Asymptotically Lacunary Statistical Equivalent Sequence Spaces Defined by Ideal Convergence and an Orlicz Function

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Abstract: In this study, we introduced the concepts asymptotically $I^p_M$-lacunary equivalence with order $\alpha$ and asymptotically $I_M$-lacunary statistical equivalence with order $\alpha$ by using a non-trivial ideal $I$, an Orlicz function $M$ and a sequence of positive real numbers $p = (p_k)$. In addition to these definitions, we also presented some inclusion theorems.

Keywords: Asymptotically equivalence, Ideal convergence, Lacunary sequence, Orlicz function.

1. Introduction
Let $s$ be the space of real valued sequences and any subspace of $s$ is also called a sequence space. $\ell_\infty$ and $c$ denote the spaces of all bounded and convergent sequences, respectively.

A lacunary sequence is an increasing sequence $\theta = (k_r)$ such that $k_0 = 0, h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = k_r/k_{r-1}$. These notations will be used throughout the paper. The sequence space of lacunary strongly convergent sequences $N_\theta$ was defined by Freedman et al. [5], as following:

$N_\theta = \{ x = (x_t) \in s : \lim_{r} h_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \text{ for some } s \}$.

Orlicz [8] used the idea of Orlicz function to construct the space $L^M$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous,
nondecreasing and convex with \( M(0) = 0, M(x) > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \).

An Orlicz function \( M \) is said to satisfy the \( \Delta_2 \)-condition for all values of \( u \), if there exists constant \( K > 0 \), such that \( M(2u) \leq KM(u) \) \((u \geq 0)\). It is also easy to see that \( K > 2 \) aways holds. The \( \Delta_2 \)-condition is equivalent to the satisfaction of inequality \( M(Lu) \leq KLM(u) \) for all values of \( u \) and \( L > 1 \).

**Remark 1.** An Orlicz function \( M \) satisfies the inequality \( M(\lambda x) < \lambda M(x) \) for all \( \lambda \) with \( 0 < \lambda < 1 \).

The following well known inequality will be used throughout the paper;

\[
|a_i + b_i|^{p_i} \leq T(|a_i|^{p_i} + |b_i|^{p_i}),
\]

where \( a_i \) and \( b_i \) are complex numbers, \( T = \max(1, 2^H-1) \), and \( H = \sup p_i < \infty. \)

Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices in [7]. Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices in [9]. Subsequently, many authors have shown their interest to solve different problems arising in this area (see [2], and [10]).

The concept of \( I \) convergence was introduced by Kostyrko et al. [5], which is a generalization of statistical convergence. Recently, Das et al. [2,3] unified these two approaches to introduce new concepts such as \( I \)-statistical convergence and \( I \)-lacunary statistical convergence and investigated some of their consequences. More investigations in this direction and more applications can be found in [1, 3, 4, 11].

In this paper we introduce the concepts asymptotically \( I^p_M \)-lacunary equivalence with order \( \alpha \) and asymptotically \( I_M \)-lacunary statistical equivalence with order \( \alpha \), by using a non-trivial ideal \( I \), an Orlicz function \( M \), and a sequence of positive real numbers \( p = (p_k) \) and also some inclusion theorems are proved.

**2. Definitions and Notations**

In this section, we recall the basic definitions and concepts. For simplicity, below and in what follows limits run to \( \infty \), that is, we use "\( \lim_k x_k \)" instead of "\( \lim_{k \to \infty} x_k \)".
Definition 2.1. Two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be asymptotically equivalent if \( \lim_{k} \frac{x_k}{y_k} = 1 \), (denoted by \( x \sim y \)).

Definition 2.2. Two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be asymptotically statistical equivalent of multiple \( L \) provided that for every \( \varepsilon > 0 \),
\[
\lim_{n} \frac{1}{n} \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} = 0
\]
(denoted by \( x S \sim y \)) and simply asymptotically statistical equivalent, if \( L = 1 \).

Definition 2.3. Two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be strong asymptotically equivalent of multiple \( L \) provided that,
\[
\lim_{n} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| = 0
\]
(denoted by \( x w \sim y \)) and simply strong asymptotically equivalent, if \( L = 1 \).

Definition 2.4. Let \( \theta \) be a lacunary sequence; the two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be asymptotically lacunary statistical equivalent of multiple \( L \) provided that for every \( \varepsilon > 0 \),
\[
\lim_{r} \frac{1}{h_r} \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} = 0
\]
(denoted by \( x S_{\theta} \sim y \)) and simply asymptotically lacunary statistical equivalent, if \( L = 1 \).

Definition 2.5. Let \( \theta \) be a lacunary sequence; the two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be strong asymptotically lacunary equivalent of multiple \( L \) provided that,
\[
\lim_{r} \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| = 0
\]
(denoted by \( x N_{\theta} \sim y \)) and simply strong asymptotically lacunary equivalent, if \( L = 1 \).

