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A Practical Quadrature - Collocation Method to Solve Frictional Contact Problems

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Abstract

The contact problem of impressing absolutely rigid punch under normal and shear forces into an elastic layer is examined in presence Coulomb friction in the contact area. It is assumed that the lower boundary of the layer is fixed or there are no normal displacements and shear stresses on it, the punch-strip system is in a condition of limit equilibrium, and the punch does not turn during deformation of the layer. First, considered contact problem is reformulated by Cauchy type singular integral equation of the second kind in which the unknown is the normal contact stresses beneath the punch, next this equation is reduced to system linear algebraic equations by using Gauss-Jacobi quadrature and collocation methods. The effects of geometrical and mechanical parameters of the materials on various subjects of interest are discussed and shown graphically and tabular form.

Keywords: Singular Integral Equation, Plane Contact Problem, Cauchy kernel, Jacobi polynomial, A system of linear algebraic equation.

1. Introduction

Problems of a layer with various boundary conditions were formulated and solved in many papers and books, e. g. [1-3]. Plane contact problems for a layer, taking into account the friction forces in the contact area, have been presented and investigated in many publications (see for example, [4-7], etc.). These problems are usually presented in the form of mixed type boundary problems in the elasticity theory, which can be reduced to singular integral equations and which can be tackled by the method originated from Muskheleshvili [8]. Another approach to the mixed problems in question suggested by Rostovtsev [9] does not involve singular integral equations. The unknown quantity in these problems is the pressure arising in the contact region. Analysis of this singular integral equation earlier performed in [2, 3] for an isotropic material, makes use of asymptotic series in a small or large relative strip thickness. Recently, an iterative method is applied to solve the plane contact problem in absence of friction for relative strip thickness [10]. Below, we consider a plane contact problem of the theory of elasticity about the interaction between a punch and a layer, taking Coulomb friction in the contact area into account. The aim of the present consideration is to obtain and analyze the results of a study of the effect of geometrical and mechanical parameters taking into account the friction forces in the contact area.

Know that, the considered contact problem can be reformulated by Cauchy type singular integral equation of the second kind in which the unknown is the normal contact stresses beneath the punch [3]. A general closed-form solution of this equation is not available; however substantial progress has been made in developing numerical methods of solution. One of these methods is a direct numerical method based on a Gaussian-type quadrature approximation of the integral parts in singular integral equation which was originally proposed by Erdogan and his colleagues [11] and extended by Krenk [12]. Recently X. Jin and his colleagues [13] proposed a practical method for singular integral equations of the second kind. This method features the standard Gauss–Jacobi quadrature and avoids the confinement on the collocation points. These advantages contribute to a convenient programming of the current algorithm.

In the present paper, we used a practical method given in [13] to solve the above mentioned singular integral equation. As is known, this method is to make the transition from a singular integral equation to system of linear algebraic equations in values of the required function in roots of appropriate Jacobi polynomials.

To illustrate the efficiency of the presented methodology, we will confine ourselves to consider the indentation of a plane punch with an elastic strip.

2. Formulation of the Contact Problem

Let, infinite elastic layer is studied under plane strain conditions and its normal intersection occupies the region $|x| < \infty, 0 \le y \le h$. The stresses vanish at the infinity.

While G denotes the shear modulus; v denotes Poisson's ratio of the elastic strip. A normal force P and a shear force $T = \mu P$ acts on the punch, and the forces of Coulomb friction with a coefficient of friction μ act in the contact area $-c \le x \le c,\ y = h$. Here, either the lower boundary of the layer $y = 0$ is fixed (Problem 1) or there are no normal displacements and shear stresses on it (Problem 2). The case of limit equilibrium is examined, and the punch does not turn during deformation of the layer.

