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Asymptotically Lacunary Statistical Equivalent Sequence Spaces Defined by Ideal Convergence and an Orlicz Function

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Abstract: In this study, we introduced the concepts asymptotically I_M^p -lacunary equivalence with order α and asymptotically I_M-lacunary statistical equivalence with order α by using a non-trivial ideal I, an Orlicz function M and a sequence of positive real numbers $p=(p_k)$. In addition to these definitions, we also presented some inclusion theorems.

Keywords: Asymptotically equivalence, Ideal convergence, Lacunary sequence, Orlicz function.

1. Introduction

Let s be the space of real valued sequences and any subspace of s is also called a sequence space. ℓ_∞ and c denote the spaces of all bounded and convergent sequences, respectively.

A lacunary sequence is an increasing sequence $\theta=(k_r)$ such that $k_0=0, h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$ .The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$ and $q_r=k_r/k_{r-1}$ .These notations will be used troughout the paper. The sequence space of lacunary strongly convergent sequences N_θ was defined by Freedman et al.[5], as following:

$$N_\theta=\{x=(x_i)\in s:\lim_r h_r^{-1}\sum_{i\in I_r}|x_i-s|=0 \text{ for some s}\}.$$

Orlicz [8] used the idea of Orlicz function to construct the space L^M.

An Orlicz function is a function $M:[0,\infty)\to[0,\infty)$,which is continuous, nondecreasing and convex with $M(0) = 0, M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists constant $K > 0$, such that $M(2u) \leq KM(u)$ $(u \geq 0)$. It is also easy to see that $K > 2$ aways holds . The Δ_2-condition is equivalent to the satisfaction of inequality $M(Lu) \leq KLM(u)$ for all values of u and $L > 1$.

Remark 1. An Orlicz function M satisfies the inequality $M(\lambda x) < \lambda M(x)$ for all λ with $0 < \lambda < 1$.

The following well known inequality will be used troughout the paper;

(1) $$|a_i + b_i|^{p_i} \leq T(|a_i|^{p_i} + |b_i|^{p_i}),$$

where a_i and b_i are complex numbers, $T = \max(1, 2^{H-1})$, and $H = \sup p_i < \infty$.

Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices in [7]. Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices in [9]. Subsequently,many authors have shown their interest to solve different problems arising in this area (see [2], and [10]).

The concept of I convergence was introduced by Kostyrko et al. [5], which is a generalization of statistical convergence. Recently, Das et al. [2,3] unified these two approaches to introduce new concepts such as I- statistical convergence and I-lacunary statistical convergence and investigated some of their consequences. More investigations in this direction and more applications can be found in [1, 3, 4, 11].

In this paper we introduce the concepts asymptotically I_M^p -lacunary equivalence with order α and asymptotically I_M-lacunary statistical equivalence with order α,by using a non-trivial ideal I , an Orlicz function M ,and a sequence of positive real numbers $p = (p_k)$ and also some inclusion theorems are proved.

2. Definitions and Notations

In this section, we recall the basic definitions and concepts. For simplicity, below and in what follows limits run to ∞, that is, we use "$\lim_k x_k$" instead of "$\lim_{k\to\infty} x_k$" .

Definition 2.1. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k} = 1$,(denoted by $x \backsim y$).

Definition 2.2. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $x \overset{S}{\backsim} y$) and simply asymptotically statistical equivalent, if $L = 1$.

Definition 2.3.Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strong asymptotically equivalent of multiple L provided that,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| = 0$$

(denoted by $x \overset{w}{\backsim} y$) and simply strong asymptotically equivalent, if $L = 1$.

Definition 2.4. Let θ be a lacunary sequence; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $x \overset{S_\theta}{\backsim} y$) and simply asymptotically lacunary statistical equivalent, if $L = 1$.

Definition 2.5. Let θ be a lacunary sequence; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strong asymptotically lacunary equivalent of multiple L provided that,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| = 0$$

(denoted by $x \overset{N_\theta}{\backsim} y$) and simply strong asymptotically lacunary equivalent, if $L = 1$.