Definition 2.6. Let \( M \) be any Orlicz function; the two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be \( M \)-asymptotically equivalent of multiple \( L \) provided that,
\[
\lim_{k} M\left( \frac{x_k}{y_k} - L \right) / \rho = 0
\]
for some $\rho > 0$, (denoted by $x^M \sim y$) and simply $M-$asymptotically equivalent, if $L = 1$.

**Definition 2.7.** Let $M$ be any Orlicz function; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strong $M$-asymptotically equivalent of multiple $L$ provided that,

$$\lim_{n} \frac{1}{n} \sum_{k=1}^{n} M\left(\left\vert \frac{x_k - L}{y_k} \right\vert / \rho \right) = 0$$

for some $\rho > 0$, (denoted by $x^M \sim w^M y$) and simply strong $M$-asymptotically lacunary equivalent, if $L = 1$.

**Definition 2.8.** Let $M$ be any Orlicz function and $\theta$ be a lacunary sequence; the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strong $M$-asymptotically lacunary equivalent of multiple $L$ provided that,

$$\lim_{r} \frac{1}{h_r} \sum_{k \in I_r} M\left(\left\vert \frac{x_k - L}{y_k} \right\vert / \rho \right) = 0$$

for some $\rho > 0$, (denoted by $x^{M,\theta} \sim y^{M,\theta}$) and simply strong $M$-asymptotically lacunary equivalent, if $L = 1$.

Let $X$ be a non-empty set. Then $P(X)$ denote the power set of $X$, that is, the space of all subsets of $X$.

**Definition 2.9.** A family $I \subseteq P(X)$ is said to be an ideal in $X$ if the following conditions hold:

(i) $\emptyset \in I$;
(ii) $A, B \in I$ imply $A \cup B \in I$ and
(iii) $A \in I, B \subseteq A$ imply $B \in I$.

**Definition 2.10.** A non-empty family $F \subseteq P(X)$ is said to be a filter in $X$ if the following conditions hold:

(i) $\emptyset \notin F$;
(ii) $A, B \in F$ imply $A \cap B \in F$ and
(iii) $A \in F, B \supseteq A$ imply $B \in F$.

An ideal $I$ is said to be non-trivial if $I \neq \{\emptyset\}$ and $X \notin I$. A non-trivial ideal $I$ is called admissible if it contains all the singleton sets. Moreover, if $I$ is a non-trivial ideal on $X$, then $F = F(I) = \{X - A : A \in I\}$ is a filter on $X$ and conversely. The filter $F(I)$ is called the filter associated with the ideal $I$. 

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Definition 2.11. Let $I \subset P(N)$ be a non-trivial ideal in $N$. A sequence $x = (x_k)$ in $X$ is said to be $I$-convergent to $L$ if for each $\varepsilon > 0$, the set

$$\{k \in N : |x_k - L| \geq \varepsilon\} \in I.$$ 

In this case, we write $I - \lim_k x_k = L$. Let $x = (x_k)$ be a sequence in $X$. Then it is said to be $I - \lim_k x_k = 0$. In this case we write $I - \lim_k x_k = 0$.

Definition 2.12. A sequence $x = (x_k)$ of numbers is said to be $I$-statistical convergent or $S(I)$-convergent to $L$, if for every $\varepsilon > 0$ and $\delta > 0$, we have

$$\left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} |x_k - L| \geq \varepsilon \right\} \geq \delta \in I.$$ 

In this case, we write $x_k \rightarrow L(S(I))$ or $S(I) - \lim_k x_k = L$.

Definition 2.13. Let $I \subset P(N)$ be a non-trivial ideal in $N$. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically equivalent of multiple $L$ with respect to the ideal $I$ provided that for each $\varepsilon > 0$,

$$\left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in I,$$

(denoted by $x \overset{I(w)}{\sim} y$) and simply strongly asymptotically equivalent with respect to the ideal $I$, if $L = 1$.

Definition 2.14. Let $I \subset P(N)$ be a non-trivial ideal in $N$ and $\theta = (k_r)$ be a lacunary sequence. The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple $L$ with respect to the ideal $I$ provided that for each $\varepsilon > 0$ and $\gamma > 0$,

$$\left\{ r \in N; \frac{1}{h_r} \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \geq \gamma \right\} \in I$$

(denoted by $x \overset{I(S_\theta)}{\sim} y$) and simply asymptotically lacunary statistical equivalent with respect to the ideal $I$, if $L = 1$.