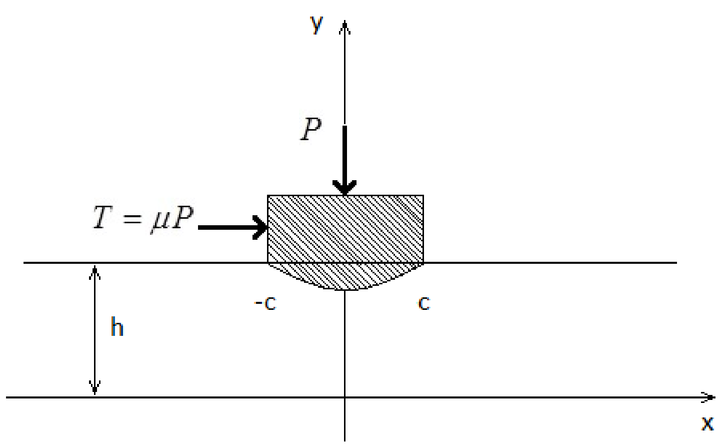

Fig. 1. The geometry of the problem

The plane contact problem outlined above as shown in Fig. 1 must be solved under the following boundary conditions:

$$\begin{aligned} &u(x,0)=0, \quad v(x,0)=0, \quad -\infty<x<\infty, \\ &\tau_{xy}(x,h)=0, \quad \sigma_y(x,h)=0, \quad |x|<c, \\ &v(x,h)=-\delta(x), \quad |x|<c, \end{aligned} \tag{2.1}$$

for Problem 1, and

$$\begin{aligned} &\tau_{xy}(x,0)=0, \quad v(x,0)=0, \quad -\infty<x<\infty, \\ &\tau_{xy}(x,h)=0, \quad \sigma_y(x,h)=0, \quad |x|<c, \\ &v(x,h)=-\delta(x), \quad |x|<c, \end{aligned} \tag{2.2}$$

for Problem 2, where u and v are the x- and y-components of the displacement vector, τ_{xy} and σ_y are shear and normal stress components, respectively. $\delta(x)$ is a function describing the shape of the base of the punch. Using integral transforms, boundary conditions for Problems 1 and 2 and by the technique originated in [10], plane elasticity equations can be converted analytically into the following singular integral equation in which the unknown is the contact pressure $q(x)$

$$\varepsilon q(x)+\frac{1}{\pi}\int_{-c}^{c}\frac{q(\xi)}{\xi-x}d\xi+\frac{1}{h}\int_{-c}^{c}q(\xi)M\left(\frac{\xi-x}{h}\right)d\xi=\theta\delta'(x), \quad -c\le x\le c \tag{2.3}$$

the kernel which can be represented in the following form is

$$M(t)=-\frac{1}{\pi}\int_0^\infty\left\{\left[1-L_1(u)\right]\sin ut+\varepsilon\left[1-L_2(u)\right]\cos ut\right\}du, \tag{2.4}$$

where $\varepsilon=\dfrac{1-2\nu}{2(1-\nu)}\mu$, $\theta=G/(1-\nu)$. The first integral in Eq. (2.3) is evaluated in the sense of Cauchy principal value. Here, for Problem 1

$$\begin{aligned} &L_1(u)=\left[2\kappa sh2u-4u\right]/\Delta_1(u) \\ &L_2(u)=\left[2\kappa(ch2u-1)-4u^2(1-2\nu)^{-1}\right]/\Delta_1(u) \\ &\Delta_1(u)=2\kappa ch2u+4u^2+1+\kappa^2, \quad \kappa=3-4\nu \end{aligned} \tag{2.5}$$

and for Problem 2

$$\begin{aligned} &L_1(u)=\left[ch2u-1\right]/\Delta_2(u), \\ &L_2(u)=\left[sh2u-2(1-2\nu)^{-1}\right]/\Delta_2(u), \\ &\Delta_2(u)=sh2u+2u. \end{aligned} \tag{2.6}$$

The singular integral equation (2.3) should be solved together with the condition of bodies' equilibrium

$$\int_{-c}^{c} q(\xi)\,d\xi = P. \tag{2.7}$$

Changing Eqs. (2.3) and (2.7) to dimensionless variables and using the notations in the formulae

$$x = ct, \quad \xi = c\tau\,,\ \varphi(t) = cq(ct)/P, \quad f(t) = \theta\delta(ct)/P, \quad \lambda = h/c\,, \tag{2.8}$$

we will have

$$\varepsilon\varphi(t) + \frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(\tau)}{\tau - t}d\tau + \frac{1}{\lambda}\int_{-1}^{1}\varphi(\tau)M\left(\frac{\tau - t}{\lambda}\right)d\tau = f'(t), \quad -1 \le t \le 1, \tag{2.9}$$

$$\int_{-1}^{1}\varphi(\tau)\,d\tau = 1, \tag{2.10}$$

Respectively. Such integral equations arise from the mixed boundary value problems of solid and fluid mechanics, and are solved frequently using direct methods based on quadrature and collocation. One of the methods for numerical solution of singular integral equation (2.9) with eq. (2.10), which is convenient for application of electronic computers, is proposed in [13] and a foundation is given briefly below.

