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The Non-commutative Geometry on the Compactification of Matrix Model

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Abstract

Using the harmonic analysis of the three and two spheres, we study the compactification of the IKKT model on this spheres surface. Like the tori and orbifolds, we show that there exists here also a possibility to see clearly the compactifications of matrix models of M-theory on non commutative geometry.

Keywords: Matrix Model; Non-commutative geometry, compactification.

1. Introduction

The matrix model formulation of M-theory is described by maximally super-symmetric U(N) gauge quantum mechanics in the large N limit [1]. This model to which we shall refer hereafter to as the BFSS model may be obtained from 10d supersymmetric Yang Mills (SYM) theory by means of dimensional reduction. The BFSS model is closely related to another basic model often known as the IKKT matrix theory [2] and obtained by reduction of 10d SYM theory to a point [3]. Recently an important development has been made in the study of toroidal compactification of M-theory using the matrix model approach where it has been shown that non commutative geometry ideas emerge in a natural way. Indeed it has been shown in [3] that the general solutions of the constraint Eqs defining the toroidal compactification of the IKKT model are related to the value of the flux of the three form potential of 11d supergravity [3, 4a, 4b], see also [5].

The general solutions go beyond the standard solutions involving commutative tori which are recovered as a special case. Using the solution of the periodicity constraints of the variables of the IKKT model compactified on, one gets a 2d SYM theory on non commutative torus [3, 4, 6]. Recent analysis concerning AdS/CFT correspondance in particular type IIB on $Ads_5{*}S^5/Z_3$ with 4d N=1 supersymmetric $su(N)^3$ gauge theory with fundamental matter [7] involve also results which seem to have some thing to do with the non commutative torus. Since the Connes et al. development, several types of toroidal and orbifold compactifications of matrix models using non commutative geometry ideas have been studied [8, 9] and general results have been obtained. The aim of this paper is to contribute to these efforts by extending the results of [3] to the compactification of the IKKT model on higher dimensional compact manifolds which have no one cycles. only. In other words we want to extend the analysis made where the role of the one cycles S^1 of the two torus $T^2=S^1xS^1$ in the Connes et al's analysis are played by the irreducible two cycles S^2 in the present study. Generalizations of this

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construction to higher 2k compact cycles involving k S^2 spheres is also possible. To address this question we have first to identify the defining constraint Eqs of the compactification of the IKKT model on F_0 . To that purpose we shall use the harmonic analysis of the three sphere S^3=SU(2) and its isometries to write down the compactification on S^2. The latter is obtained from S^3 by gauging out the Cartan subsymmetry of SU(2). The compactification on S^2*S^2 is then obtained by using S^3*S^3 = SU(2) x SU(2) and gauging out the subgroup U(1)*U(1). The presentation of this paper is as follows : In section 2 we review briefly the main lines of the toroidal compactification of the IKKT model. This study, which is also valid for the BFSS matrix theory, allows us to give a reformulation of the defining constraints Eq of Banks et al, useful when discussing the compactification on S^2. In section 3 we give the solution of the constraints of periodicity in presence of winding numbers. In section 4 we consider the compactification on S^2 by using the above mentioned harmonic analysis. We first derive the compactification constraints Eqs on S^3; then we give their solution for both S^3 and S^2.

2. Compactification of Matrix Theory

As mentioned earlier we focus our attention in this section on the compact-ification of the IKKT model on the two torus. First we review the defining constraint Eqs of the toroidal compactification as well as the solutions involving winding numbers. This analysis is useful when we discuss the compactification on S^2. To start recall that the IKKT model is a zero dimen-sional supersymmetric field theory reproducing the usual Banks et al. matrix model of M-theory up to a compactification on S^1. The physical degrees of freedom of the IKKT model are given by a 10d, N=1 supersymmetric multiple $(X^{\mu},\psi_A);\mu=1,....,10;A=1,...,16$ in the adjoint representation of the u(N) Lie algebra. The action $S(X,\Psi)$ of this model reads as:

$$S(X,\Psi)=R\,Tr\begin{pmatrix}\left[X_{\mu},X_{\nu}\right]\left[X^{\mu},X^{\nu}\right]\\+2\psi\Gamma^{\mu}\left[X_{\mu},\psi\right]\end{pmatrix} \tag{1}$$