Definition 2.15. Let $I \subset P(N)$ be a non-trivial ideal in $N$ and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically lacunary equivalent of multiple $L$
with respect to the ideal $I$ provided that for $\varepsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in I$$

(denoted by $x \overset{I(N_\theta)}{\sim} y$) and simply asymptotically lacunary equivalent with respect to the ideal $I$, if $L = 1$.

Quite recently Bilgin [1] and Savas [11] have given the following definitions.

**Definition 2.16.** Let $I \subset P(N)$ be a non-trivial ideal in $N$, $M$ be any Orlicz function, $\theta = (k_r)$ be a lacunary sequence and $p = (p_k)$ be a sequence of positive real numbers. Two sequences $x = (x_k)$ and $y = (y_k)$ are said to be $(M,p)$-asymptotically lacunary equivalent of multiple $L$ with respect to the ideal $I$ provided that for each $\varepsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[ M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right]^{p_k} \geq \varepsilon \right\} \in I$$

for some $\rho > 0$, (denoted by $x \overset{I(N_\theta(M,p))}{\sim} y$) and simply $(M,p)$-asymptotically lacunary equivalent with respect to the ideal $I$, if $L = 1$.

**Definition 2.17.** Let $I \subset P(N)$ be a non-trivial ideal in $N$, and $\theta = (k_r)$ be a lacunary sequence. Two sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $I$-lacunary statistical equivalent of order $\alpha$, where $0 < \alpha \leq 1$, to multiple $L$ provided that for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \geq \delta \right\} \in I.$$ 

In this case we write $x \overset{S_{I\alpha}(\theta)}{\sim} y$.

**Definition 2.18.** Let $I \subset P(N)$ be a non-trivial ideal in $N$, $\theta = (k_r)$ be a lacunary sequence and $p = (p_k)$ be a sequence of positive real numbers. Two sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically $I$-lacunary equivalent of order $\alpha$, where $0 < \alpha \leq 1$, to multiple $L$ for the sequence $p$ provided that for any $\varepsilon > 0$,
\(\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right|^{p_k} \geq \varepsilon \} \in I\)

for some \(\rho > 0\), (denoted by \(x^{N^L_p(I)^\alpha}_\theta \bowtie y\)).

3. Main Results

In this section we shall give some new definitions and also prove some inclusion relations.

We begin with the following definitions.

**Definition 3.1.** Let \(I \subset P(N)\) be a non-trivial ideal in \(N\), \(M\) be any Orlicz function, \(\theta = (k_r)\) be a lacunary sequence. Two number sequences \(x = (x_k)\) and \(y = (y_k)\) are said to be asymptotically \(I_M\)-lacunary statistical equivalent of order \(\alpha\), where \(0 < \alpha \leq 1\), to multiple \(L\) provided that for any \(\varepsilon > 0\) and \(\delta > 0\).

\[\{ r \in N; \frac{1}{h_r} \left\{ k \in I_r : M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \geq \varepsilon \right\} \geq \delta \} \in I\]

for some \(\rho > 0\). In this case we write \(x^{I_M(S^\theta_\alpha)} \bowtie y\). If we put \(M(x) = x\) for \(x \geq 0\), we write \(x^{I(S^\theta_\alpha)} \bowtie y\). Hence \(x^{I(S^\theta_\alpha)} \bowtie y\) is the same as the \(x^{S^L_{\theta^1}(I)^\alpha} \bowtie y\) of Savas [11].

**Definition 3.2.** Let \(I \subset P(N)\) be a non-trivial ideal in \(N\), \(M\) be any Orlicz function, \(\theta = (k_r)\) be a lacunary sequence and \(p = (p_k)\) be a sequence of positive real numbers. Two sequences \(x = (x_k)\) and \(y = (y_k)\) are said to be asymptotically \(I_p\)-lacunary equivalent of order \(\alpha\), where \(0 < \alpha \leq 1\), to multiple \(L\) for the sequence \(p\) provided that for any \(\varepsilon > 0\),

\[\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[ M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right)^{p_k} \right] \geq \varepsilon \} \in I\]

for some \(\rho > 0\), (denoted by \(x^{I_p(M)^\alpha(N^\theta_\alpha)} \bowtie y\)). If we take \(\alpha = 1\), we write \(x^{I_p(M)^\alpha(N^\theta_\alpha)} \bowtie y\) instead of \(x^{I_p(M)^\alpha(N^\theta_\alpha)} \bowtie y\). Hence \(x^{I_p(M)^\alpha(N^\theta_\alpha)} \bowtie y\) is the same as the \(x^{I(M^{\alpha, p})} \bowtie y\) of Bilgin [1].