3. Discretization

Equation (2,9) is Cauchy-type integral equation of second kind for unknown pressure under the punch in the contact area. The unbounded solution at the both-end points of Eqs. (2.9) and (2.10) may be expressed as the product of a bounded function $g(t)$, and a fundamental function $w(t)$,

$$\varphi(t) = w(t)g(t) \tag{3.1}$$

where the fundamental function is given by

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta}\,. \tag{3.2}$$

The numbers α and β are determined as $\alpha = -\frac{1}{2} + \frac{1}{\pi}arctg(\varepsilon)$ and $\beta = -\frac{1}{2} - \frac{1}{\pi}arctg(\varepsilon)$ [8, 11-14].

Let $H(x,t)$ denotes the function obtained by

$$H(t,\tau) = \begin{cases} \dfrac{g(\tau) - g(t)}{\tau - t}, & \tau \ne t \\ g'(\tau), & \tau = t \end{cases} \tag{3.3}$$

If the Eqs. (3.1) and (3.2) are used in Eq. (2.9), and the singularity in it is removed, then via Eq. (3.3), last equation can be converted to

$$\frac{1}{\pi}\int_{-1}^{1} w(\tau)\left[H(t,\tau) + \frac{\pi}{\lambda}M\left(\frac{\tau - t}{\lambda}\right)g(\tau)\right]d\tau = f'(t), \quad -1 \le t \le 1, \tag{3.4}$$

Numerical discretization of the integral eq. (3.4) may be accomplished by using a numerical integration rule to evaluate the integral of the left-hand side. Gauss-Jacobi quadrature rule is an appropriate choice for the Jacobi weight function indicated by Eq. (3.2). According to [13], at the collocation points t_i, Eq. (3.4) is reduced as

$$\sum_{j=1}^{n} A_j^{(\alpha,\beta)}\left[H(t_i,\tau_j)+\frac{\pi}{\lambda}M\left(\frac{\tau_j-t_i}{\lambda}\right)g(\tau_j)\right]\cong \pi f'(t_i), \tag{3.5}$$

where n is the total number of the integration points, the values $A_j^{(\alpha,\beta)}$ ($j=1,\ldots,n$) are the weights of the standard Gauss-Jacobi quadrature, and abscissas t_j ($j=1,\ldots,n$) are roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ of degree n corresponding to the function $w(t)$.

In the current algorithm proposed by X. Jin et al., the selection of the collocation points is flexible. However, it is important that the collocation points should not coincide with any of the integration points. The values $g(t_i)$ in $H(t_i,\tau_j)$ may be represented by linear combination of $g(\tau_j)$ through Lagrangian interpolation

$$g(t_i)=\sum_{j=1}^{n}\lambda_{i,j}g(\tau_j) \tag{3.6}$$

where

$$\lambda_{i,j}=\prod_{k=1,\quad k\neq j}^{n}\frac{t_i-\tau_k}{\tau_j-\tau_k}. \tag{3.7}$$

By virtue of Eq. (3.6), Eq. (3.5) may be converted to a system of linear algebraic equations

$$\sum_{j=1}^{n} c_{ij}g(\tau_j)\approx\pi f'(t_i),\quad i=1,\ldots,n-1, \tag{3.8}$$

$$c_{ij}=A_j^{(\alpha,\beta)}\left[\frac{1}{\tau_j-t_i}+\frac{\pi}{\lambda}M\left(\frac{\tau_j-t_i}{\lambda}\right)\right]-\lambda_{ij}\sum_{k=1}^{n}\frac{A_k^{(\alpha,\beta)}}{\tau_k-t_i}. \tag{3.9}$$

The additional condition (2.8) may be expressed as

$$\sum_{j=1}^{n} A_j^{(\alpha,\beta)}g(\tau_j)\approx 1. \tag{3.10}$$

Eqs. (3.8) and (3.10) provide n linear algebraic equations to determine $g(\tau_j)$, ($j=1\ldots n$).

