subset of X. A is said to be α-e^*-almost compact if each fuzzy e^*-open α-shading $\mathcal{U}$ of A has a finite e^*-proximate α-subshading, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{e^*clU : U \in \mathcal{U}_0\}$ is again an α-shading of A. If, in particular $A = X$, we get the definition of α-e^*-almost compact space X.

4. α^{e^*}-Closed Set: Some Properties

Let us now introduce a class of crisp sets in an fts X, as follows.

Definition 4.1. Let (X,τ) be an fts and $A \subseteq X$. A point $x \in X$ is said to be an α^{e^*}-limit point of A if for every fuzzy e^*-open set U in X with $U(x) > \alpha$, there exists $y \in A \setminus \{x\}$ such that $(e^*clU)(y) > \alpha$. The set of all α^{e^*}-limit points of A will be denoted by $[A]^{\alpha}_{e^*}$.

The α^{e^*}-closure of A, to be denoted by α^{e^*}-clA, is defined by α^{e^*}-c$lA = A \bigcup [A]^{\alpha}_{e^*}$.

Definition 4.2. A crisp subset A of an fts X is said to be α^{e^*}-closed if it contains all its α^{e^*}-limit points. Any subset B of X is called α^{e^*}-open if $X \setminus B$ is α^{e^*}-closed.

Remark 6.3. It is clear from Definition 4.1 that for any set $A \subseteq X$, $A \subseteq \alpha^{e^*}$-clA and α^{e^*}-$clA = A$ if and only if $[A]^{\alpha}_{e^*} \subseteq A$. Again it follows from Definition 4.1 that A is α^{e^*}-closed if and only if α^{e^*}-$clA = A$. It is also clear that $A \subseteq B \subseteq X \Rightarrow [A]^{\alpha}_{e^*} \subseteq [B]^{\alpha}_{e^*}$.

Theorem 4.4. An α^{e^*}-closed subset A of an α-e^*-almost compact space X is α-e^*-almost compact.

PROOF. Let $A(\subseteq X)$ be an α^{e^*}-closed set in an α-e^*-almost compact space X. Then for any $x \notin A$, there is a fuzzy e^*-open set U_x in X such that $U_x(x) > \alpha$, and $(e^*clU_x)(y) \leq \alpha$, for every $y \in A$. Consider the collection $\mathcal{U} = \{U_x : x \notin A\}$. For proving A to be α-e^*-almost compact, consider a fuzzy e^*-open α-shading $\mathcal{V}$ of A. Clearly $\mathcal{U} \bigcup \mathcal{V}$ is a fuzzy e^*-open α-shading of X. Since X is α-e^*-almost compact, there exists a finite subcollection $\{V_1, V_2, ..., V_n\}$ of $\mathcal{U} \bigcup \mathcal{V}$ such that for every $t \in X$, there exists $V_i (1 \leq i \leq n)$ such that $(e^*clV_i)(t) > \alpha$. For every member U_x of $\mathcal{U}$, $(e^*clU_x)(y) \leq \alpha$, for every $y \in A$. So if this subcollection contains any member of $\mathcal{U}$, we omit it and hence we get the result.

To achieve the converse of Theorem 4.4, we define the following.

Definition 4.5. An fts (X,τ) is said to be α-e^*-Urysohn if for any two distinct points x, y of X, there exist a fuzzy open set U and a fuzzy e^*-open set V in X with $U(x) > \alpha$, $V(y) > \alpha$ and $min((e^*clU)(z), (e^*clV(z)) \leq \alpha$, for each $z \in X$.

Theorem 4.6. An α-e^*-almost compact set in an α-e^*-Urysohn space X is α^{e^*}-closed.

Proof. Let A be an α-e^*-almost compact set and $x \in X \setminus A$. Then for each $y \in A$, $x \neq y$. As X is α-e^*-Urysohn, there exist a fuzzy open set U_y and a fuzzy e^*-open set V_y in X such that $U_y(x) > \alpha, V_y(y) > \alpha$ and $min((e^*clU_y)(z), (e^*clV_y)(z)) \leq \alpha$, for all $z \in X$ ... (1).

Then $\mathcal{U}=\{V_y : y\in A\}$ is a fuzzy e^*-open α-shading of A and so by α-e^*-almost compactness of A, there exist finitely many points $y_1,y_2,...,y_n$ of A such that $\mathcal{U}_0=\{e^*clV_{y_1},e^*clV_{y_2},...,e^*clV_{y_n}\}$ is again an α-shading of A. Now $U=U_{y_1}\bigcap...\bigcap U_{y_n}$ being a fuzzy open set is a fuzzy e^*-open set in X such that $U(x)>\alpha$. In order to show that A to be α^{e^*}-closed, it now suffices to show that $(e^*clU)(y)\le\alpha$, for each $y\in A$. In fact, if for some $z\in A$, we assume $(e^*clU)(z)>\alpha$, then as $z\in A$, we have $(e^*clV_{y_k})(z)>\alpha$, for some k $(1\le k\le n)$. Also $(e^*clU_{y_k})(z)>\alpha$. Hence $min((e^*clU_{y_k})(z),(e^*clV_{y_k})(z))>\alpha$, contradicting (1).

