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Nonparametric Estimation of the Limiting Interval Reliability for Stationary Dependent Sequences

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Abstract

In this paper, we consider the nonparametric estimation of the limiting interval reliability of a repairable system when the sequences of failure and repair times are generated by stationary dependent sequence of random variables. The proposed nonparametric estimator is shown to be consistent and asymptotically normal. A simulation study is also conducted to assess the performance of the proposed estimator using a first-order exponential autoregressive process.

Keywords: limiting interval reliability, availability, repairable systems, exponential autoregressive process

1. Introduction

Consider a one-unit repairable system which is at any time either in operation or under repair after failure. Suppose that the system starts to operate at time $t = 0$. If we define $\xi(t)$ as the state of the system at time 't', we have

$$\xi(t) = \begin{cases} 1 & \text{if the system is operating at time } t \\ 0 & \text{otherwise} \end{cases}$$

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Based on $\xi(t)$, a number of useful measures of the system availability may be constructed. The point availability, $A(t) = P[\xi(t) = 1]$ and the limiting availability, $A = \lim_{t\to\infty} A(t)$ are two commonly used availability measures.

In the context of repairable system, another important measure of successful performance of a system is the interval reliability. The interval reliability, $R(x,t)$, is defined as the probability that at a specified time 't' the system is operating and will continue to operate for an interval of duration 'x'[3]. That is, $R(x,t) = P[\xi(s) = 1,\ t \le s \le t + x]$. The interval reliability becomes simply reliability when $t = 0$ and point availability at time 't' as $x \to 0$. Thus, the interval reliability is one of the most important measures of system performance from the viewpoint of reliability and availability, and it is useful in many practical situations. Since it is difficult to obtain an explicit expression for the interval reliability except for few simple cases, in the literature more attention is being paid to its limiting measure $R(x) = \lim_{t\to\infty} R(x,t)$. The limiting interval reliability, $R(x)$, is a useful measure when one may be interested in knowing the extent to which the system will survive an interval of duration after it has been run for a long time. The properties of these measures are usually studied using the successive failure times $\{X_n\}$ and repair times $\{Y_n\}$ of the system.

If $\{X_n\}$ and $\{Y_n\}$ are independent and identically distributed (i.i.d) non-negative random variables with marginal distribution functions $F_X(.)$ and $F_Y(.)$ respectively, then

$$R(x,t) \to R(x) = \upsilon(x)/(\mu_X + \mu_Y),$$

where $\mu_X = E(X_1)$ and $\mu_Y = E(Y_1)$ and $\upsilon(x) = \int_x^\infty \bar{F}_X(u)du = \int_0^\infty (u - x) I_{(u>x)} dF_X(u)$.

If we define $\psi(.) = \bar{F}_X(.)/\mu_X$ as the density of the asymptotic recurrence time of a renewal process governed by the distribution function $F_X(.)$, the limiting interval reliability can be expressed as

$$R(x) = \frac{\mu_X}{\mu_X + \mu_Y}[1 - \Psi(x)],$$

where $\Psi(.)$ is the distribution function with density $\psi(.)$. Thus, $R(x)$ is the product of the limiting probability that the system is available at some point and the limiting probability that it survives an interval of duration at least 'x' [5, 11].

One of the major limitations of the existing approaches is the assumption of independence among the successive sequences of failure and repair times. When the system is operating in a random environment it is natural to observe dependence among successive sequence of failure and repair times. Several non-Gaussian time series models such as first-order random coefficient autoregressive models are discussed in the literature for modeling life time data [6, 9]. In the case of repairable systems, the study of availability measures for stationary dependent sequences is not discussed much, except those considered by Abraham and Balakrishna [1], Balakrishna and Mathew [2] and Mathew [10].

This paper is organized as follows. In section 2, we consider the nonparametric estimation of the limiting interval reliability of a repairable system when the failure and repair times are generated by stationary dependent sequence of random variables. A simulation study is presented in section 3 using first-order exponential autoregressive process. The conclusion will be given in section 4.