It is obvious that, the approximate solution of the Eqs. (2.9) and (2.10) occurs as follows

$$\varphi_n(t)=\sum_{i=1}^{n}\frac{g(\tau_i)P_n^{(\alpha,\beta)}(t)}{(t-\tau_i)P_n^{(\alpha,\beta)\prime}(\tau_i)}, \tag{3.11}$$

where $\varphi(t_i)=\varphi_n(t_i)$ ($i=1\ldots n$) are valid.

The function $\varphi_n(t)$ will be defined if we find the number $g(\tau_i)$ (i=1... n) from the convenient system of n linear algebraic Eqs (3.8) and (3.10).

The absolute error between approximate and exact solutions will be estimated by a theorem given in [14].

4. Numerical Calculations

We will confine ourselves to consider the case of a plane punch, which interacts with the boundary of a strip y=h. In this case $\delta(x) = \delta_0, -c \le x \le c$, consequently $f'(t) = 0, -1 \le t \le 1$. The parameter δ_0 denotes the displacement of the punch in the vertical direction.

Table 1. The values $M_*, t_*, \varphi(t_*), \varphi(-0.9)$ and $\varphi(0.9)$ obtained by various values of λ and ν for the Problem 1

λ	ν	M_*	t_*	$\varphi(t_*)$	$\varphi(-0.9)$	$\varphi(0.9)$
μ=0.5						
0.5	0.1	-0.037	-0.118	0.461	0.555	0.495
0.5	0.45	-0.106	-0.824	0.334	0.350	0.605
1	0.1	-0.053	0.356	0.406	0.644	0.553
1	0.45	0.075	-0.668	0.377	0.464	0.658
2	0.1	-0.078	0.075	0.353	0.745	0.600
2	0.45	0.036	-0.252	0.368	0.601	0.706
4	0.1	-0.102	0.084	0.324	0.811	0.609
4	0.45	0.007	-0.054	0.335	0.690	0.722
8	0.1	-0.118	0.105	0.314	0.844	0.601
8	0.45	-0.009	-0.009	0.323	0.730	0.717
μ=0.9						
0.5	0.1	-0.064	-0.544	0.462	0.568	0.464
0.5	0.45	0.182	-0.830	0.280	0.292	0.737
1	0.1	-0.093	0.479	0.399	0.665	0.508
1	0.45	0.134	-0.711	0.330	0.396	0.744
2	0.1	-0.136	0.142	0.347	0.784	0.539
2	0.45	0.066	-0.381	0.359	0.559	0.746
4	0.1	-0.177	0.148	0.313	0.870	0.531
4	0.45	0.013	-0.097	0.335	0.676	0.734
8	0.1	-0.206	0.182	0.299	0.918	0.511
8	0.45	-0.017	-0.016	0.323	0.734	0.712

We will use the notations $M_* = M/(Pc)$. The collocation points in the discussion below are specified as the mid-point of two successive integration points.

Tables 1 and 2 give the values of $M_*, t_*, \varphi(t_*), \varphi(-0.9)$ and $\varphi(0.9)$ for Problem 1 and 2, where $x_* = ct_*$ is the point of the contact area with the minimum contact stresses for certain values λ, ν and μ.

Figs 2 and 3 show the influence of Poisson's ratio on the dimensionless contact stresses $\varphi(t)$ in the contact area [-1, 1] for Problem 1 when $\lambda = 2$, $\mu = 0.8$ and for Problem 2 when $\lambda = 3$, $\mu = 0.7$ respectively.

