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Some Identities for a Family of Fibonacci and Lucas Numbers

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Abstract

In this work, we prove some properties of a family of Fibonacci numbers and a family of Lucas numbers. Also,we give some identities between the family of Fibonacci numbers and family of Lucas numbers.

Keywords: Fibonacci Numbers, Generalized Fibonacci Numbers, Lucas Numbers.

Introduction

Fibonacci numbers and their generalizations have many important applications to various fields of science (e.g. see [9]). Also, we see application of Fibonacci numbers in many branches of mathematics in [1, 2, 3, 4, 6, 7, 8, 10-18.]. In present paper, we give some properties of a family $\boldsymbol{k}$-Fibonacci numbers and relationship between the family of $\boldsymbol{k}$-Fibonacci and $\boldsymbol{k}$-Lucas numbers.

The Fibonacci numbers $\boldsymbol{F_n}$ are the terms of the sequence 1,1,2,3,5,8,13,21,34,55,89,144,… . Every Fibonacci number, except the first two, is the sum of the two previous Fibonacci numbers. The numbers $\boldsymbol{F_n}$ satisfy the second order linear recurrence relation

$$\boldsymbol{F_n = F_{n-1} + F_{n-2}}, \qquad \boldsymbol{n = 2, 3, 4, \dots}$$

with the initial values $\boldsymbol{F_0 = 0, F_1 = 1}$.

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It is well known that the Fibonacci numbers are defined by Binet's formula

$$\boldsymbol{F_n} \coloneqq \frac{\mathbf{1}}{\sqrt{\mathbf{5}}}\left(\boldsymbol{\alpha}^{\boldsymbol{n}+\mathbf{1}} - \boldsymbol{\beta}^{\boldsymbol{n}+\mathbf{1}}\right), \qquad \boldsymbol{n} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$$

where $\boldsymbol{\alpha} = \left(\mathbf{1} + \sqrt{\mathbf{5}}\right)/\mathbf{2}$ and $\boldsymbol{\beta} = \left(\mathbf{1} - \sqrt{\mathbf{5}}\right)/\mathbf{2}$.

Definition: Let $\boldsymbol{n}$ and $\boldsymbol{k} \neq \mathbf{0}$ be natural numbers, then there exist unique numbers $\boldsymbol{m}$ and $\boldsymbol{r}$ such that $\boldsymbol{n} = \boldsymbol{mk} + \boldsymbol{r}(\mathbf{0} \leq \boldsymbol{r} < k)$. The generalized $\boldsymbol{k}$-Fibonacci numbers $\boldsymbol{F_n^{(k)}}$ are defined by

$$\boldsymbol{F_n^{(k)}} = \frac{\mathbf{1}}{\left(\sqrt{\mathbf{5}}\right)^k}\left(\boldsymbol{\alpha}^{\boldsymbol{m}+\mathbf{2}} - \boldsymbol{\beta}^{\boldsymbol{m}+\mathbf{2}}\right)^{\boldsymbol{r}}\left(\boldsymbol{\alpha}^{\boldsymbol{m}+\mathbf{1}} - \boldsymbol{\beta}^{\boldsymbol{m}+\mathbf{1}}\right)^{\boldsymbol{k}-\boldsymbol{r}}, \qquad \boldsymbol{n} = \boldsymbol{mk} + \boldsymbol{r}$$

where $\boldsymbol{\alpha} = \left(\mathbf{1} + \sqrt{\mathbf{5}}\right)/\mathbf{2}$ and $\boldsymbol{\beta} = \left(\mathbf{1} - \sqrt{\mathbf{5}}\right)/\mathbf{2}$.

The first few numbers of the family for $\boldsymbol{k} = \mathbf{2}, \mathbf{3}, \mathbf{4}$ are as follows:

$$\left\{\boldsymbol{F_n^{(2)}}\right\}_{\boldsymbol{n}=\mathbf{0}}^{\mathbf{10}} = \{\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{9}, \mathbf{15}, \mathbf{25}, \mathbf{40}, \mathbf{64}\}$$

$$\left\{\boldsymbol{F_n^{(3)}}\right\}_{\boldsymbol{n}=\mathbf{0}}^{\mathbf{11}} = \{\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{12}, \mathbf{18}, \mathbf{27}, \mathbf{45}, \mathbf{75}\}$$

$$\left\{\boldsymbol{F_n^{(4)}}\right\}_{\boldsymbol{n}=\mathbf{0}}^{\mathbf{12}} = \{\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{16}, \mathbf{24}, \mathbf{36}, \mathbf{54}, \mathbf{81}\}.$$