Also if we put \(M(x) = x\) for \(x \geq 0\), we write \(x^{I_p(M)^\alpha(N^\theta_\alpha)} \bowtie y\) instead of \(x^{I_p(N^\theta_\alpha)} \bowtie y\). Hence \(x^{I_p(N^\theta_\alpha)} \bowtie y\) is the same as the \(x^{N^L_{\theta^1}(I)^\alpha} \bowtie y\) of Savas [11].
If we take \( \frac{1}{n^r} \) instead of \( \frac{1}{n^s} \), denoted by \( x \overset{I}{\sim}_{M} y \) and simply strongly Cesaro \( I_{M} \)-asymptotically equivalent of order \( \alpha \) for \( L = 1 \).

If we take \( p_{k} = 1 \) for all \( k \in N \), we write \( x \overset{I}{\sim}_{M} y \) instead of \( x \overset{I}{\sim}_{M} y \).

We now prove some inclusion theorems.

**Theorem 3.1.** Let \( I \subset P(N) \) be a non-trivial ideal in \( N \), \( M \) be any Orlicz function, \( \theta = (k_{r}) \) be a lacunary sequence, \( p = (p_{k}) \) be a sequence of positive real numbers and \( 0 < \alpha_{1} \leq \alpha_{2} \leq 1 \) then \( x \overset{I}{\sim}_{M} y \) implies \( x \overset{I}{\sim}_{M} y \).

**Proof.** Let \( 0 < \alpha_{1} \leq \alpha_{2} \leq 1 \) and \( x \overset{I}{\sim}_{M} y \). Since \( h_{r} = k_{r} - k_{r-1} \rightarrow \infty \) as \( r \rightarrow \infty \), we can actually choose \( r \), so that \( h_{r}^{\alpha_{1}} \leq h_{r}^{\alpha_{2}} \) and \( \frac{1}{h_{r}^{\alpha_{1}}} \leq \frac{1}{h_{r}^{\alpha_{2}}} \). Hence

\[
\frac{1}{h_{r}^{\alpha_{2}}} \sum_{k \in I_{r}} \left[ M \left( \left\| \frac{x_{k}}{y_{k}} - L \right\| / \rho \right) \right]^{p_{k}} \leq \frac{1}{h_{r}^{\alpha_{1}}} \sum_{k \in I_{r}} \left[ M \left( \left\| \frac{x_{k}}{y_{k}} - L \right\| / \rho \right) \right]^{p_{k}}.
\]

And so,

\[
\left\{ r \in N; \frac{1}{h_{r}^{\alpha_{2}}} \sum_{k \in I_{r}} \left[ M \left( \left\| \frac{x_{k}}{y_{k}} - L \right\| / \rho \right) \right]^{p_{k}} \geq \varepsilon \right\} \subset \left\{ r \in N; \frac{1}{h_{r}^{\alpha_{1}}} \sum_{k \in I_{r}} \left[ M \left( \left\| \frac{x_{k}}{y_{k}} - L \right\| / \rho \right) \right]^{p_{k}} \geq \varepsilon \right\}.
\]

Finally, we have

\[
\left\{ r \in N; \frac{1}{h_{r}^{\alpha_{2}}} \sum_{k \in I_{r}} \left[ M \left( \left\| \frac{x_{k}}{y_{k}} - L \right\| / \rho \right) \right]^{p_{k}} \geq \varepsilon \right\} \in I
\]

Hence \( x \overset{I}{\sim}_{M} y \).

One can have the following result by setting \( \alpha_{2} = 1 \) in Theorem 3.1.

**Corollary.** Let \( I \subset P(N) \) be a non-trivial ideal in \( N \), \( M \) be any Orlicz function, \( \theta = (k_{r}) \) be a lacunary sequence, \( p = (p_{k}) \) be a sequence of positive real numbers, and \( 0 < \alpha \leq 1 \), then \( x \overset{I}{\sim}_{M} y \) implies \( x \overset{I}{\sim}_{M} y \).

**Theorem 3.2.** Let \( I \subset P(N) \) be a non-trivial ideal in \( N \), \( M_{1} \), \( M_{2} \) be any Orlicz functions, \( \theta = (k_{r}) \) be a lacunary sequence, \( p = (p_{k}) \) be a
sequence of positive real numbers and $0 < \alpha \leq 1$. Then $x \sim^{l_{M_1 \cap M_2}(N_\theta \alpha)} y$ implies $x \sim^{l_{(M_1 + M_2)(N_\theta \alpha)}} y$.