Definition 2.6. Let M be any Orlicz function; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be M-asymptotically equivalent of multiple L provided that,

$$\lim_k M(\left| \frac{x_k}{y_k} - L \right| / \rho) = 0$$

for some $\rho > 0$, (denoted by $x \overset{M}{\backsim} y$) and simply $M-$asymptotically equivalent, if $L = 1$.

Definition 2.7. Let M be any Orlicz function; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strong M-asymptotically equivalent of multiple L provided that,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} M(\left|\frac{x_k}{y_k} - L\right| /\rho) = 0$$

for some $\rho > 0$, (denoted by $x \overset{w^M}{\backsim} y$) and simply strong $M-$asymptotically equivalent, if $L = 1$.

Definition 2.8. Let M be any Orlicz function and θ be a lacunary sequence; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strong M-asymptotically lacunary equivalent of multiple L provided that,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} M(\left|\frac{x_k}{y_k} - L\right| /\rho) = 0$$

for some $\rho > 0$, (denoted by $x \overset{N_\theta^M}{\backsim} y$) and simply strong M-asymptotically lacunary equivalent, if $L = 1$.

Let X be a non-empty set. Then $P(X)$ denote the power set of X, that is, the space of all subsets of X.

Definition 2.9. A family $I \subseteq P(X)$ is said to be an ideal in X if the following conditions hold:

(i) $\emptyset \in I$;
(ii) $A, B \in I$ imply $A \cup B \in I$ and
(iii) $A \in I, B \subset A$ imply $B \in I$.

Definition 2.10. A non-empty family $F \subseteq P(X)$ is said to be a filter in X if the following conditions hold:

(i) $\emptyset \notin F$;
(ii) $A, B \in F$ imply $A \cap B \in F$ and
(iii) $A \in F, B \supset A$ imply $B \in F$.

An ideal I is said to be non-trivial if $I \neq \{\emptyset\}$ and $X \notin I$. A non-trivial ideal I is called admissible if it contains all the singleton sets. Moreover, if I is a non-trivial ideal on X, then $F = F(I) = \{X - A : A \in I\}$ is a filter on X and conversely. The filter$F(I)$ is called the filter associated with the ideal I.

Definition 2.11. Let $I \subset P(N)$ be a non-trivial ideal in N. A sequence $x = (x_k)$ in X is said to be I-convergent to L if for each $\varepsilon > 0$, the set

$$\{k \in N : |x_k - L| \geq \varepsilon\} \in I.$$

In this case, we write $I - \lim_k x_k = L$. Let $x = (x_k)$ be a sequence in X. Then it is said to be $I - null$ if $L = 0$. In this case we write $I - \lim_k x_k = 0$.

Definition 2.12. A sequence $x = (x_k)$ of numbers is said to be I-statistical convergent or $S(I)$-convergent to L, if for every $\varepsilon > 0$ and $\delta > 0$, we have

$$\left\{ n \in N; \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| \geq \delta \right\} \in I.$$

In this case, we write $x_k \to L(S(I))$ or $S(I)$ - $\lim_k x_k = L$.

Definition 2.13 Let $I \subset P(N)$ be a non-trivial ideal in N. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically equivalent of multiple L with respect to the ideal I provided that for each $\varepsilon > 0$,

$$\left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in I,$$

(denoted by $x \overset{I(w)}{\backsim} y$) and simply strongly asymptotically equivalent with respect to the ideal I, if $L = 1$.

Definition 2.14. Let $I \subset P(N)$ be a non-trivial ideal in N and $\theta = (k_r)$ be a lacunary sequence. The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L with respect to the ideal I provided that for each $\varepsilon > 0$ and $\gamma > 0$,

$$\left\{ r \in N; \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in I$$

(denoted by $x \overset{I(S_\theta)}{\backsim} y$) and simply asymptotically lacunary statistical equivalent with respect to the ideal I, if $L = 1$.

Definition 2.15. Let $I \subset P(N)$ be a non-trivial ideal in N and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically lacunary equivalent of multiple L

with respect to the ideal I provided that for $\varepsilon > 0$,

$$\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in I$$

(denoted by $x \overset{I(N_\theta)}{\backsim} y$) and simply asymptotically lacunary equivalent with respect to the ideal I, if $L = 1$.