Corollary 4.7. In an α-e^*-almost compact, α-e^*-Urysohn space X, a subset A of X is α-e^*-almost compact if and only if it is α^{e^*}-closed.

Theorem 4.8. In an α-e^*-almost compact space X, every cover of X by α^{e^*}-open sets has a finite subcover.

Proof. Let $\mathcal{U}=\{U_i : i\in\Lambda\}$ be a cover of X by α^{e^*}-open sets. Then for each $x\in X$, there exists $U_x\in\mathcal{U}$ such that $x\in U_x$. Since $X\setminus U_x$ is α^{e^*}-closed, there exists a fuzzy e^*-open set V_x in X such that $V_x(x)>\alpha$ and $(e^*clV_x)(y)\le\alpha$, for each $y\in X\setminus U_x$ ... (1).

Then $\{V_x : x\in X\}$ forms a fuzzy e^*-open α-shading of the α-e^*-almost compact space X. Thus there exists a finite subset $\{x_1,x_2,...,x_n\}$ of X such that $\{e^*clV_{x_i} : i=1,2,...,n\}$ is an α-shading of X ... (2).

We claim that $\{U_{x_1},U_{x_2},...,U_{x_n}\}$ is a finite subcover of $\mathcal{U}$. If not, then there exists $y\in X\setminus\bigcup_{i=1}^{n}U_{x_i}=\bigcap_{i=1}^{n}(X\setminus U_{x_i})$. Then by (1), $(e^*clV_{x_i})(y)\le\alpha$, for $i=1,2,...,n$ and so $(\bigcup_{i=1}^{n}e^*clV_{x_i})(y)\le\alpha$, contradicting (2).

Theorem 4.9. Let (X,τ) be an fts. If X is α-e^*-almost compact, then every collection of α^{e^*}-closed sets in X with finite intersection property has non-empty intersection.

Proof. Let $\mathcal{F}=\{F_i : i\in\Lambda\}$ be a collection of α^{e^*}-closed sets in an α-e^*-almost compact space X having finite intersection property. If possible, let $\bigcap_{i\in\Lambda}F_i=\phi$. Then $X\setminus\bigcap_{i\in\Lambda}F_i=\bigcup_{i\in\Lambda}(X\setminus F_i)=X\Rightarrow\mathcal{U}=\{X\setminus F_i : i\in\Lambda\}$ is an α^{e^*}-open cover of X. Then by Theorem 4.8, there is a finite subset Λ_0 of Λ such that $\bigcup_{i\in\Lambda_0}(X\setminus F_i)=X\Rightarrow\bigcap_{i\in\Lambda_0}F_i=\phi$, a contradiction.

5. α^{e^*}-Continuity: Some Applications

In this section, we now introduce a class of functions under which α-e^*-almost compactness remains invariant.

Definition 5.1. Let X, Y be fts's. A function $f : X \to Y$ is said to be α^{e^*}-continuous if for each point $x \in X$ and each fuzzy e^*-open set V in Y with $V(f(x)) > \alpha$, there exists a fuzzy e^*-open set U in X with $U(x) > \alpha$ such that $e^*clU \le f^{-1}(e^*clV)$.

Theorem 5.2. If $f : X \to Y$ is α^{e^*}-continuous (where X, Y are, as usual, fts's), then the following are true :

(a) $f([A]^{\alpha}_{e^*}) \subseteq [f(A)]^{\alpha}_{e^*}$, for every $A \subseteq X$.

(b) $[f^{-1}(A)]^{\alpha}_{e^*} \subseteq f^{-1}([A]^{\alpha}_{e^*})$, for every $A \subseteq Y$.

(c) For each α^{e^*}-closed set A in Y, $f^{-1}(A)$ is α^{e^*}-closed in X.

(d) For each α^{e^*}-open set A in Y, $f^{-1}(A)$ is α^{e^*}-open in X.

Proof (a). Let $x \in [A]^{\alpha}_{e^*}$ and U be any fuzzy e^*-open set in Y with $U(f(x)) > \alpha$. Then there is a fuzzy e^*-open set V in X with $V(x) > \alpha$ and $e^*clV \le f^{-1}(e^*clU)$. Now $x \in [A]^{\alpha}_{e^*}$ and V is a fuzzy e^*-open set in X with $V(x) > \alpha \Rightarrow e^*clV(x_0) > \alpha$, for some $x_0 \in A \setminus \{x\}$ $\Rightarrow \alpha < e^*clV(x_0) \le (f^{-1}(e^*clU))(x_0) = (e^*clU)(f(x_0))$ where $f(x_0) \in f(A) \setminus \{f(x)\} \Rightarrow f(x) \in [f(A)]^{\alpha}_{e^*}$. Thus (a) follows.