Table 2. The values $M_*, t_*, \varphi(t_*), \varphi(-0.9)$ and $\varphi(0.9)$ obtained by various values of λ and ν for the Problem 2

λ	ν	M_*	t_*	$\varphi(t_*)$	$\varphi(-0.9)$	$\varphi(0.9)$
μ=0.5						
0.5	0.1	0.008	0.360	0.470	0.500	0.549
0.5	0.45	0.285	-0.823	0.216	0.228	0.934
1	0.1	-0.031	-0.297	0.405	0.620	0.585
1	0.45	0.141	-0.634	0.337	0.432	0.800
2	0.1	-0.068	0.029	0.351	0.736	0.617
2	0.45	0.061	-0.241	0.346	0.597	0.767
4	0.1	-0.097	0.072	0.323	0.807	0.617
4	0.45	0.017	-0.072	0.328	0.686	0.745
8	0.1	-0.116	0.010	0.314	0.842	0.604
8	0.45	-0.005	-0.018	0.321	0.726	0.726
$\mu = 0.9$						
0.5	0.1	0.014	-0.710	0.402	0.470	0.553
0.5	0.45	0.492	-0.931	0.711	0.072	1.300
1	0.1	-0.056	-0.378	0.405	0.618	0.557
1	0.45	0.253	-0.746	0.257	0.299	0.960
2	0.1	-0.119	0.060	0.347	0.765	0.564
2	0.45	0.110	-0.393	0.334	0.528	0.834
4	0.1	-0.170	0.126	0.314	0.861	0.542
4	0.45	0.032	-0.129	0.327	0.661	0.767
8	0.1	-0.202	0.174	0.299	0.914	0.515
8	0.45	-0.009	-0.033	0.321	0.725	0.725

The results of calculations given in Tables 1 and 2 and also Figs. 2 and 3 are enable us to draw a number of important conclusions. When the Poisson's ratio increases or the coefficient friction reduces assuming the value of the relative thickness of the strip is fixed, the quantity t_* is displaced in the negative direction of the *x* axis. In these cases the nature of the contact pressure distributions in the vicinity of the contact zones ends also changes: while the values of $\varphi(t)$ drop in the vicinity of the point *t=-1*, it grows in the vicinity of the point *t=1*.

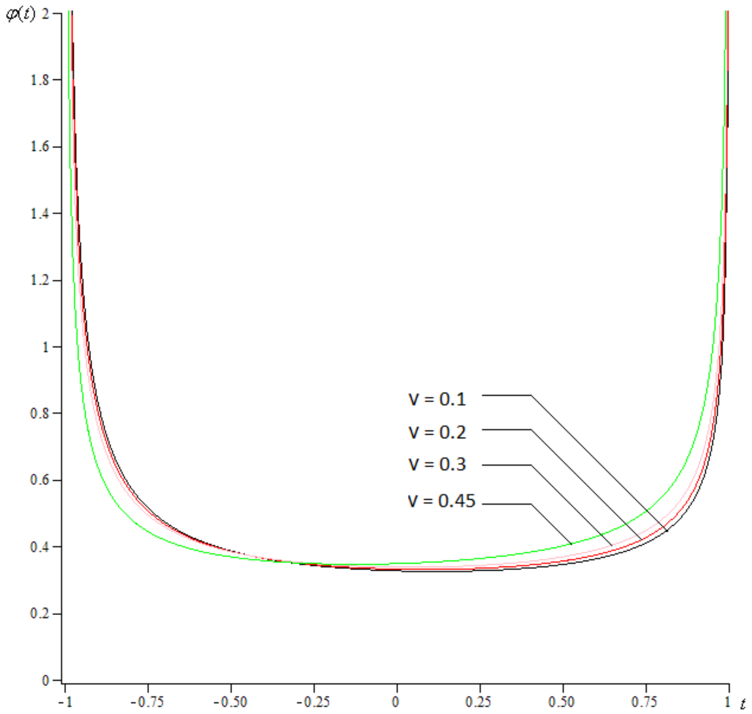

Fig. 2. The influence of the v on the $\varphi\,(t)$ for the Problem 1 for the case $\lambda = 2$, $\mu = 0.8$.