Quite recently Bilgin [1] and Savas[11] have given the following definitions.

Definition 2.16. Let $I \subset P(N)$ be a non-trivial ideal in N, M be any Orlicz function, $\theta = (k_r)$ be a lacunary sequence and $p = (p_k)$ be a sequence of positive real numbers. Two sequences $x = (x_k)$ and $y = (y_k)$ are said to be (M,p)-asymptotically lacunary equivalent of multiple L with respect to the ideal I provided that for each $\varepsilon > 0$,

$$\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} \geq \varepsilon \right\} \in I$$

for some $\rho > 0$, (denoted by $x \overset{I(N_\theta^{(M,p)})}{\backsim} y$) and simply (M,p) -asymptotically lacunary equivalent with respect to the ideal I, if $L = 1$.

Definition 2.17. Let $I \subset P(N)$ be a non-trivial ideal in N ,and $\theta = (k_r)$ be a lacunary sequence. Two sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically I-lacunary statistical equivalent of order α, where $0 < \alpha \leq 1$, to multiple L provided that for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ r \in N; \frac{1}{h_r^\alpha} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in I.$$

In this case we write $x \overset{S_\theta^L(I)^\alpha}{\backsim} y$.

Definition 2.18. Let $I \subset P(N)$ be a non-trivial ideal in N , $\theta = (k_r)$ be a lacunary sequence and $p = (p_k)$ be a sequence of positive real numbers. Two sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically I -lacunary equivalent of order α, where $0 < \alpha \leq 1$, to multiple L for the sequence p provided that for any $\varepsilon > 0$,

$$\left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right|^{p_k} \geq \varepsilon \right\} \in I$$

for some $\rho > 0$, (denoted by $x \overset{N_\theta^{L_p}(I)^\alpha}{\backsim} y$).

3. Main Results

In this section we shall give some new definitions and also prove some inclusion relations.

We begin with the following definitions.

Definition 3.1. Let $I \subset P(N)$ be a non-trivial ideal in N , M be any Orlicz function ,and $\theta = (k_r)$ be a lacunary sequence. Two number sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically I_M-lacunary statistical equivalent of order α, where $0 < \alpha \leq 1$, to multiple L provided that for any $\varepsilon > 0$ and $\delta > 0$.

$$\left\{ r \in N; \frac{1}{h_r^\alpha} \left| \left\{ k \in I_r : M(\left| \frac{x_k}{y_k} - L \right| /\rho) \geq \varepsilon \right\} \right| \geq \delta \right\} \in I$$

for some $\rho > 0$. In this case we write $x \overset{I_M(S_\theta^\alpha)}{\backsim} y$. if we put $M(x) = x$ for $x \geq 0$, we write $x \overset{I(S_\theta^\alpha)}{\backsim} y$. Hence $x \overset{I(S_\theta^\alpha)}{\backsim} y$ is the same as the $x \overset{S_\theta^L(I)^\alpha}{\backsim} y$ of Savas [11].

Definition 3.2. Let $I \subset P(N)$ be a non-trivial ideal in N , M be any Orlicz function, $\theta = (k_r)$ be a lacunary sequence and $p = (p_k)$ be a sequence of positive real numbers. Two sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically I_M^p -lacunary equivalent of order α, where $0 < \alpha \leq 1$, to multiple L for the sequence p provided that for any $\varepsilon > 0$,

$$\left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| /\rho) \right]^{p_k} \geq \varepsilon \right\} \in I$$

for some $\rho > 0$, (denoted by $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$). If we take $\alpha = 1$, we write $x \overset{I_M^p(N_\theta)}{\backsim} y$ instead of $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$. Hence $x \overset{I_M^p(N_\theta)}{\backsim} y$ is the same as the $x \overset{I(N_\theta^{(M,p)})}{\backsim} y$ of Bilgin [1].

Also if we put $M(x) = x$ for $x \geq 0$, we write $x \overset{I^p(N_\theta^\alpha)}{\backsim} y$ instead of $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$. Hence $x \overset{I^p(N_\theta^\alpha)}{\backsim} y$ is the same as the $x \overset{N_\theta^{L_p}(I)^\alpha}{\backsim} y$ of Savas [11].