2. Nonparametric Estimation of the Limiting Interval Reliability

Suppose that $\{X_n\}$ and $\{Y_n\}$ are two mutually independent strictly stationary and strong mixing sequence of non-negative random variables with mixing coefficients $\alpha_X(h)$ and $\alpha_Y(h)$ respectively. When the observations on n failure times $X_1, X_2, ..., X_n$ and n repair times $Y_1, Y_2, ..., Y_n$ are recorded for a repairable one-unit system, a natural nonparametric estimator for the limiting interval availability $R(x)$ is

$$\hat{R}_n(x) = \frac{\bar{U}_n}{\bar{X}_n + \bar{Y}_n}, \tag{1}$$

where $\bar{X}_n = \sum_{i=1}^{n} X_i \Big/ n$, $\bar{Y}_n = \sum_{i=1}^{n} Y_i \Big/ n$, $\bar{U}_n = \sum_{i=1}^{n} U_i \Big/ n$ with $U_i = (X_i - w) I(X_i > w)$.

Since $\{X_n\}$ and $\{Y_n\}$ are strictly stationary, we have $\bar{X}_n \to \mu_X$, $\bar{Y}_m \to \mu_Y$, and $\bar{U}_n \to \upsilon(x)$

almost surely as $n \to \infty$ and hence we conclude that $\hat{R}_n(x) \to R(x)$ almost surely as $n \to \infty$.

In order to establish the asymptotic normality of $\hat{R}_n(x)$, we assume that for some $\delta > 0$,

$$E(X_1^{2+\delta}) < \infty,\ E(Y_1^{2+\delta}) < \infty,\ \sum_{h=1}^{\infty} \alpha_X^{\delta/(2+\delta)}(h) < \infty \text{ and } \sum_{h=1}^{\infty} \alpha_Y^{\delta/(2+\delta)}(h) < \infty .$$

Since $\{X_n\}$ and $\{Y_n\}$ are strictly stationary and strong mixing, under the above assumptions, by the central limit theorem for such sequences [7, p. 346] we have as $n \to \infty$

$$\sqrt{n}\left(\bar{Y}_n - \mu_Y\right) \xrightarrow{D} N\left(0,\ \sigma_{YY}\right) \text{and}$$

$$\sqrt{n}\left(\bar{X}_n - \mu_X,\ \bar{U}_n - \upsilon(x)\right) \xrightarrow{D} N_2\left(0,\ \Sigma_2\right),$$

where $N_2(0, \Sigma_2)$ is a 2-variate normal vector with mean $\mathbf{0} = (0,0)'$ and dispersion matrix

$$\Sigma_2 = \begin{pmatrix} \sigma_{XX} & \sigma_{XU} \\ \sigma_{XU} & \sigma_{UU} \end{pmatrix},$$

with $\sigma_{XX} = \operatorname{var}(X_1) + 2\sum_{h=2}^{\infty} \operatorname{cov}(X_1, X_h)$, $\sigma_{YY} = \operatorname{var}(Y_1) + 2\sum_{h=2}^{\infty} \operatorname{cov}(Y_1, Y_h)$,

$$\sigma_{UU} = \operatorname{var}(U_1) + 2\sum_{h=2}^{\infty} \operatorname{cov}(U_1, U_h) \text{ and } \sigma_{XU} = \operatorname{cov}(X_1, U_1) + \sum_{h=2}^{\infty} \operatorname{cov}(X_1, U_h) + \sum_{h=2}^{\infty} \operatorname{cov}(X_h, U_1).$$