**Proof.** Now suppose that $x \sim^{l_{(M_1 \cap M_2)(N_\theta \alpha)}} y$ and $\varepsilon > 0$. Let the set

$$A = \left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[ M_1 \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} < \varepsilon / 2 \right\}$$

and

$$B = \left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[ M_2 \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} < \varepsilon / 2 \right\}$$

be given for some $\rho > 0$. Then we have,

$$\left[ (M_1 + M_2) \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} = \left[ M_1 \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} + \left[ M_2 \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} \leq T \left\{ \left[ M_1 \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} + \left[ M_2 \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} \right\},$$

with $T = \max(1, 2^H - 1)$, and $H = \sup p_i < \infty$. So

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[ (M_1 + M_2) \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} \leq T \left\{ \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[ M_1 \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} + \right.\left. \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[ M_2 \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} \right\}.$$

It follows that for any $\varepsilon > 0$,

$$\left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[ (M_1 + M_2) \left( \frac{x_k}{y_k} - L \right) \right]^{p_k} < \varepsilon \right\} \in F(I)$$

which yields that $x \sim^{l_{(M_1 + M_2)(N_\theta \alpha)}} y$. 

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The next theorem shows relationship between the strongly $I^p_M$–asymptotically equivalence and the $I^p_M$–asymptotically lacunary equivalence with respect to the ideal $I$.

**Theorem 3.3.** Let $I \subset P(N)$ be a non-trivial ideal in $N$, $M$ be any Orlicz function, $\theta = (k_r)$ be a lacunary sequence, $p = (p_k)$ be a sequence of positive real numbers and $0 < \alpha \leq 1$. Then following propositions are true.

(i) If $\sup_r \frac{1}{k_r-1} \sum_{m=1}^{r} (k_m - k_{m-1})^\alpha = B$ (say) $< \infty$ then $x \overset{I^p_M(\alpha \theta)}{\sim} y$ implies $x \overset{I^p_M(w^\alpha)}{\sim} y$,

(ii) If $\sup_r \frac{k_r}{k_{r-1}} = C$ (say) $< \infty$ then $x \overset{I^p_M(w^\alpha)}{\sim} y$ implies $x \overset{I^p_M(\alpha \theta)}{\sim} y$.

**Proof.** (i): Now suppose that $x \overset{I^p_M(\alpha \theta)}{\sim} y$ and $\varepsilon > 0$. Let

$$A = \left\{ r \in N; \frac{1}{h^\alpha_{n-1}} \sum_{k \in I_r} \left[ M\left( \frac{x_k}{y_k} - L / \rho \right) \right]^{p_k} < \varepsilon \right\},$$

for some $\rho > 0$. Hence, for all $j \in A$ and for some $\rho > 0$, we have

$$H_j = \frac{1}{h^\alpha_j} \sum_{k \in I_j} \left[ M\left( \frac{x_k}{y_k} - L / \rho \right) \right]^{p_k} < \varepsilon.$$

Choose $n$ is any integer with $k_r \geq n > k_{r-1}$ where $r \in A$. Now write

$$\frac{1}{n^\alpha} \sum_{k=1}^{n} \left[ M\left( \frac{x_k}{y_k} - L / \rho \right) \right]^{p_k} \leq \frac{1}{k_r^\alpha} \sum_{k=1}^{k_r} \left[ M\left( \frac{x_k}{y_k} - L / \rho \right) \right]^{p_k}$$

$$= \frac{1}{k_r^\alpha} \left\{ \sum_{k \in I_1} \left[ M\left( \frac{x_k}{y_k} - L / \rho \right) \right]^{p_k} + \sum_{k \in I_r} \left[ M\left( \frac{x_k}{y_k} - L / \rho \right) \right]^{p_k} \right\}$$

$$= \frac{1}{k_r^\alpha} \left\{ k_1^\alpha \sum_{k \in I_1} \left[ M\left( \frac{x_k}{y_k} - L / \rho \right) \right]^{p_k} + \frac{(k_r - k_{r-1})^\alpha}{h^\alpha_r} \sum_{k \in I_r} \left[ M\left( \frac{x_k}{y_k} - L / \rho \right) \right]^{p_k} \right\}$$

$$= \frac{1}{k_r^\alpha} \left\{ k_1^\alpha H_1 + (k_2 - k_1)^\alpha H_2 + ... + (k_r - k_{r-1})^\alpha H_r \right\}$$
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\[
\frac{1}{k_r^{\alpha}} \sum_{m=1}^{r} k_m - k_{m-1} \sum_{k \in I_m} M(\frac{x_k}{y_k} - L \mid \rho)^{p_k} \\
= \frac{1}{k_r^{\alpha}} \sum_{m=1}^{r} (k_m - k_{m-1})^\alpha \sup_{j \in A} H_j \\
\leq B \sup_{j \in A} H_j \\
< B \varepsilon = \varepsilon'.
\]

It follows that for any \( \varepsilon' > 0 \),

\[
\left\{ n \in N; \frac{1}{n^{\alpha}} \sum_{k=1}^{n} M\left( \left| \frac{x_k}{y_k} - L \right| \rho \right)^{p_k} < \varepsilon' \right\} \in F(I)
\]
since for any set

\[
A \in F(I), \cup \{ n : k_{r-1} < n < k_r, r \in A \} \in F(I).
\]

which yields that \( x \overset{I_p(w_\alpha)}{\sim} y \).