For more details on Eq(1) see [2]. Note in passing that X_{μ} and Ψ_A are respectively ten components and sixteen ones of the SO(10) vector and Weyl spinor representations. Note also that in addition to the 10d =1 supersymmetry and SO(10) symmetry; Eq(1) has moreover a U(N) gauge invariance acting as an automorphism group on the algebra. Thus, under U(N) a gauge transformation g, the sand X_s^{μ} and ψ_A^s transform as:

$$X^{\mu}\to gX^{\mu}g^{-1};\Psi_A\to g\psi_A g^{-1} \tag{2}$$

In practice g may be expressed in terms of the u(N) Lie algebra generators

$$X^{\mu}\to gX^{\mu}g^{-1}\left\{T^a;a=0,1,...,(N^2-1\right\}$$

as:

$$g=\exp(i\alpha^a T^a) \tag{3}$$

where the constants $\left\{\alpha^0,\vec{\alpha}\right\}$ are the U(N)=U(1)XSU(N) group parameters.

For the simplicity of the exposé, we shall focuss ourselves hereafter on the analysis of the bosonic sector of the IKKT model; the inclusion of fermions is a priori straightforward. The compactification of the matrix model (1) on a two torus of radii R_1 and R_2 is achieved by identifying $X_i + 2\pi R_i ; i = 1, 2$ X_i+2 with the X_s^i themselves up to some U(N) gauge transformations. Introducing the winding numbers n_1 and n_2 on the two one cycles of the two torus T^2, the defining constraint Eqs of the compactification on T^2 reads as:

$$\begin{aligned} &X_1 + 2\pi n_1 R_1 = g X_1 g_1^{-1} \\ &X_2 + 2\pi n_2 R_2 = g_2 X_2 g_2^{-1} \\ &X_\mu = g_i X_\mu g_i^{-1}; \mu \neq i \end{aligned} \tag{4}$$

In Eq(4), g_1 and g_2 are U(N) gauge transformations of respective groups parameters $\{\alpha_1^a\}$ and $\{\alpha_2^a\}$ gas in Eq (3) but with some details to be specified later on. Before considering the solution of Eqs (4), we want to note the three following: (a) Finite u(N) matrices cannot satisfy Eqs (4) as shown by taking the trace on both sides. (b) Eqs (4) involve two gauge transformations g_1 and g_2. In general the number g_i^s is exactly as the number of irreducible cycles of the compact manifold on which the matrix model is compactified on. (c) The constraint Eqs (4) may be rewritten in the following remarkable form:

$$\begin{aligned} &[X_1, g_1] = -2\pi n_1 R_1 g_1 \\ &[X_2, g_2] = -2\pi n_2 R_2 g_2 \\ &[X_\mu, g_i] = 0; \mu \neq i; i = 1, 2 \end{aligned} \tag{5}$$

The above formulation of constraints of the compactification of the IKKT model on T^2 shows that the g_1 and g_2 gauge transformations are eigenvectors of the adjoint action of the X_s^μ and so is any monomial operator of the form:

$$g = g_{N_1, N_2; M_1, M_2} = g_1^{N_1} g_2^{N_2} g_1^{-M_1} g_2^{-M_2} \tag{6}$$

Where N_1, M_1, N_2 and M_2 are arbitrary integers. Using Eqs (5), one can easily check that the composite U(N) gauge transformation (6) is also an eigenvector of the adjoint action of the X_s^μ of eigenvalue $\{-\delta_{\mu i}(n_i (N_i - M_i) R_i\}$, namely:

$$[X_\mu, g] = -n_i (N_i - M_i) R_i g \delta_{\mu i} \tag{7}$$

Observe that Eq (7) is just a condensate way of writing Eqs(5). Another property of Eq (7) is that in the special case where *Ni=Mi* that is for the monomial

$$g_{N_1, N_2} = g_1^{N_1} g_2^{N_2} g_1^{-M_1} g_2^{-M_2} \tag{8}$$

the rhs of Eq(7) vanishes identically and consequently g_{N_1, N_2} should belong to the center of the U(N) automorphism group. In other words we should have:

$$g_{N_1,N_2} = \lambda_{N_1 N_{12}} I \tag{9}$$

where $\lambda_{N_1 N_{12}}$ are complex parameters which in the present case they are unimodular. Actually Eqs(8-9) seem to go beyond the defining constraint Eqs of the non commutative torus of [3] where n_1; N_1; n_2 and N_2 are taken equal to one. Although one can usually put Eqs (8-9) into the form $G_1G_2 = \lambda G_2G_1$ by making the change $G_1 = g_1^{N_1}.G_2 = g_2^{N_2}$, the physical interpretation shows that our solutions are general as they carry manifestly the solitonic effects of topological charges n_1 and n_2. On the other hand our way of doing shows that one needs at least two irreducible one cycles in order to get non commutative geometry. Taking $n_2=0$ for instance, Eqs (8-9) become trivial. This feature is one of the motivations behind our interest in looking for higher dimensional non-commutative geometries involving higher dimensional irreducible cycles. An other motivation is that the Connes et al. non commutative analysis is expected to hold for general reducible k cycles other than tori. We suspect also that the non commutative analysis may be also extended to local Calabi Yau manifolds. In the conclusion section we make a comment on this issue. Other comments are also given in the following sections.