Now, by the Cramer-Wold device [4, p.49], we have as $n \to \infty$,

$$\sqrt{n}\left(\bar{X}_n - \mu_X,\ \bar{Y}_n - \mu_Y,\ \bar{U}_n - \upsilon(x)\right) \xrightarrow{L} N_3(\mathbf{0}, \Sigma_3),$$

where $N_3(0, \Sigma_3)$ is a 3-variate normal vector with mean $\mathbf{0} = (0,0,0)'$ and dispersion matrix

$$\Sigma_3 = \begin{pmatrix} \sigma_{XX} & 0 & \sigma_{XU} \\ 0 & \sigma_{YY} & 0 \\ \sigma_{XU} & 0 & \sigma_{UU} \end{pmatrix}.$$

If we define $f(x, y, z) = \dfrac{z}{x+y}$, then $f(\bar{X}_n, \bar{Y}_n, \bar{U}_n) = \hat{R}_n(x)$ and $f(\mu_X, \mu_Y, \upsilon(x)) = R(x)$ and hence, employing a Taylor series expansion, we can write

$$\hat{R}_n(x) \cong R(x) - \frac{\upsilon(x)}{(\mu_X+\mu_Y)^2}(\overline{X}_n - \mu_X) - \frac{\upsilon(x)}{(\mu_X+\mu_Y)^2}(\overline{Y}_n - \mu_Y) + \frac{1}{\mu_X+\mu_Y}(\overline{U}_n - \upsilon(x))$$

Now, by using the results from Serfling [12, p.122] we can show that as $n \to \infty$

$$\sqrt{n}\left(\hat{R}_n(x) - R(x)\right) \xrightarrow{D} N\left(0,\, \tau^2(x)\right),$$

where

$$\tau^2(x) = \frac{\upsilon^2(x)}{(\mu_X+\mu_Y)^4}(\sigma_{XX}+\sigma_{YY}) - \frac{2\upsilon(x)}{(\mu_X+\mu_Y)^3}\sigma_{XU} + \frac{1}{(\mu_X+\mu_Y)^2}\sigma_{UU}\,. \tag{2}$$

Thus we proved the following theorem.

Theorem 1: *If $\{X_n\}$ and $\{Y_n\}$ are two mutually independent strictly stationary and strong mixing sequence of non-negative random variables such that for some $\delta > 0$, $E(X_1^{2+\delta}) < \infty$, $E(Y_1^{2+\delta}) < \infty$, $\sum_{h=1}^{\infty} \alpha_X^{\delta/(2+\delta)}(h) < \infty$ and $\sum_{h=1}^{\infty} \alpha_Y^{\delta/(2+\delta)}(h) < \infty$, then $\hat{R}_n(x)$ is a consistent and asymptotically normal (CAN) estimator for the limiting interval relaibility $R(x)$.*

A consistent estimator $\hat{\tau}^2(x)$ of $\tau^2(x)$ can be obtained by replacing $\mu_X, \mu_Y, \upsilon(x)$, $\sigma_{XX}, \sigma_{YY}, \sigma_{UU}$ and σ_{XU} with their corresponding consistent estimators in (2). Obviously $\overline{X}_n, \overline{Y}_n$, and $\overline{U}_n$ are the consistent estimators for μ_X, μ_Y, and $\upsilon(x)$ respectively. In order to construct consistent estimators for $\sigma_{XX}, \sigma_{YY}, \sigma_{UU}$ and σ_{XU}, we use the moving-block jackknife method for variance estimation with dependent data [8]. The moving-block jackknife estimators for $\sigma_{XX}, \sigma_{YY}, \sigma_{UU}$ and σ_{XU} respectively are

$$\hat{\sigma}^2_{XX,l} = \frac{l}{n-l+1}\sum_{i=1}^{n-l+1}\left(\overline{X}_i^{(l)} - (n+l-1)^{-1}\sum_{j=1}^{n-l+1}\overline{X}_j^{(l)}\right)^2,$$

$$\hat{\sigma}^2_{ZZ,l} = \frac{l}{n-l+1}\sum_{i=1}^{n-l+1}\left(\overline{Y}_i^{(l)} - (n+l-1)^{-1}\sum_{j=1}^{n-l+1}\overline{Y}_j^{(l)}\right)^2,$$