If we take $(\frac{1}{n^\alpha})$ instead of $(\frac{1}{h_r^\alpha})$,denoted by $x \overset{I_M^p(w^\alpha)}{\backsim} y$ and simply strongly Cesaro I_M^p -asymptotically equivalent of order α, if $L = 1$.

If we take $p_k = 1$ for all $k \in N$, we write $x \overset{I_M(N_\theta^\alpha)}{\backsim} y$ instead of $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$.

We now prove some inclusion theorems.

Theorem 3.1. Let $I \subset P(N)$ be a non-trivial ideal in N , M be any Orlicz function , $\theta = (k_r)$ be a lacunary sequence, $p = (p_k)$ be a sequence of positive real numbers and $0 < \alpha_1 \leq \alpha_2 \leq 1$ then $x \overset{I_M^p(N_\theta^{\alpha 1})}{\backsim} y$ implies $x \overset{I_M^p(N_\theta^{\alpha 2})}{\backsim} y$.

Proof. Let $0 < \alpha_1 \leq \alpha_2 \leq 1$ and $x \overset{I_M^p(N_\theta^{\alpha 1})}{\backsim} y$. Since $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$,we can actually choose r, so that $h_r^{\alpha_1} \leq h_r^{\alpha_2}$ and $\frac{1}{h_r^{\alpha 2}} \leq \frac{1}{h_r^{\alpha 1}}$. Hence

$$\frac{1}{h_r^{\alpha_2}} \sum_{k \in I_r} \left[M(\left|\frac{x_k}{y_k} - L\right| / \rho) \right]^{p_k} \leq \frac{1}{h_r^{\alpha_1}} \sum_{k \in I_r} \left[M(\left|\frac{x_k}{y_k} - L\right| / \rho) \right]^{p_k}.$$

And so,

$$\left\{ r \in N; \frac{1}{h_r^{\alpha_2}} \sum_{k \in I_r} \left[M(\left|\frac{x_k}{y_k} - L\right| / \rho) \right]^{p_k} \geq \varepsilon \right\} \subset \left\{ r \in N; \frac{1}{h_r^{\alpha_1}} \sum_{k \in I_r} \left[M(\left|\frac{x_k}{y_k} - L\right| / \rho) \right]^{p_k} \geq \varepsilon \right\}.$$

Finally, we have

$$\left\{ r \in N; \frac{1}{h_r^{\alpha_2}} \sum_{k \in I_r} \left[M(\left|\frac{x_k}{y_k} - L\right| / \rho) \right]^{p_k} \geq \varepsilon \right\} \in I$$

Hence $x \overset{I_M^p(N_\theta^{\alpha 2})}{\backsim} y$.

One can have the following result by setting $\alpha_2 = 1$ in Theorem 3.1.

Corollary. Let $I \subset P(N)$ be a non-trivial ideal in N , M be any Orlicz function , $\theta = (k_r)$ be a lacunary sequence, $p = (p_k)$ be a sequence of positive real numbers, and $0 < \alpha \leq 1$, then $x \overset{I_M(N_\theta^\alpha)}{\backsim} y$ implies $x \overset{I_M(N_\theta)}{\backsim} y$.

Theorem 3.2. Let $I \subset P(N)$ be a non-trivial ideal in N, M_1, M_2 be any Orlicz functions, $\theta = (k_r)$ be a lacunary sequence, $p = (p_k)$ be a sequence of positive real numbers and $0 < \alpha \leq 1$. Then $x \overset{I_{M_1 \cap M_2}((N_\theta^\alpha)}{\backsim} y$ implies $x \overset{I_{(M_1+M_2)}(N_\theta^\alpha)}{\backsim} y$.