3. Solving the Constraint Eqs

The solving of the compactification constraint Eqs (5) depends on the way we solve the consistency conditions (8) which in turn depend on the unimodular parameter . Moreover since λ is a C- number it is not difficult to see, by help of Eq (3) and the adjoint action of the automorphism group on the Lie algebra u(N), that the solution of the gauge transformations g_1 and g_2 appearing in Eq (5) should have the form $g_1 = \exp(i\alpha_1^0 T_1^0)$ and $g_2 = \exp(i\alpha_2^0 T_2^0)$ that we write for convenience as follows:

$$\begin{aligned} g_1 &= \exp(\alpha_1 Q_1) \\ g_1 &= \exp(\alpha_1 Q_1) \end{aligned} \tag{10}$$

where the charge operators $\alpha_1 Q_1$ and $\alpha_2 Q_2$ will be specified later on. Putting back Eqs (10) into the relation $g_1 g_2 = \lambda g_2 g_1$ one see that is a priori solved by:

$$\lambda = \exp(-2\pi i \Delta) \tag{11}$$

with:

$$\begin{aligned} \left[\alpha_1 Q_1, \alpha_2 Q_2\right] &= 2\pi i \Delta \\ &= \alpha_1 \left[Q_1, \alpha_2\right] Q_2 + \alpha_2 Q_1 \left[\alpha_1, Q_2\right] \\ &+ \alpha_1 \alpha_2 \left[Q_1, Q_2\right] \\ \left[\Delta, \alpha_1 Q_1\right] &= \left[\Delta, \alpha_2 Q_2\right] = 0 \end{aligned} \tag{12}$$

Having at hand these informations on Q_i's the let us turn now to solve Eqs (5). Putting back Eqs (10) into (5), one finds

$$\begin{aligned}
&\left[X_1, \alpha_1 Q_1\right] = 2\pi n_1 R_1 \\
&\left[X_2, \alpha_2 Q_2\right] = 2\pi n_2 R_2 \\
&\left[X_\mu, Q_i\right] = 0; \quad \mu \neq i = 1,2
\end{aligned} \tag{13}$$

Moreover acting on these equations by $ad(X_i)$ and $ad(\alpha_i Q_i)$ one obtains the following identity

$$\begin{aligned}
&\left[\alpha_1 Q_1, \left[X_1, X_2\right]\right] = 0 \\
&\left[\alpha_1 Q_1, \left[X_1, X_2\right]\right] = 0 \\
&\left[X_\mu, \left[\alpha_1 Q_1, \alpha_2 Q_2\right]\right] = 0
\end{aligned} \tag{14}$$

suggesting that the commutator $[X_1,X_2]$ is proportional to Δ that is

$$\left[X_1, X_2\right] \approx \left[\alpha_1 Q_1, \alpha_2 Q_2\right] \approx \Delta \tag{15}$$

In what follows we discuss briefly the two cases $\lambda_1 = 1$ and $\lambda_1 \neq 1$.

3.1 The abelian case $\lambda_1 = 1$ and $\Delta = 0$

In this case the g_i*'s* and X_μ are realised as operators on the space of 2d fields $\Phi(s_1, s_2)$ on T^2 as follows:

$$\begin{aligned}
(g_j \Phi)(s_1, s_2) &= \exp(2i\pi n_j R_j s_j)\Phi(s_1, s_2) \\
X_\mu &= i\delta_\mu^j \partial_j + A_\mu(s_1, s_2) \\
&= \delta_\mu^j P_j + A_\mu(s_1, s_2)
\end{aligned} \tag{16}$$

where $A_1(s_1,s_2)$ and $A_2(s_1,s_2)$ are 2d gauge connections and $A_1(s_1,s_2)$ are 2d scalar fields all of them commuting with the g_i*'s*. Note by the way that particular solutions *Pj* of the *Xi'*s are just the translation operators along the two one cycle directions *sj* and satisfy

$$\begin{aligned}
&\left[P_1, \alpha_1 Q_1\right] = -2\pi n_1 R_1 \\
&\left[P_2, \alpha_2 Q_2\right] = -2\pi n_2 R_2 \\
&\left[P_1, \alpha_2 Q_2\right] = \left[P_2, \alpha_1 Q_1\right] = 0 \\
&\left[P_1, P_2\right] = \left[X_1, X_2\right] = 0
\end{aligned} \tag{17}$$