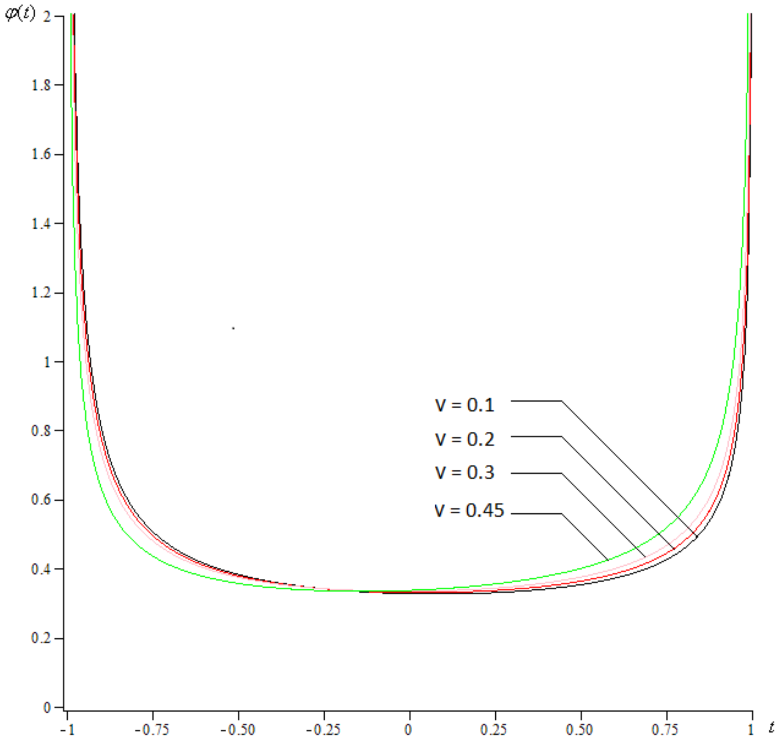

Fig. 3. The influence of the v on the $\varphi\,(t)$ for the Problem 2 for the case $\lambda = 3$, $\mu = 0.7$.

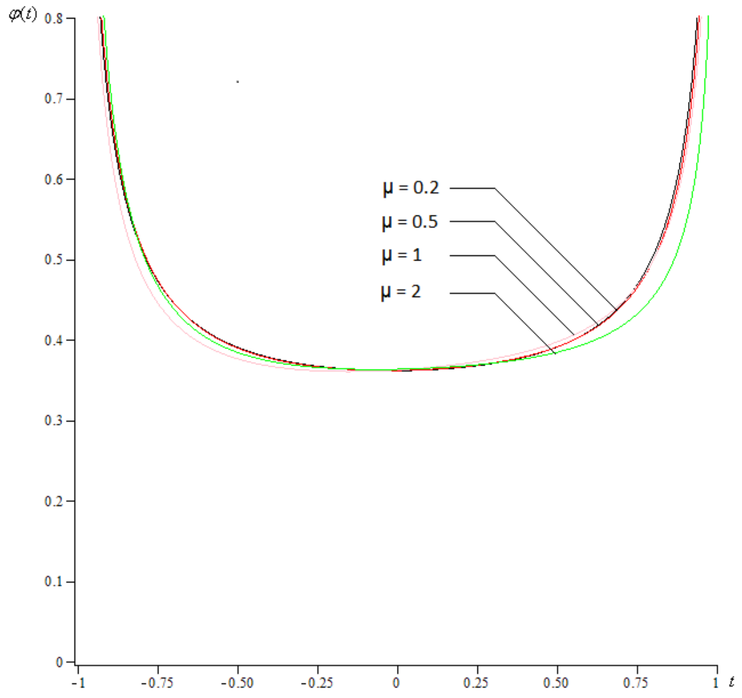

Fig. 4. The graphs of the dependencies between φ (t) and μ for the Problem 1 for the case λ = 2, ν = 0.3.