Proof. Now suppose that $x \overset{I_{(M_1 \cap M_2)}(N_\theta^\alpha)}{\backsim} y$ and $\varepsilon > 0$. Let the set

$$A = \left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M_1(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} < \varepsilon/2 \right\}$$

and

$$B = \left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M_2(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} < \varepsilon/2 \right\}$$

be given for some $\rho > 0$. Then we have,

$$\left[(M_1 + M_2)(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} = [M_1(\left| \frac{x_k}{y_k} - L \right| / \rho) + M_2(\left| \frac{x_k}{y_k} - L \right| / \rho)]^{p_k}$$

$$\leq T\{ \left[M_1(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} + \left[M_2(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} \},$$

with $T = \max(1, 2^{H-1})$, and $H = \sup p_i < \infty$. So

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[(M_1 + M_2)(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} \leq T\{ \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M_1(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} +$$

$$+ \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M_2(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} \}$$

$$< T\varepsilon = \varepsilon^{'}.$$

It follows that for any $\varepsilon^{'} > 0$,

$$\left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[(M_1 + M_2)(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} < \varepsilon^{'} \right\} \in F(I)$$

which yields that $x \overset{I_{(M_1+M_2)}(N_\theta^\alpha)}{\backsim} y$.

The next theorem shows relationship between the strongly I_M^p-asymptotically equivalence and the I_M^p-asymptotically lacunary equivalence with respect to the ideal I.

Theorem 3.3. Let $I \subset P(N)$ be a non-trivial ideal in N, M be any Orlicz function, $\theta = (k_r)$ be a lacunary sequence, $p = (p_k)$ be a sequence of positive real numbers and $0 < \alpha \leq 1$. Then following propositions are true.

(i) If $\sup_r \frac{1}{k_{r-1}^\alpha} \sum\limits_{m=1}^{r} (k_m - k_{m-1})^\alpha = B(say) < \infty$ then $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$ implies $x \overset{I_M^p(w^\alpha)}{\backsim} y$,

(ii) If $\sup_r \frac{k_r}{h_r^\alpha} = C(say) < \infty$ then $x \overset{I_M^p(w^\alpha)}{\backsim} y$ implies $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$.

Proof. (i): Now suppose that $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$ and $\varepsilon > 0$. Let

$$A = \left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} < \varepsilon \right\},$$

for some $\rho > 0$. Hence,for all $j \in A$ and for some $\rho > 0$, we have

$$H_j = \frac{1}{h_j^\alpha} \sum_{k \in I_j} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} < \varepsilon.$$

Choose n is any integer with $k_r \geq n > k_{r-1}$ where $r \in A$. Now write

$$\frac{1}{n^\alpha} \sum_{k=1}^{n} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} \leq \frac{1}{k_{r-1}^\alpha} \sum_{k=1}^{k_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k}$$

$$= \frac{1}{k_{r-1}^\alpha} \left\{ \sum_{k \in I_1} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} + ... + \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} \right\}$$

$$= \frac{1}{k_{r-1}^\alpha} \left\{ \frac{k_1^\alpha}{h_1^\alpha} \sum_{k \in I_1} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} + ... + \frac{(k_r - k_{r-1})^\alpha}{h_r^\alpha} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} \right\}$$

$$= \frac{1}{k_{r-1}^\alpha} \left\{ k_1^\alpha H_1 + (k_2 - k_1)^\alpha H_2 + ... + (k_r - k_{r-1})^\alpha H_r \right\}$$

$$= \frac{1}{k_{r-1}^{\alpha}} \sum_{m=1}^{r} \frac{k_m - k_{m-1}}{h_m^{\alpha}} \sum_{k \in I_m} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k}$$

$$= \frac{1}{k_{r-1}^{\alpha}} \sum_{m=1}^{r} (k_m - k_{m-1})^{\alpha} \sup_{j \in A} H_j$$

$$\leq B \sup_{j \in A} H_j$$

$$< B\varepsilon = \varepsilon^{\prime}.$$

It follows that for any $\varepsilon^{\prime} > 0$,

$$\left\{ n \in N; \frac{1}{n^{\alpha}} \sum_{k=1}^{n} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} < \varepsilon^{\prime} \right\} \in F(I)$$

since for any set

$$A \in F(I), \cup \{ n : k_{r-1} < n < k_r, r \in A \} \in F(I).$$

which yields that $x \overset{I_M^p(w^{\alpha})}{\backsim} y$.