3.2 The non commutative case $\lambda_1 = 1$ and $\Delta \neq 0$

Following [3], projective module solutions of Eqs(13-14) are realized on the space of functions de.ned on *RxZq* where *Zq* is the usual cyclic group *Z/Zq*; *q* a positive integer. In the particular *q=1* case where the Connes et al. solutions are reduced to the Schwartz space *S(R)* of regular functions on *R*, the g_j*'s* are realized as:

$$\begin{aligned}
(g_1\Phi)(s) &= \exp(2i\pi n_1 R_1 s)\Phi(s) \\
(g_2\Phi)(s) &= \exp(2i\pi n_2 R_2 \frac{\partial}{\partial s})\Phi(s)
\end{aligned} \tag{18}$$

Comparing with Eq (12), one sees that

$$\begin{aligned}\alpha_1 Q_1 &= 2i\pi n_1 R_1 sI \\ \alpha_2 Q_2 &= 2i\pi n_2 R_2 \frac{\partial}{\partial s}\end{aligned} \tag{19}$$

Moreover since Δ is non zero, one can easily obtain the solution for $X'_\mu s$ the by help of Eqs(15) and (13). We have:

$$\begin{aligned}X_1 &= i\frac{\partial}{\partial s} + A_1(g_1, g_2) \\ X_2 &= \alpha_1 Q_1 + A_2(g_1, g_2) \\ X_1 &= A_1(g_1, g_2)\end{aligned} \tag{20}$$

suggesting that X_1 is the covariant translation operator along the s direction whereas X_2 is a generalized position operator are functions valued in the algebra of operators commuting with g_1 and g_2. For more details see [3].

4. Compactification on S^2

Since the two sphere S^2 is an irreducible two cycle, the compactification of the IKKT model on S^2 will technically differ a little bit from the toroidal one, although general but quite similar results will be obtained. To address the compactification on S^2, we have two ways to follow: either use local features of S^2 as generally done in 10d type II superstrings compactifications on local Calabi Yau manifolds especially in the geometrical engineering of *4d N=2* supersymmetric QFT [10] or to use global features of S^2. We shall follow the second way. In this regards we shall make use of some SU(2) group theoretical features, as well as similarities with compactification on one cycles to study this problem. Our strategy is then as follows: First we study the compactification of the matrix model on S^3 which is isomorphic to SU(2). Then we derive the results of the compactification on S^2 by imposing appropriate constraints on the algebra of functions on S^3.

4.1 Compactification on S^3

To study the compactification of the IKKT model on S^3, we need to identify first the analogue of the constraint Eqs(4) for the three sphere S^3. To that purpose it is instructif to start by recalling the two following clue points in the toroidal compactification. The first point is that the solutions (16) of the constraint Eqs(4) of the compactification on T^2 involve the dual torus parametrized by the local coordinates S^1 and S^2, satisfying the identification

$$\begin{aligned}S_1 &= S_1 + \frac{1}{R_1} \\ S_2 &= S_2 + \frac{1}{R_{21}}\end{aligned} \tag{21a}$$

as well as the gauge transformations which read for the commutative torus as:

$$g_1 = \exp(2i\pi n_1 R_1 s_1), \qquad n_1 \in Z$$

$$g_2 = \exp(2i\pi n_2 R_2 s_{21}), \qquad n_2 \in Z$$

For the non commutative torus; see Eqs (18) and (19).The second thing is that the compactified IKKT coordinates *Xi* are either *U(1)xU(1)* gauge covavariant derivatives for $\lambda = 1$ and or operators obeying a Heisenberg algebra for $\lambda \neq 1$ as described by Eqs (15), (17) and (19). In particular, we shall show that the solving of the problem of compactification of the IKKT model on S^3, and then S^2, involve considering the dual sphere $\hat{S}^3$ exactly as for the toroidal compactification.