(ii): Let \( x \overset{I_p(w_\alpha)}{\sim} y \). Let us take \( \varepsilon > 0 \) and define the set,

\[
A = \left\{ k_r \in N; \frac{1}{k_r^{\alpha}} \sum_{k=1}^{k_r} M\left( \left| \frac{x_k}{y_k} - L \right| \rho \right)^{p_k} < \varepsilon \right\},
\]

for some \( \rho > 0 \). We have \( A \in F(I) \), which is the filter of the ideal I. For each \( k_r \in A \), we have, for some \( \rho > 0 \),

\[
\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} M\left( \left| \frac{x_k}{y_k} - L \right| \rho \right)^{p_k} = \frac{1}{h_r^{\alpha}} \sum_{k=1}^{k_r} M\left( \left| \frac{x_k}{y_k} - L \right| \rho \right)^{p_k} - \frac{1}{h_r^{\alpha}} \sum_{k=1}^{k_r-1} M\left( \left| \frac{x_k}{y_k} - L \right| \rho \right)^{p_k} \\
\leq \frac{1}{h_r^{\alpha}} \sum_{k=1}^{k_r} M\left( \left| \frac{x_k}{y_k} - L \right| \rho \right)^{p_k} \\
\leq k_r \frac{1}{h_r^{\alpha}} \sum_{k=1}^{k_r} M\left( \left| \frac{x_k}{y_k} - L \right| \rho \right)^{p_k}
\]
It follows that for any $\varepsilon' > 0$,

$$\left\{ r \in N; \frac{1}{h_r^a} \sum_{k \in I_r} \left[ M \left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right]^{p_k} < \varepsilon' \right\} \in F(I)$$

which yields that $x_{I^p(N^a_\theta)} \sim y$.

Now we give relation between asymptotically $I$-lacunary statistical equivalence and $I^p_M$-asymptotically lacunary equivalence of order $\alpha$.

Theorem 3.4. Let $I \subset P(N)$ be a non-trivial ideal in $N$, $M$ be any Orlicz function, $\theta = (k_r)$ be a lacunary sequence, $p = (p_k)$ be a sequence of positive real numbers and $0 < \alpha \leq 1$. If $h = \inf p_k \leq \sup p_k = H < \infty$, then $x_{I^p(N^a_\theta)} \sim y$ implies $x_{I(S^a_\theta)} \sim y$.

Proof. Take $\varepsilon > 0$ and let $\sum_1$ denote the sum over $k \in I_r$, with $\left| \frac{x_k}{y_k} - L \right| \geq \varepsilon$. Then,

$$\frac{1}{h_r^a} \sum_{k \in I_r} \left[ M \left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right]^{p_k} \geq \frac{1}{h_r^a} \sum_{1} \left[ M \left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right]^{p_k}$$

$$\geq \frac{1}{h_r^a} \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} M(\varepsilon/\rho)^{p_k}$$

$$\geq \frac{1}{h_r^a} \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \min \{ M(\varepsilon/\rho)^h, M(\varepsilon/\rho)^H \}$$

and

$$\left\{ r \in N; \frac{1}{h_r^a} \sum_{k \in I_r} \left[ M \left( \left| \frac{x_k}{y_k} - L \right| \right) \right]^{p_k} \geq \gamma \min \{ M(\varepsilon/\rho)^h, M(\varepsilon/\rho)^H \} \right\} \in I.$$ 

But then, by definition of an ideal, later set belongs to I, and therefore $x_{I(S^a_\theta)} \sim y$. 

Theorem 3.5. Let $I \subset P(N)$ be a non-trivial ideal in $N$, $M$ be any Orlicz function, $\theta = (k_r)$ be a lacunary sequence, $p = (p_k)$ be a sequence of positive real numbers and $0 < \alpha \leq 1$. If $h = \inf p_k \leq \sup p_k = H < \infty$, then $x^{I_M(N^\alpha)} y$ implies $x^{I_M(S^\alpha)} y$.