It is well known that the relation of the generalized $\boldsymbol{k}$-Fibonacci and Fibonacci numbers is

$$\boldsymbol{F_n^{(k)}} = (\boldsymbol{F_m})^{\boldsymbol{k}-\boldsymbol{r}}(\boldsymbol{F_{m+1}})^{\boldsymbol{r}}$$

where $\boldsymbol{n} = \boldsymbol{mk} + \boldsymbol{r}$. Consider the case $\boldsymbol{k} = \mathbf{1}$ in last equation, we get that $\boldsymbol{m} = \boldsymbol{n}$ and $\boldsymbol{r} = \mathbf{0}$ so $\boldsymbol{F_n^{(1)}} = \boldsymbol{F_n}$.

The Lucas numbers $\boldsymbol{L_n}$ are defined

$$\boldsymbol{L_n} = \boldsymbol{L_{n-1}} + \boldsymbol{L_{n-2}}\,, \qquad \boldsymbol{n} = \mathbf{2}, \mathbf{3}, \mathbf{4}, \ldots$$

with initial conditions $\boldsymbol{L_0} = \mathbf{2}$, $\boldsymbol{L_1} = \mathbf{1}$.

The first a few Lucas numbers are 2,1,3,4,7,11,18,29,47,76,123,199,322, … . The Binet’s formula for the Lucas numbers $\boldsymbol{L_n}$ is

$$L_n = \alpha^n + \beta^n, \quad n = 0, 1, 2, \ldots$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

We see that the Lucas numbers and Fibonacci numbers are related by

$$L_n = F_n + F_{n-2} = \frac{F_{2n-1}}{F_{n-1}}.$$

Definition: Let n and $k \neq 0$ be natural numbers, then there exist unique numbers m and r such that $n = mk + r (0 \leq r < k)$. The generalized k-Lucas numbers $L_n^{(k)}$ are defined

$$L_n^{(k)} = \left(\alpha^{m+1} + \beta^{m+1}\right)^r (\alpha^m + \beta^m)^{k-r}, \qquad n = mk + r$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

It is well known that the relation of the generalized k-Lucas and Lucas numbers is

$$L_n^{(k)} = (L_m)^{k-r}(L_{m+1})^r$$

where $n = mk + r$.

The first few numbers of the family for $k = 2, 3, 4$ are as follows:

$$\left\{L_n^{(2)}\right\}_{n=0}^{9} = \{4, 2, 1, 3, 9, 12, 16, 28, 49, 77\}$$
$$\left\{L_n^{(3)}\right\}_{n=0}^{10} = \{8, 4, 2, 1, 3, 9, 27, 36, 48, 64, 112\}$$

Some Identities For Fibonacci And Lucas Numbers

The following identities for Fibonacci and Lucas numbers are given in [5] and [9]

$$F_{n+1}^3 - F_n^3 - F_{n-1}^3 = 3F_{n+1} \cdot F_n \cdot F_{n-1} \tag{1}$$

$$\sum_{t=1}^{n} F_t F_{3t} = F_n F_{n+1} F_{2n+1} \tag{2}$$

$$F_{n-1}^6 + F_n^6 + F_{n+1}^6 = 2\left[2F_n^2 + (-1)^n\right]^3 + 3F_{n-1}^2 F_n^2 F_{n+1}^2 \quad (3)$$

$$5F_n = L_{n+2} - L_{n-2} \quad (4)$$

$$5F_{2n} = (L_{n+1})^2 - (L_n)^2 \quad (5)$$

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2 = F_n L_n \quad (6)$$

$$F_{3n} = 5(F_n)^3 + 3(-1)^n F_n \quad (7)$$

$$L_n^2 - F_n^2 = 4F_{n-1}F_{n+1} \quad (8)$$

$$L_n L_{n+2} + 4(-1)^n = 5F_{n-1}F_{n+3} \quad (9)$$

$$(F_{n+1})^3 = F_n^3 + F_{n-1}^3 + 3F_{n-1}F_nF_{n+1} \quad (10)$$

Main Results

Theorem 1. Let $n \in \{1, 2, \ldots\}$. For fixed n, the generalized 2-Fibonacci numbers satisfy