$$\hat{\sigma}^2_{UU,l} = \frac{l}{n-l+1}\sum_{i=1}^{n-l+1}\left(\bar{U}_i^{(l)} - (n+l-1)^{-1}\sum_{j=1}^{n-l+1}\bar{U}_j^{(l)}\right)^2, \text{ and}$$

$$\hat{\sigma}_{XU,l} = \frac{l}{n-l+1}\sum_{i=1}^{n-l+1}\left(\bar{X}_i^{(l)} - (n+l-1)^{-1}\sum_{j=1}^{n-l+1}\bar{X}_j^{(l)}\right)\left(\bar{U}_i^{(l)} - (n+l-1)^{-1}\sum_{j=1}^{n-l+1}\bar{U}_j^{(l)}\right),$$

where $\bar{X}_i^{(l)} = l^{-1}\sum_{j=i}^{i+l-1} X_j$, $\bar{Y}_i^{(l)} = l^{-1}\sum_{j=i}^{i+l-1} Y_j$, $\bar{U}_i^{(l)} = l^{-1}\sum_{j=i}^{i+l-1} U_j$ and l is the block size.

If we assume that for some $\delta > 0$, $E\left[|X_1|^{6+\delta}\right] < \infty$, $E\left[|Y_1|^{6+\delta}\right] < \infty$, $\sum k^2 \alpha_X(k)^{\delta/(6+\delta)} < \infty$ and $\sum k^2 \alpha_Y(k)^{\delta/(6+\delta)} < \infty$, then the estimators $\hat{\sigma}^2_{XX,l}$, $\hat{\sigma}^2_{YY,l}$, $\hat{\sigma}^2_{UU,l}$ and $\hat{\sigma}_{XU,l}$ converge almost surely to σ_{XX}, σ_{YY}, σ_{UU} and σ_{XU} respectively if $l = o(n)$, and $l \to \infty$ [8].

Then, it is easy to see that

$$\hat{\tau}^2(x) \to \tau^2(x) \quad \text{almost surely as } n \to \infty.$$

Thus, given a significance level $\alpha \in (0,1)$, an approximate large sample $100(1-\alpha)\%$ confidence interval for the limiting interval reliability $R(x)$ is

$$\hat{R}_n(x) - z_{\alpha/2}\frac{\hat{\tau}(x)}{\sqrt{n}} \le R(x) \le \hat{R}_n(x) + z_{\alpha/2}\frac{\hat{\tau}(x)}{\sqrt{n}}.$$

3. Simulation Study

A simulation study is conducted in this section to assess the performance of the proposed estimator and to compare their efficiencies with corresponding estimator in the i.i.d. set-up. Here, we assume that the failure and repair times are generated using two independent first-orderexponential autoregressive (EAR(1)) processes [6] given by,

$$X_n = \begin{cases} 0.5X_{n-1} & \text{with probability } 0.5, \\ 0.5X_{n-1} + \varepsilon_n & \text{with probability } 0.5. \end{cases} \text{ and}$$

$$Y_n = \begin{cases} 0.25Y_{n-1} & \text{with probability } 0.25, \\ 0.25Y_{n-1} + \eta_n & \text{with probability } 0.75. \end{cases}, \quad n = 1, 2, 3, \ldots,$$

where $\{\varepsilon_n\}$ and $\{\eta_n\}$ are two independent i.i.d. exponential sequences with parameters $\lambda_1 = 1/6$ and $\lambda_2 = 1/2$ respectively. Thus $\{X_n\}$ and $\{Y_n\}$ have exponential marginal distributions with mean failure time $\mu_X = 6$ and mean repair time $\mu_Y = 2$ respectively.

We consider the limiting interval reliability $R(x)$ at $x = 0, 0.25, 0.50,$ and 0.75 for the simulation study. In order to compare the performance of the estimator of $R(x)$ in the stationary dependent case (EAR(1) model) with that of the i.i.d. exponential case, we compute the empirical coverage probabilities in the case of EAR(1) model and the i.i.d exponential model seperately.