More specifically, we shall use a harmonic coordinate system for $\hat{S}^3$ where its global features are manifest and where formal analogy with the torus analysis of previous sections is parent. To guide the reader in identifying the above-mentioned analogy from the begining, let us anticipate on giving some useful relations which may help in following our way of doing. For a rigourous derivation of these relations, see Eqs(37-39), (44,45) and (48,49). The dual sphere $\hat{S}^3$ is defined by help of the harmonic variables $u_a^{\pm}$ as:

$$u^{+a} u_a^- = 1$$

$$u^{+a} u_a^+ = u^{-a} u_a^- = 0$$

where u^{+a} and $u_a^- = (u^{+a})^*$ are su(2) doublets. Eqs (23) has the symmetry

$$u^+ = u^+ + \lambda^{++} u^- \tag{22}$$

where λ^{++} the parameter is a priori defined as

$$\lambda^{++} = u_{(a}^+ u_{b)}^+ \, \lambda^{(a,b)} + u_{(a}^+ u_b^+ u_{c)}^+ \, \lambda^{(a,b,c)} + ... \tag{23}$$

However, up on imposing the compactification constraints, the expansion of reduces λ^{++} to the leading one, namely

$$\lambda^{++} = u_{(a}^+ u_{b)}^+ \, \lambda^{(a,b)} \tag{24}$$

the other terms appear as pure gauge degrees of freedom and are used to introduce the gauge fields A^{++}, A^{--} and A^0 see Eqs(46-48). There are also other relations; but the relevent things for our analysis that one should retain from these relations are: (a) The parameter

$$\lambda^0 = u_{(a}^+ u_{b)}^+ \, \lambda^{(a,b)} \quad \lambda^- = u_{(a}^- u_{b)}^- \, \lambda^{(a,b)} \tag{25}$$

These parameters live altogether with the sphere and turn out to play a crucial role in our solutions for the compactification S^3 and S^2. (b) As far as formal analogy with Eqs (21) and (22) is concerned, Eq (24) should be compared with Eq (21a); the isotriplet $\lambda^{(a,b)}$ with the inverse radius 1/R; the isovector $x_{(a,b)} = u_{(}^+ u_{b)}^-$ with the parameter *s* and the product $x_{(a,b)} = \lambda^{(a,b)}$ with *s=1/R*. Similar identifications may be done with Eq (22); see for example Eq (35). A priori these two features maight be enough to postulate the constraints for the compactification on S^3; nevertheless it could be more interesting to try to obtain the constraint Eqs in a way more a less rigourous. This is what we want to do now. To derive the constraint Eqs of the compactification on S^3, we use the group theoretical feature $S^3=SU(2)$, the *U(N)* invariance of the IKKT model and moreover the harmonic coordinate system of the dual three sphere $\widehat{S}^3$ where its global properties are manifestly exhibited. But first of all, decompose the ten IKKT coordinates $\left\{X_\mu; \mu=1,....,10\right\}$ into two subset $\left\{X^i\right\}$ and $\left\{X^I\right\}$ of three and seven coordinates respectively. The ; *i=1,2,3*, form an su(2) isotriplet which may be written by help of the *2x2* Pauli matrices $(\sigma_i)^{a,b} a,b=1,2,$ as $X^{(a,b)}=(\sigma_i \in)^{ab} X^i$. To get the compactification on S^3, we require that the *Xi's* should obey a su(2) algebra, in addition to the U(N) gauge transformations (2), as shown herebelow :

$$\left[X^i, X^j\right]=i \in^{ijk} X^k \tag{26}$$

$$X_g^i = gX^i g^{-1} \tag{27}$$

$$X^i = X_g^i = gX^i g^{-1} \tag{28}$$

where g is a gauge transformation as in Eqs (3) and where X_g^μ stands for the gauge change of X^μ.

Note that we have used only one gauge transformation for the three components of the triplet. This is related with the irreducibility of the two cycle S^2 as already remarked. Note also that compatibility between Eqs(28.a) and (28.b) requires, amongst others, that we should also have:

$$\begin{aligned}\left[X^i, X^j\right]&=\left[gX^i g^{-1}, gX^j g^{-1}\right]\\ &=\left[X^i, gX^i g^{-1}\right]=0\end{aligned} \tag{29}$$