(ii): Let $x \overset{I_M^p(w^{\alpha})}{\backsim} y$. Let us take $\varepsilon > 0$ and define the set,

$$A = \left\{ k_r \in N; \frac{1}{k_r} \sum_{k=1}^{k_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} < \varepsilon \right\},$$

for some $\rho > 0$. We have $A \in F(I)$, which is the filter of the ideal I. For each $k_r \in A$, we have, for some $\rho > 0$,

$$\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} = \frac{1}{h_r^{\alpha}} \sum_{k=1}^{k_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} - \frac{1}{h_r^{\alpha}} \sum_{k=1}^{k_{r-1}} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k}$$

$$\leq \frac{1}{h_r^{\alpha}} \sum_{k=1}^{k_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k}$$

$$\leq \frac{k_r}{h_r^{\alpha}} \frac{1}{k_r} \sum_{k=1}^{k_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k}$$

$$< C\varepsilon = \varepsilon'$$

It follows that for any $\varepsilon' > 0$,

$$\left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} < \varepsilon' \right\} \in F(I)$$

which yields that $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$.

Now we give relation between asymptotically I-lacunary statistical equivalence and I_M^p-asymptotically lacunary equivalence of order α.

Theorem 3.4. Let $I \subset P(N)$ be a non-trivial ideal in N, M be any Orlicz function , $\theta = (k_r)$ be a lacunary sequence, $p = (p_k)$ be a sequence of positive real numbers and $0 < \alpha \leq 1$. If $h = \inf p_k \leq \sup p_k = H < \infty$, then $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$ implies $x \overset{I(S_\theta^\alpha)}{\backsim} y$.

Proof. Take $\varepsilon > 0$ and let $\sum_1$ denote the sum over $k \in I_r$, with $\left| \frac{x_k}{y_k} - L \right| \geq \varepsilon$. Then,

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} \geq \frac{1}{h_r^\alpha} \sum_1 \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k}$$

$$\geq \frac{1}{h_r^\alpha} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| M(\varepsilon/\rho)^{p_k}$$

$$\geq \frac{1}{h_r^\alpha} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \min\{M(\varepsilon/\rho)^h, M(\varepsilon/\rho)^H\}$$

and

$$\left\{ r \in N; \frac{1}{h_r^\alpha} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\}$$

$$\subseteq \left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right|) \right]^{p_k} \geq \gamma \min\{M(\varepsilon/\rho)^h, M(\varepsilon/\rho)^H\} \right\} \in I.$$

But then, by definition of an ideal, later set belongs to I, and therefore $x \overset{I(S_\theta^\alpha)}{\backsim} y$.

Theorem 3.5. Let $I \subset P(N)$ be a non-trivial ideal in N, M be any Orlicz function, $\theta = (k_r)$ be a lacunary sequence, $p = (p_k)$ be a sequence of positive real numbers and $0 < \alpha \le 1$. If $h = \inf p_k \le \sup p_k = H < \infty$, then $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$ implies $x \overset{I_M(S_\theta^\alpha)}{\backsim} y$,

Proof. Take $\varepsilon > 0$ and let $\sum_1$ denote the sum over $k \in I_r$ with $M(\left|\frac{x_k}{y_k} - L\right|/\rho) \ge \varepsilon$. Then,

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r}\left[M(\left|\frac{x_k}{y_k} - L\right|/\rho)\right]^{p_k} \ge \frac{1}{h_r^\alpha}\sum_1\left[M(\left|\frac{x_k}{y_k} - L\right|/\rho)\right]^{p_k}$$

$$\ge \frac{1}{h_r^\alpha}\left|\left\{k \in I_r : M(\left|\frac{x_k}{y_k} - L\right|/\rho) \ge \varepsilon\right\}\right|(\varepsilon)^{p_k}$$

$$\ge \frac{1}{h_r^\alpha}\left|\left\{k \in I_r : M(\left|\frac{x_k}{y_k} - L\right|/\rho) \ge \varepsilon\right\}\right|\min\{\varepsilon^h, \varepsilon^H\}$$

and

$$\left\{r \in N; \frac{1}{h_r^\alpha}\left|\left\{k \in I_r : M(\left|\frac{x_k}{y_k} - L\right|/\rho) \ge \varepsilon\right\}\right| \ge \gamma\right\}$$

$$\subseteq \left\{r \in N; \frac{1}{h_r^\alpha}\sum_{k\in I_r}\left[M(\left|\frac{x_k}{y_k} - L\right|/\rho)\right]^{p_k} \ge \gamma\min\{\varepsilon^h, \varepsilon^H\}\right\} \in I.$$

But then, by definition of an ideal, later set belongs to I and therefore $x \overset{I_M(S_\theta^\alpha)}{\backsim} y$.