Table 1: Simulation results of limiting interval relaibility

x	$R(x)$	n	$\bar{\hat{R}}(x)$	EAR(1) Model		i.i.d. Case	
				$\bar{\hat{\tau}}^2(x)$	CP	$\bar{\hat{\tau}}_*^2(x)$	CP^*
		25	0.75098	0.17619	0.9367	0.08013	0.6179
			(0.0459)	(0.0218)		(0.0283)	
0.00	0.75000	75	0.74986	0.16823	0.9392	0.07382	0.6243
			(0.0361)	(0.0156)		(0.0179)	
		150	0.75013	0.16672	0.9413	0.07259	0.6338
			(0.0213)	(0.0124)		(0.0131)	
		25	0.71816	0.19433	0.9326	0.08512	0.6020
			(0.0437)	(0.0269)		(0.0318)	
0.25	0.71939	75	0.72017	0.18996	0.9341	0.07962	0.6215
			(0.0319)	(0.0183)		(0.0243)	
		150	0.71998	0.18854	0.9392	0.07946	0.6298
			(0.0198)	(0.0112)		(0.0167)	
		25	0.70143	0.22198	0.9284	0.09117	0.5919
			(0.0447)	(0.0328)		(0.0329)	
0.50	0.69003	75	0.70019	0.21316	0.9331	0.08494	0.6012
			(0.0306)	(0.0235)		(0.0228)	
		150	0.68918	0.21194	0.9403	0.08322	0.6194
			(0.0184)	(0.0192)		(0.0173)	
		25	0.65902	0.24129	0.9421	0.09514	0.6226
			(0.0491)	(0.0311)		(0.0287)	
0.75	0.66187	75	0.66867	0.23417	0.9439	0.09192	0.6354
			(0.0384)	(0.0204)		(0.0187)	
		150	0.66091	0.23273	0.9503	0.09015	0.6546
			(0.0176)	(0.0156)		(0.0133)	

The results of the simulation study are summarized in Table 1. The notations $\bar{\hat{R}}(x)$, $\bar{\hat{\tau}}^2(x)$ and *CP* denote the average of the estimated value of $R(x)$, its asymptotic variance $\tau^2(x)$ and the empirical coverage probability of 95% confidence interval for $R(x)$ over 750 repetitions in the stationary dependent case. The same quantities are also computed by assuming the stationary dependent failure and repair times as i.i.d. exponential observations ignoring the autocorrelations present in the data. Let $\bar{\hat{\tau}}_*^2(x)$ and CP^* denote the average of the asymptotic variance and the empirical coverage probabilities in the i.i.d. case. Note that the estimated value of $R(x)$ is the same for both the stationary dependent and i.i.d. case. The values within the parenthesis represent the *MSE* of the estimator.

From the Tables, we can see that the estimated asymptotic variance of the estimator in the stationary dependent case is approximately twice of that in the i.i.d. case and hence the confidence interval of the estimator in the i.i.d. case is shorter than that in the stationary dependent case. Also, the empirical coverage probabilities of the estimators corresponding to 95% confidence interval in the i.i.d. set-up is around 0.60-0.65 and that in the case of stationary dependent model is around 0.90- 0.95. This suggests that when the successive failure and repair times are dependent, the ignorance of dependence present in the data will lead to poor coverage probabilities and this may lead to erroneous interpretations in the inference procedure.

4. Conclusions

In this paper,we discussed the nonparametric estimation of the limiting interval reliability when the failure and repair times are generated by two mutually independent strictly stationary dependent sequences of random variables. The proposed estimators were shown to be consistent and asymptotically normal. A simulation study was conducted to assess the performance of the proposed estimator in the stationary dependent casewith the corresponding estimator in the i.i.d. set-up. The simulation study showed that if the true process is generated from stationary dependent sequences of random variables, the ignorance of dependence among successive observations leads to poor coverage probabilities and results in erroneous conclusions.

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