For later use we prefer to rewrite these constraint Eqs in an equivalent form by working in the su(2) Cartan basis; that is :

$$\left[X^{++}, X^{--}\right]=-iX^0 \tag{30}$$

$$\left[X^0, X^{++}\right]=-2iX^{++} \tag{31}$$

$$\left[X^0, X^{--}\right]=+2iX^{--} \tag{32}$$

together with

$$X_g^{++}=gX^{++}g^{-1} \tag{33}$$

$$X_g^{--}=gX^{--}g^{-1} \tag{34}$$

$$X_g^0=gX^0g^{-1} \tag{35}$$

and

$$\begin{aligned}\left[X^{++}, X^{++}\right]&=\left[gX^{++}g^{-1}, gX^{++}g^{-1}\right]\\ &=\left[X^{++}, gX^{++}g^{-1}\right]=0\end{aligned} \tag{36}$$

$$\left[X^{--}, X^{--}\right]=\left[X^{--}, gX^{--}g^{-1}\right]=0 \tag{37}$$

$$\left[X^{0}, X^{0}\right]=\left[X^{0}, gX^{0}g^{-1}\right]=0 \tag{38}$$

Before going ahead let us remark that, although having different form, the constraints Eqs (28-29) or equivalently (30-32) can be put into a form that has a formal analogy with the constraints Eqs (4) and (5) on the torus. To exhibit this property, let us consider Eqs (30-31) and show that X_g^{++}, X_g^{--} and X_g^{0} can be rewritten as:

$$\begin{aligned} X_g^{++} &= X^{++}+\Gamma^{++}=gX^{++}g^{-1} \\ X_g^{--} &= X^{--}+\Gamma^{--}=gX^{--}g^{-1} \\ X_g^{0} &= X^{0}+\Gamma^{0}=gX^{0}g^{-1} \end{aligned} \tag{39}$$

To do so, one starts from Eqs (30) and note that they may be solved on the dual sphere as follows:

$$\begin{aligned} X^{++} &= iD^{++} \\ X^{--} &= iD^{--} \\ X^{0} &= iDg^{0} \end{aligned} \tag{40}$$

where D^{++}, D^{--} and D^{0} are harmonic derivatives on $\hat{S}^3$ to be spec-field later on. Then, like for Eqs(10), we realize the abelian gauge transformation *g*, as:

$$g=\exp(i\Gamma Q) \tag{41}$$

where now the gauge parameter Γ is a function on $\hat{S}^3$. Putting these realisations back into Eqs (30), one discovers Eqs(33) with the identifications

$$\begin{aligned} \Gamma^{++} &= D^{++}(\Gamma Q) \\ \Gamma^{--} &= D^{--}(\Gamma Q) \\ X^{0} &= D^{0}(\Gamma Q) \end{aligned} \tag{42}$$

At this level, one should note that Γ is not an arbitrary function on $\hat{S}^3$ since we still have to solve the compatibility constraints (32). We will turn to this in a moment; for the time being we shall make a break where we describe the dual sphere $\hat{S}^3$ and its global features. This is the second step in our approach for styding the compactification of the IKKT model on S^3. This approach consists to solve the constraint Eqs (30-36) by help of the $\hat{S}^3$ harmonic analysis. ($\hat{S}^3$ is the dual sphere of S^3 involved in Eqs (23-27); for simplicity we shall drop, in what follows, the hat carried by the dual sphere). In addition to the harmonic realization of the dual sphere, this method give us a tricky and powerful way to describe the global features of the two and three spheres. Recall that the S^3 harmonic analysis was successfully used in different occasions in particular in the study of *4d N=2* supersymmetric quantum field theory [11] and related models [12, 13]. The main idea of this harmonic analysis is to realize the SU(2) group 2x2 matrix elements U as follows:

$$U=\begin{pmatrix} u_1^{+}\ldots u_2^{+} \\ u_1^{-}\ldots u_2^{-} \end{pmatrix} \tag{43}$$

where, for reference, the harmonic variables $u_a^{\pm}; a=1,2$; are related to the usual spherical coordinates R,θ,ψ and ϕ ,with R constant, which can be set to one considers a unit sphere $\widehat{S}^3$, as:

$$
\begin{aligned}
u_1^+ &= R\cos(\frac{1}{2}\theta)\exp(\frac{1}{2}i(\psi+\varphi)) \\
u_1^+ &= R\cos(\frac{1}{2}\theta)\exp(\frac{1}{2}i(\psi+\varphi)) \\
u_1^- &= R\sin(\frac{1}{2}\theta)\exp(-\frac{1}{2}i(\psi-\varphi)) \\
u_2^- &= Rcon(\frac{1}{2}\theta)\exp(-\frac{1}{2}i(\psi-\varphi))
\end{aligned}
\tag{44}
$$