Proof. Take $\varepsilon > 0$ and let $\sum_1^\infty$ denote the sum over $k \in I_r$ with $M(\left| \frac{x_k}{y_k} - L \right| / \rho) \geq \varepsilon$. Then,

$$
\frac{1}{H_r^\alpha} \sum_{k \in I_r} \left( M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right)^{p_k} \geq \frac{1}{H_r^\alpha} \sum_1^\infty \left( M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right)^{p_k}
$$

$$
\geq \frac{1}{H_r^\alpha} \left\{ k \in I_r : M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \geq \varepsilon \right\} (\varepsilon)^{p_k}
$$

and

$$
\left\{ r \in N; \frac{1}{H_r^\alpha} \left\{ k \in I_r : M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \geq \varepsilon \right\} \geq \gamma \right\}
$$

$$
\subseteq \left\{ r \in N; \frac{1}{H_r^\alpha} \sum_{k \in I_r} \left( M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right)^{p_k} \geq \gamma \min\{\varepsilon^h, \varepsilon^H\} \right\} \in I.
$$

But then, by definition of an ideal, later set belongs to $I$ and therefore $x^{I_M(S^\alpha)} y$.

Theorem 3.6. Let $I \subset P(N)$ be a non-trivial ideal in $N$, $M$ be any Orlicz function that satisfy the $\Delta_2$-condition, $\theta = (k_r)$ be a lacunary sequence, and $0 < \alpha \leq 1$. Then $x^{I_M(S^\alpha)} y$ implies $x^{I_M(S^\alpha)} y$.

Proof. Let $x^{I_M(S^\alpha)} y$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that $M(u) < \varepsilon/2$ for every $u$ with $0 \leq u \leq \delta$.

Let $\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \leq \delta$, then $M(\left| \frac{x_k}{y_k} - L \right| / \rho) < \varepsilon/2$. For $\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) > \delta$ we use the fact that $\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) < (\left| \frac{x_k}{y_k} - L \right| / \rho) / \delta < 1 + (\left| \frac{x_k}{y_k} - L \right| / \rho) / \delta$.

Since $M$ is non-decreasing and convex, it follows that
\[ M\left(\frac{x_k}{y_k} - L \middle| \rho \right) < M(1 + \left(\frac{x_k}{y_k} - L \middle| \rho \right)/\delta) < \frac{1}{2} M(2 + \frac{1}{2} M(2\left(\frac{x_k}{y_k} - L \middle| \rho \right)/\delta). \]

Since \( M \) satisfies the \( \Delta_2 \)-condition, therefore
\[ M\left(\frac{x_k}{y_k} - L \middle| \rho \right) < \frac{1}{2} K((\frac{x_k}{y_k} - L \middle| \rho \delta) M(2) + \frac{1}{2} K((\frac{x_k}{y_k} - L \middle| \rho \delta) M(2) \]
\[ = K(\frac{x_k}{y_k} - L \middle| \rho \delta) M(2). \]

Hence,
\[ M\left(\frac{x_k}{y_k} - L \middle| \rho \right) \leq K\left(\frac{x_k}{y_k} - L \middle| \rho \delta\right) M(2) + \varepsilon/2. \]

Thus
\[ \left\{ r \in N; \frac{1}{h_r^\alpha} \left\{ k \in I_r : M\left(\frac{x_k}{y_k} - L \middle| \rho \delta\right) \geq \varepsilon \right\} \geq \gamma \right\} \]
\[ \subset \left\{ r \in N; \frac{1}{h_r^\alpha} \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \middle| \rho \delta\right) \geq \varepsilon/2 K M(2) \right\} \geq \gamma \right\}. \]

Since \( x^{I(\theta^\alpha)} \) \( y \) it follows the later set, and hence, the first set in above expression belongs to \( I \). This proves that \( x^{I_{\alpha,\theta}} \) \( y \).

Let \( p_k = p \) for all \( k \), \( t_k = t \) for all \( k \) and \( 0 < p \leq t \). Then we can give following theorem.

**Theorem 3.7.** Let \( I \subset P(N) \) be a non-trivial ideal in \( N \), \( M \) be an Orlicz function, \( \theta = (k_r) \) be a lacunary sequence and \( 0 < \alpha \leq 1 \) then \( x^{I_{\alpha,\theta}} \) \( y \) implies \( x^{I_{\alpha,\theta}} \) \( y \).

**Proof.** Let \( x^{I_{\alpha,\theta}} \) \( y \). It follows from Holder’s inequality,
\[
\frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[ M\left(\frac{x_k}{y_k} - L \middle| \rho \right) \right]^p \leq \left( \frac{1}{h_r^\alpha} \sum_{k \in I_r} \left[ M\left(\frac{x_k}{y_k} - L \middle| \rho \right) \right]^t \right)^{p/t}
\]
and
\[
\left\{ r \in N; \frac{1}{h_r^\alpha} \sum_{k \in I_r} M\left(\frac{x_k}{y_k} - L \middle| \rho \right) \right\}^p \geq \varepsilon \right\}
\]
Then, we have, \( z \) for \( 0 < \lambda \) and so \( \theta \) Orlicz function, \( x \) we have \( \varepsilon^{t/p} \). Let \( \text{Asymptotically Lacunary Statistical Equivalent Sequence Spaces defined by ideal } /p \). Let \( \text{Theorem 3.8} \). We now consider that \( (p_k) \) and \( (t_k) \) are not constant sequences.