Theorem 3.6. Let $I \subset P(N)$ be a non-trivial ideal in N, M be any Orlicz function that satisfy the Δ_2-condition, $\theta = (k_r)$ be a lacunary sequence, and $0 < \alpha \le 1$. Then $x \overset{I(S_\theta^\alpha)}{\backsim} y$ implies $x \overset{I_M(S_\theta^\alpha)}{\backsim} y$.

Proof. Let $x \overset{I(S_\theta^\alpha)}{\backsim} y$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that $M(u) < \varepsilon/2$ for every u with $0 \le u \le \delta$.

Let $(\left|\frac{x_k}{y_k} - L\right|/\rho) \le \delta$, then $M(\left|\frac{x_k}{y_k} - L\right|/\rho) < \varepsilon/2$. For $(\left|\frac{x_k}{y_k} - L\right|/\rho) > \delta$ we use the fact that $(\left|\frac{x_k}{y_k} - L\right|/\rho) < (\left|\frac{x_k}{y_k} - L\right|/\rho)/\delta < 1 + (\left|\frac{x_k}{y_k} - L\right|/\rho)/\delta$. Since M is non-decreasing and convex, it follows that

$M(\left|\frac{x_k}{y_k}-L\right|/\rho)<M(1+(\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta)<\frac{1}{2}M(2)+\frac{1}{2}M(2(\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta).$
Since M satisfies the Δ_2-condition, therefore
$M(\left|\frac{x_k}{y_k}-L\right|/\rho)<\frac{1}{2}K((\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta)\ M(2)+\frac{1}{2}K((\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta)M(2)$
$=K(\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta)M(2).$
Hence.

$$M(\left|\frac{x_k}{y_k}-L\right|/\rho)\leq K(\left|\frac{x_k}{y_k}-L\right|/\rho)/\delta)M(2)+\varepsilon/2.$$

Thus

$$\left\{r\in N;\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:M(\left|\frac{x_k}{y_k}-L\right|/\rho)\geq\varepsilon\right\}\right|\ \geq\gamma\right\}$$

$$\subset\left\{r\in N;\frac{1}{h_r^\alpha}\left|\left\{k\in I_r:\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\rho\delta/2KM(2)\right\}\right|\ \geq\gamma\right\}.$$

Since $x \overset{I(S_\theta^\alpha)}{\backsim} y$ it follows the later set, and hence, the first set in above expression belongs to I. This proves that $x \overset{I_M(w^\alpha)}{\backsim} y$.

Let $p_k=p$ for all k , $t_k=t$ for all k and $0<p\leq t$. Then we can give following theorem.

Theorem 3.7. Let $I\subset P(N)$ be a non-trivial ideal in N, M be an Orlicz function, $\theta=(k_r)$ be a lacunary sequence and $0<\alpha\leq 1$ then $x \overset{I_M^t(N_\theta^\alpha)}{\backsim} y$ implies $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$.

Proof. Let $x \overset{I_M^t(N_\theta^\alpha)}{\backsim} y$. It follows from Holder's inequality,

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r}\left[M(\left|\frac{x_k}{y_k}-L\right|/\rho)\right]^p\leq(\frac{1}{h_r^\alpha}\sum_{k\in I_r}\left[M(\left|\frac{x_k}{y_k}-L\right|/\rho)\right]^t)^{p/t}$$

and

$$\left\{r\in N;\frac{1}{h_r^\alpha}\sum_{k\in I_r}\left[M(\left|\frac{x_k}{y_k}-L\right|/\rho)\right]^p\ \geq\varepsilon\right\}$$

$\subseteq \left\{ r \in N; \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^t \geq \varepsilon^{t/p} \right\} \in I$. Thus we have $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$.

We now consider that (p_k) and (t_k) are not constant sequences.