Forgetting about the realisation (38) as we shall use $u_a^{\pm}$ as our basic variables to parameterize S^3, and solving the SU(2) group constraints namely unitary condition $U^+U=UU^+=1$, and unimodularity, $detU=1$, in terms of the $U_a^{\pm}$,one discovers the defining Eqs of the $S^3=SU(2)$ sphere, namely:

$$
\begin{aligned}
u^{+a}u_a^- &= 1 \\
u^{+a}u_a^+ &= u^{-a}u_a^- = 0
\end{aligned}
\tag{45}
$$

The algebra of harmonic functions on S^3 and then S^2, may be exposed in very nice way within this formalism. It has been described first in [11] and was exploited intensively in differents areas of supersymmetric quantum field theories with eight supercharges. For more informations we invite the reader to consult the literature on extended supersymmetric theories. Harmonic analysis on S^3 and S^2 relevant for our present study will be presented at the proper time.

After this digression on the S^3 harmonic analysis, we turn now to the constraints Eq defining the compactification on S^3. Eqs(30.a) are solved on the dual sphere Eqs(39) by the help of the su(2) harmonic derivatives $D^{++};D^{--}$ and D^0 as

$$
\begin{aligned}
-iX^{++} &= D^{++} = u^{+a}\frac{\partial}{\partial u^{-a}} \\
-iX^{--} &= D^{--} = u^{-a}\frac{\partial}{\partial u^{+a}} \\
-iX^{0} &= D^{0} = u^{+a}\frac{\partial}{\partial u^{+a}} - u^{-a}\frac{\partial}{\partial u^{-a}}
\end{aligned}
\tag{46}
$$

A direct check shows that D^{++}, D^0 and D^{--} generate indeed an su(2) algebra:

$$
\begin{aligned}
\left[D^{++},D^{--}\right] &= D^0 \\
\left[D^{0},D^{++}\right] &= 2D^{++} \\
\left[D^{0},D^{--}\right] &= -2D^{--}
\end{aligned}
\tag{47}
$$

Now using the solutions Eqs (34) and (40) of the constraint Eqs (30), we can rewrite the constraints Eqs (31) as follows:

$$D_g^{++} = gD^{++}g^{-1}$$
$$D_g^{--} = gD^{--}g^{-1} \tag{48}$$
$$D_g^{0} = gD^{0}g^{-1}$$

or equivalently by setting $g = \exp(i\Gamma Q)$

$$D_g^{++} = D^{++} - iD^{++}(\Gamma Q)$$
$$D_g^{--} = D^{--} - iD^{--}(\Gamma Q) \tag{49}$$
$$D_g^{0} = D^{0} - iD^{0}(\Gamma Q)$$

where Γ is a function of the $u_a^{\pm}s$ valued in the u(N) algebra. Putting the solutions (40) back into Eqs(32), one gets for instance

$$\left[D^{++}, D^{++} - iD^{++}(\Gamma Q)\right] = -iD^{++\cdot}(\Gamma Q) = 0 \tag{50}$$

Eqs(44) implies in turns that is in fact an isotriplet given by:

$$\Gamma = u_{(a}^{+}u_{b)}^{-}\Gamma^{(a,b)} \tag{51}$$

Note that the identifications (43) may be put into equalities by introducing three gauge fields A^{++}, A^{--} and A^0, on the three sphere, and replacing the harmonic derivatives D^{++},D^{--} and D^0 by the following covariants ones:

$$\nabla^{++} = D^{++} + iA^{++}$$
$$\nabla^{--} = D^{--} + iA^{--} \tag{52}$$
$$\nabla^{0} = D^{0} + iA^{0}$$

In this case, the gauge transformation *g* acts on and as follows:

$$\nabla^{++} = g\nabla^{++}g^{-1}$$
$$\nabla^{--} = g\nabla^{--}g^{-1} \tag{53}$$
$$\nabla^{0} = g\nabla^{0}g^{-1}$$

together with

$$A^{++} = A^{++} + D^{++}(\Gamma Q)$$
$$A^{--} = A^{--} + D^{--}(\Gamma Q) \tag{54}$$
$$A^{0} = A^{0} + D^{0}(\Gamma Q)$$

where now Γ is an arbitrary field on S^3.In connection with Eqs (41-43), we would like to note that the harmonic variables $u_a^{\pm}$ are isospinors which are related as $\overline{u}^{+a} = -u_a^{+}\ \overline{u}^{+a} = -u_a^{+}$ and behave under the Cartan conjugaison (*) as $(u_a^{\pm})^{*} = \pm u_a^{\mp}$; so that one has $(\overline{u}_a^{\pm})^{*} = u_a^{\pm}$. In *4d N=2* supersymmetric quantum field theory in harmonic superspace [11] and related models, one uses the combined conjugaison (*,-) leaving stable the Cartan charges carried by the $u_a^{\pm}$ as the involution. In other words, one considers u^{+a} and u^{-a} as independent variables. Here also we shall use this combined conjugaison and we shall refer to it just by (.) for simplicity. The defining Eqs (39) of the three sphere S^3 is invariant under the change