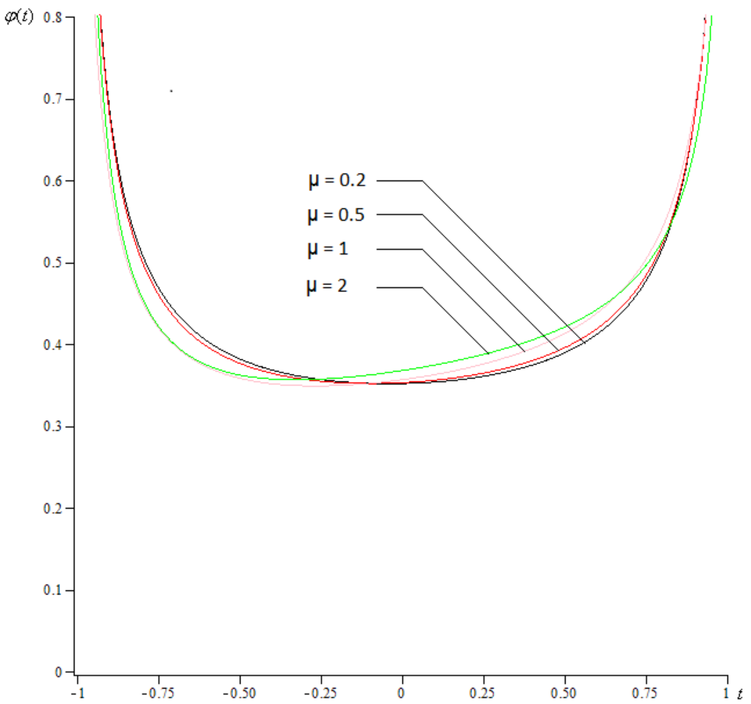

Fig. 5. The graphs of the dependencies between φ (t) and μ for the Problem 2 for the case λ = 2, ν = 0.3.

The graphs given in Figs. 4 and 5 show the influence of the coefficient of friction on the dimensionless contact pressure under λ = 2, ν = 0.3 for Problem 1 and Problem 2. As is seen, the intensity of pressure grow at the left semi contact zone, and drops at the right semi contact zone by increasing the coefficient of friction.

Figs 6 and 7 show the effect of the relative thickness λ on the normalized φ (t) for Problem 1 and when Problem 2 v = 0.3, μ = 0.6. It might be observed that the increase of intensivity of pressure occurs in the left semi contact zone, when the opposite in the right semi contact zone.

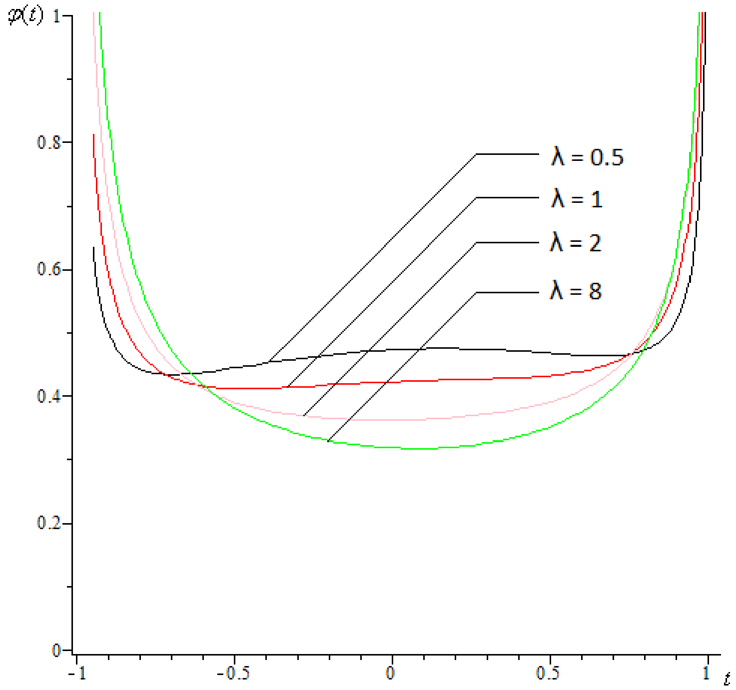

Fig. 6. The dependencies between φ (t) and λ for the Problem 1 for the case v = 0.3, μ = 0.6.