**Theorem 3.8.** Let \( I \subset P(N) \) be a non-trivial ideal in \( N \), \( M \) be an Orlicz function, \( \theta = (k_r) \) be a lacunary sequence, \( 0 < p_k \leq t_k \) for all \( k \), \( (t_k/p_k) \) be bounded, and \( 0 < \alpha \leq 1 \), then \( x \varepsilon^{t/p} y \) implies \( x \varepsilon^{t/p} y \).

**Proof.** Let \( x \varepsilon^{t/p} y \). \( z_k = \left[ M\left( \frac{x_k}{y_k} - L \right) / \rho \right]^{t_k} \) and \( \lambda_k = (p_k / t_k) \), so that for \( 0 < \lambda \leq \lambda_k \leq 1 \), we define the sequences \( (u_k) \) and \( (v_k) \) as follows. For \( z_k \geq 1 \); let \( u_k = z_k \) and \( v_k = 0 \) and for \( z_k < 1 \); let \( v_k = z_k \) and \( u_k = 0 \). Then, we have \( z_k = u_k + v_k \); \( z_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k} \). Now it follows that \( u_k^{\lambda_k} \leq u_k \) \( \leq z_k \) and \( v_k^{\lambda_k} \leq v_k^{\lambda_k} \). Therefore,

\[
\frac{1}{h_r^\alpha} \sum_{k \in I_r} z_k^{\lambda_k} = \frac{1}{h_r^\alpha} \sum_{k \in I_r} (u_k^{\lambda_k} + v_k^{\lambda_k}) \\
\leq \frac{1}{h_r^\alpha} \sum_{k \in I_r} z_k + \frac{1}{h_r^\alpha} \sum_{k \in I_r} v_k^{\lambda_k}.
\]

Now for each \( r \);

\[
\frac{1}{h_r^\alpha} \sum_{k \in I_r} v_k^{\lambda_k} = \sum_{k \in I_r} \left( \frac{1}{h_r^\alpha} v_k \right)^\lambda \left( \frac{1}{h_r^\alpha} \right)^{1-\lambda} \\
\leq \left( \sum_{k \in I_r} \left( \frac{1}{h_r^\alpha} v_k \right)^\lambda \right)^{\lambda} \left( \sum_{k \in I_r} \left( \frac{1}{h_r^\alpha} \right)^{1-\lambda} \right)^{1/\lambda} \\
< \left( \frac{1}{h_r} \sum_{k \in I_r} v_k \right)^\lambda.
\]

and so

\[
\frac{1}{h_r^\alpha} \sum_{k \in I_r} M\left( \frac{x_k}{y_k} - L \right) / \rho \right]^{p_k} = \frac{1}{h_r^\alpha} \sum_{k \in I_r} z_k^{\lambda_k} \leq \frac{1}{h_r^\alpha} \sum_{k \in I_r} z_k + \left( \frac{1}{h_r^\alpha} \sum_{k \in I_r} v_k \right)^\lambda \\
= \left\{ \begin{array}{ll}
\frac{1}{h_r^\alpha} \sum_{k \in I_r} z_k & , z_k \geq 1 \\
\frac{1}{h_r^\alpha} \sum_{k \in I_r} z_k + \left( \frac{1}{h_r^\alpha} \sum_{k \in I_r} z_k \right)^\lambda & , z_k < 1
\end{array} \right\}
\]
If 
\[
\frac{1}{h^m} \sum_{k \in I_r} \left[ M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right]^{p_k} \geq \varepsilon
\]
then
\[
\left\{ \begin{array}{l}
\frac{1}{h^m} \sum_{k \in I_r} \left[ M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right]^{t_k} \geq \varepsilon, z_k \geq 1 \\
\frac{1}{h^m} \sum_{k \in I_r} \left[ M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right]^{t_k} \geq \left( \frac{\varepsilon}{7} \right)^{1/\lambda}, z_k < 1
\end{array} \right. 
\]

Hence
\[
\left\{ r \in N; \frac{1}{h^m} \sum_{k \in I_r} \left[ M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right]^{p_k} \geq \varepsilon \right\} 
\]
\[
\subseteq \left\{ r \in N; \frac{1}{h^m} \sum_{k \in I_r} \left[ M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right]^{t_k} \geq \min \left\{ \varepsilon, \left( \frac{\varepsilon}{2} \right)^{1/\lambda} \right\} \right\} \in I.
\]

Thus we have \( x \overset{p_r(N^p_m)}{\sim} y \).

References


Asymptotically Lacunary Statistical Equivalent Sequence Spaces defined by ideal convergence and an Orlicz Function


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