$$\begin{aligned} u^+ &\to u^+ + \lambda^{++} u^- \\ u^- &\to u^- \end{aligned} \tag{55}$$

where λ^{++} is an arbitrary function on S^3 which should be compared to the gauge parameter of Eq(30). Put differently; the isometries of the S^3 sphere appear then as gauge parameters of abelian symmetries of the *U(N)* invariance of the IKKT model on S^3. In summary the solutions of the constraints Eqs(30-32) are given by the following u(1) gauge covariant derivatives on S^3.

$$\begin{aligned} -iX^{++} &= \nabla^{++} \\ -iX^{--} &= \nabla^{--} \\ -iX^{0} &= \nabla^{0} \quad X^I = A^I(u^\pm) \end{aligned} \tag{56}$$

and obey the su(2) algebra

$$\begin{aligned} \left[\nabla^{++}, \nabla^{--}\right] &= \nabla^0 \\ \left[\nabla^{0}, \nabla^{++}\right] &= 2\nabla^{++} \\ \left[\nabla^{0}, \nabla^{--}\right] &= -2\nabla^{--} \end{aligned} \tag{57}$$

Putting Eqs (46) back into Eqs (51), we get by help of Eqs (41) the relations:

$$A^0 = D^{++}A^{--} - D^{--}A^{++} \tag{58}$$

$$2A^{++} = D^0 A^{++} - D^{++}A^0 \tag{59}$$

$$-2A^{--} = D^0 A^{--} - D^{--}A^0 \tag{60}$$

aserting that the gauge fields A^{++}; A^{--} and A^0 are not independent fields. They form a su(2) triplet.

4.2 Compactification on S^2

The solutions of the compactification of the IKKT model on S^2 may be derived from those of the compactifcation on S^3 by constraining the gauge degrees of freedom associated with the U(1) Cartan subsymmetry of $su(2)=S^3$. The latter is just the *U(1)* Cartan invariance of the S^3 acting on the harmonics as:

$$u_a^\pm \to e^{\pm i\theta} u_a^\pm \,;\ \theta \text{ a real parameter} \tag{61}$$

The constraint of Eqs defining the compactification on S^2 are then given by Eqs (30-32) with the extra requirement that all harmonic functions defined on S^3 are subject to the extra constraint

$$\left[D^0, F^q\right] = qF^q \tag{62}$$

or equivalently

$$F^{q}(u^{\pm}) = \sum u^{+}_{a_{n+q}} ......u^{+}_{a}, u^{-}_{b_1}, ...u^{-}_{b_2} F^{(a_1.....a_{n+q}b_1......b_n)} \tag{63}$$

Starting from the solutions (50) and (51) established for the compactification on S^3, the requirement of conservation of the *U(1)* charge of the *su(2)* Cartan subsymmetry implies that $\nabla^0 = D^0; i.e\ A^0 = 0$. Then Eqs (51-52) should be replaced by:

$$\begin{aligned} &\left[\nabla^{++}, \nabla^{--}\right] = D^0 \\ &\left[D^0, \nabla^{++}\right] = 2\nabla^{++} \\ &\left[D^0, \nabla^{--}\right] = -2\nabla^{--} \end{aligned} \tag{64}$$

and moreover:

$$\begin{aligned} &D^{++}A^{--} - D^{--}A^{++} = 0 \\ &D^0 A^{++} = 2A^{++} \\ &D^0 A^{--} = -2A^{--} \end{aligned} \tag{65}$$

in agreement with Eq (52) and (54). Eqs (57) are in turn solved as:

$$\begin{aligned} &A^{++} = D^{++}V \\ &A^{--} = D^{--}V \end{aligned} \tag{66}$$

where V is an arbitrary function on S^2. Moreover as a consequence of Eqs (54) $V(u^{\pm})$ and $A_I(u^{\pm})$ have an harmonic expansions type Eq (55).

5. Conclusion

In this paper we have studied the compactification of the IKKT model on the three and two spheres. This method can be done by other way such that the Hirzebruch complex surface F_0 geometry for the next works.

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