Theorem 3.8. Let $I \subset P(N)$ be a non-trivial ideal in N , M be an Orlicz function, $\theta = (k_r)$ be a lacunary sequence, $0 < p_k \leq t_k$ for all k, (t_k/p_k) be bounded, and $0 < \alpha \leq 1$, then $x \overset{I_M^t(N_\theta^\alpha)}{\backsim} y$ implies $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$.

Proof. Let $x \overset{I_M^t(N_\theta^\alpha)}{\backsim} y$. $z_k = \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{t_k}$ and $\lambda_k = (p_k / \ t_k)$, so that for $0 < \lambda \leq \lambda_k \leq 1$, we define the sequences (u_k) and (v_k) as follows. For $z_k \geq 1$;let $u_k = z_k$ and $v_k = 0$ and for $z_k < 1$; let $v_k = z_k$ and $u_k = 0$. Then, we have, $z_k = u_k + v_k$; $z_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that $u_k{}^{\lambda_k} \leq u_k \leq z_k$ and $v_k^{\lambda_k} \leq v_k^{\lambda}$. Therefore,

$$\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} z_k^{\lambda_k} = \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} (u_k^{\lambda_k} + v_k^{\lambda_k})$$
$$\leq \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} z_k + \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} v_k^{\lambda}.$$

Now for each r;

$$\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} v_k^{\lambda} = \sum_{k \in I_r} (\frac{1}{h_r^{\alpha}} v_k)^{\lambda} (\frac{1}{h_r^{\alpha}})^{1-\lambda}$$

$$\leq (\sum_{k \in I_r} \left[(\frac{1}{h_r^{\alpha}} v_k)^{\lambda} \right]^{1/\lambda})^{\lambda} (\sum_{k \in I_r} \left[(\frac{1}{h_r^{\alpha}})^{1-\lambda} \right]^{1/1-\lambda})^{1-\lambda}$$

$$< (\frac{1}{h_r} \sum_{k \in I_r} v_k)^{\lambda}.$$

and so

$$\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} \left[M(\left| \frac{x_k}{y_k} - L \right| / \rho) \right]^{p_k} = \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} z_k^{\lambda_k} \leq \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} z_k + (\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} v_k)^{\lambda}$$

$$= \left\{ \begin{array}{ll} \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} z_k & , z_k \geq 1 \\ \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} z_k + (\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} z_k)^{\lambda} & , z_k < 1 \end{array} \right\}$$

$$\leq \left\{ \begin{array}{ll} \frac{1}{h_r^\alpha}\sum\limits_{k\in I_r} z_k & , z_k \geq 1 \\ 2(\frac{1}{h_r^\alpha}\sum\limits_{k\in I_r} z_k)^\lambda & , z_k < 1 \end{array} \right\}.$$

If

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r}\left[M(\left|\frac{x_k}{y_k}-L\right|/\rho)\right]^{p_k} \geq \varepsilon$$

then

$$\left\{ \begin{array}{ll} \frac{1}{h_r^\alpha}\sum\limits_{k\in I_r}\left[M(\left|\frac{x_k}{y_k}-L\right|/\rho)\right]^{t_k} \geq \varepsilon & , z_k \geq 1 \\ \frac{1}{h_r^\alpha}\sum\limits_{k\in I_r}\left[M(\left|\frac{x_k}{y_k}-L\right|/\rho)\right]^{t_k} \geq \left(\frac{\varepsilon}{2}\right)^{1/\lambda} & , z_k < 1 \end{array} \right\}.$$

Hence

$$\left\{r\in N; \frac{1}{h_r^\alpha}\sum_{k\in I_r}\left[M(\left|\frac{x_k}{y_k}-L\right|/\rho)\right]^{p_k} \geq \varepsilon\right\}$$

$$\subseteq \left\{r\in N; \frac{1}{h_r^\alpha}\sum_{k\in I_r}\left[M(\left|\frac{x_k}{y_k}-L\right|/\rho)\right]^{t_k} \geq \min\left\{\varepsilon, \left(\frac{\varepsilon}{2}\right)^{1/\lambda}\right\}\right\} \in I.$$

Thus we have $x \overset{I_M^p(N_\theta^\alpha)}{\backsim} y$.

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