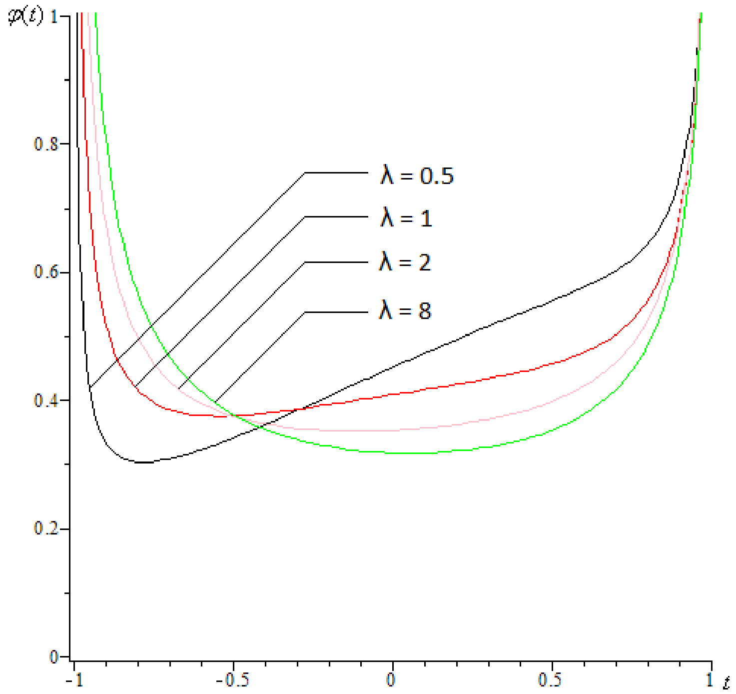

Fig. 7. The dependencies between φ (t) and λ for the Problem 2 for the case where v = 0.3, μ = 0.6.

5. Conclusion

In this paper, the contact of a punch with an elastic isotropic layer, taking Coulomb friction in the contact region into account, is considered. It is assumed that either the lower boundary of the layer is fixed or three are no normal displacements and shear stresses on it, and that normal and shear forces are acting on the punch. The contact problem is reformulated by Cauchy type singular integral equation of the second kind in which the unknown is the normal contact stresses beneath the punch. A practical quadrature-collocation method is adapted to transmission of the singular integral equation to the system of the algebraic linear equations. The indentation of a plane punch to layer is examined and the effects of geometrical and mechanical parameters of the materials on distribution contact pressure are studied in order to demonstrate the efficiency of the developed methodology.

References

[1]. Ufliand I. S. Integral Transforms in the Problem of the Theory of Elastisity, Leningrad, Nauka, 1967.

[2]. Vorovich, I.I., Aleksandrov, V.I. And Babeshko, V.A., Nonclassical Mixed Problems of the Theory of Elasticity. Moscow, Nauka, 1974

[3]. Aleksandrov, V.M. and Kovalenko, Ye.V., Problems of Continuum Mechanics with Mixed Boundary Conditions, Nauka, Moscow, 1986.

[4]. Aleksandrov, V.M., On Plane contact problems of the theory of elasticity in the presence of adhesion and friction. Journal of Applied Mathematics and Mechanics vol. 34/2, (1970), p.232-243.

[5]. Jing, H.-S., Liao, M.-L., An improved finite element scheme for elastic contact problems with friction. Computers and Structures 35 (1990), 571-578.

[6]. L.F. Ma, A.M. Korsunsky Fundamental formulation for frictional contact problems of coated systems, International Journal of Solids and Structures 41 (2004) 2837-2854

[7]. Chebakov M. I. Asymptotic solution of contact problems for a relatively thick elastic layer when there are friction forces in the contact area, Journal of Applied Mathematics and Mechanics 69 (2005) 296-304.

[8]. N.I. Muskhelishvili, Singular Integral Equations, Noordhoff, Leiden, 1953.

[9]. Rostovstev N.A., On some cases of the contact problems, Journal of Ukrainian Mathematics (in Russian), Vol 6 (3), 1954.

[10]. Erbaş B, Yusufoğlu E, Kaplunov J. A plane contact problem for an elastic orthotropic strip. J. Engrg. Math. 70: 399-409, 2011.

[11]. F.E. Erdogan, G.D. Gupta, and T.S. Cook, The numerical solutions of singular integral equations, in Mechanics of Fracture. Volume 1. Methods of Analysis and Solution of Crack Problems, pp. 368-425, Noordhoff Intern. Publ., Leyden, 1973.

[12]. S. Krenk, On quadrature formulas for singular integral equations of the first and second kind, Quarterly of applied mathematics, 33 (1975) 225-232.

[13]. Xiaoqing Jin, Leon M. Keer, Qian Wang, A practical method for singular integral equations of the second kind, Engineering Fracture Mechanics 75 (2008) 1005-1014.

[14]. Belocerkovskiy S.M. and Lifanov I.K., Numerical methods in Singular integral Equations, Nauka, Moskow (1985).

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