(b) By (a), $f([f^{-1}(A)]^{\alpha}_{e^*}) \subseteq [ff^{-1}(A)]^{\alpha}_{e^*} \subseteq [A]^{\alpha}_{e^*} \Rightarrow [f^{-1}(A)]^{\alpha}_{e^*} \subseteq f^{-1}([A]^{\alpha}_{e^*})$.

(c) We have $[A]^{\alpha}_{e^*} = A$. By (b), $[f^{-1}(A)]^{\alpha}_{e^*} \subseteq f^{-1}([A]^{\alpha}_{e^*}) = f^{-1}(A) \Rightarrow [f^{-1}(A)]^{\alpha}_{e^*} = f^{-1}(A) \Rightarrow$ $f^{-1}(A)$ is α^{e^*}-closed set in X.

(d) Follows from (c).

Theorem 5.3. Let X, Y be fts's and $f : X \to Y$ be fuzzy α^{e^*}-continuous function. If $A (\subseteq X)$ is α-e^*-almost compact set in X, then so is $f(A)$ in Y.

Proof. Let $V = \{V_i : i \in \Lambda\}$ be a fuzzy e^*-open α-shading of $f(A)$, where A is α-e^*-almost compact set in X. For each $x \in A$, $f(x) \in f(A)$ and so there exists $V_x \in \mathcal{V}$ such that $V_x(f(x)) > \alpha$. As f is fuzzy α^{e^*}-continuous, there exists a fuzzy e^*-open set U_x in X such that $U_x(x) > \alpha$ and $f(e^*clU_x) \le e^*clV_x$. Then $\{U_x : x \in A\}$ is a fuzzy e^*-open α-shading of A. By α-e^*-almost compactness of A, there are finitely many points $a_1, a_2, ..., a_n$ in A such that $\{e^*clU_{a_i} : i = 1, 2, ..., n\}$ is again an α-shading of A.

We claim that $\{e^*clV_{a_i} : i = 1, 2, ..., n\}$ is an α-shading of $f(A)$. In fact, $y \in f(A) \Rightarrow$ there exists $x \in A$ such that $y = f(x)$. Now there is a U_{a_j} (for some $j, 1 \le j \le n$) such that $(e^*clU_{a_j})(x) > \alpha$ and hence $(e^*clV_{a_j})(y) \ge f(e^*clU_{a_j})(y) \ge e^*clU_{a_j}(x) > \alpha$.

We now introduce a function under which α^{e^*}-closedness of a set remains invariant.

Definition 5.4. Let X, Y be fts's. A function $f : X \to Y$ is said to be fuzzy e^*-open if $f(A)$ is fuzzy e^*-open in Y whenever A is fuzzy e^*-open in X.

Remark 5.5. For a fuzzy e^*-open function $f : X \to Y$, for every fuzzy e^*-closed set A in X, $f(A)$ is fuzzy e^*-closed in Y.

Theorem 5.6. If $f : (X, \tau) \to (Y, \tau_1)$ is a bijective fuzzy e^*-open function, then the image of a α^{e^*}-closed set in (X, τ) is α^{e^*}-closed in (Y, τ_1).

Proof. Let A be a α^{e^*}-closed set in (X, τ) and let $y \in Y \setminus f(A)$. Then there exists a unique $z \in X$ such that $f(z) = y$. As $y \notin f(A)$, $z \notin A$. Now, A being α^{e^*}-closed in X, there exists a fuzzy e^*-open set V in X such that $V(z) > \alpha$ and $e^*clV(p) \le \alpha$, for each $p \in A$ ... (1).

As f is fuzzy e^*-open, $f(V)$ is a fuzzy e^*-open set in Y, and also $(f(V))(y) = V(z) > \alpha$. Let $t \in f(A)$. Then there is a unique $t_0 \in A$ such that $f(t_0) = t$. As f is bijective and fuzzy e^*-open, by Remark 5.5, $e^*clf(V) \le f(e^*clV)$. Then $(e^*clf(V))(t) \le f(e^*clV)(t) = e^*clV(t_0) \le \alpha$, by (1). Thus y is not an α^{e^*}-limit point of $f(A)$. Hence the proof.

From Theorem 5.2 (c) and Theorem 5.6, it follows that

Corollary 5.7. Let $f : X \to Y$ be a fuzzy α^{e^*}-continuous, bijective and fuzzy e^*-open function. Then A is α^{e^*}-closed in Y if and only if $f^{-1}(A)$ is α^{e^*}-closed in X.

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