$$F_{2n+2}^{(2)} + F_{2n}^{(2)} = 2\, F_{2n+1}^{(2)} + F_{2n-2}^{(2)}\,.$$

Proof. Bythe (1), we may write

$$F_{n+1}^3 - F_n^3 = F_{n-1}^3 + 3F_{n+1}.F_n.F_{n-1}$$

$$(F_{n+1} - F_n)\left(F_{n+1}^2 + F_nF_{n+1} + F_n^2\right) = F_{n-1}(F_{n-1}^2 + 3F_n.F_{n+1})$$

$$F_{n-1}\left(F_{2n+2}^{(2)} + F_{2n+1}^{(2)} + F_{2n}^{(2)}\right) = F_{n-1}\left(F_{2n-2}^{(2)} + 3\, F_{2n+1}^{(2)}\right)$$

$$F_{2n+2}^{(2)} + F_{2n+1}^{(2)} + F_{2n}^{(2)} = 3\, F_{2n+1}^{(2)} + F_{2n-2}^{(2)}$$

$$F_{2n+2}^{(2)} + F_{2n}^{(2)} = 2\, F_{2n+1}^{(2)} + F_{2n-2}^{(2)}$$

Theorem 2. Let $n \in \{1, 2, \ldots\}$. For fixed n, the generalized 2-Fibonacci numbers satisfy

$$\sum_{i=1}^{n} F_i F_{3i} = F_{2n+1}^{(2)}\left(F_{2n+3}^{(2)} - F_{2n-1}^{(2)}\right)$$

Proof. Using (2) and $F_{2n+1} = F_{n+1}^2 + F_n^2$, we have

$$\sum_{i=1}^{n} F_i F_{3i} = F_n F_{n+1}(F_{n+2}F_{n+1} - F_n.F_{n-1})$$

$$= F^{(2)}_{2n+1}\left(F^{(2)}_{2n+3} - F^{(2)}_{2n-1}\right)$$

Theorem 3: Let $n \in \{1, 2, ...\}$. For fixed n, the generalized 2-Fibonacci numbers satisfy

$$\left(F^{(2)}_{2n-2}\right)^3 + \left(F^{(2)}_{2n}\right)^3 + \left(F^{(2)}_{2n+2}\right)^3 = 2\left[2F^{(2)}_{2n} + (-1)^n\right]^3 + 3F^{(2)}_{2n-2}F^{(2)}_{2n}F^{(2)}_{2n+2}\,.$$

Proof. We get from (3)

$$\begin{aligned}\left(F^{(2)}_{2n-2}\right)^3 + \left(F^{(2)}_{2n}\right)^3 + \left(F^{(2)}_{2n+2}\right)^3 &= \left(F^2_{n-1}\right)^3 + \left(F^2_{n}\right)^3 + \left(F^2_{n+1}\right)^3 \\ &= (F_{n-1})^6 + (F_n)^6 + (F_{n+1})^6 \\ &= 2\left[2F^2_n + (-1)^n\right]^3 + 3F^2_{n-1}F^2_nF^2_{n+1} \\ &= 2[2F^{(2)}_{2n} + (-1)^n]^3 + 3F^{(2)}_{2n-2}F^{(2)}_{2n}F^{(2)}_{2n+2}\,.\end{aligned}$$

Theorem 4. Let $n \in \{1, 2, 3, ...\}$. For fixed n, we have a relation among the generalized 2-Lucas numbers as follows

$$L^{(2)}_{2n+2} - L^{(2)}_{2n} = L^{(2)}_{2n+1} + L^{(2)}_{2n} - L^{(2)}_{2n-3} - L^{(2)}_{2n-4}\,.$$

Proof. We have

$$L^{(2)}_{2n+1} = L_nL_{n+1}$$

$$L^{(2)}_{2n} = (L_n)^2$$

$$L^{(2)}_{2n-3} = L_{n-1}L_{n-2}$$

$$L^{(2)}_{2n-4} = (L_{n-2})^2$$

then we get from (4), (5) and (6)

$$\begin{aligned}L^{(2)}_{2n+1} + L^{(2)}_{2n} - L^{(2)}_{2n-3} - L^{(2)}_{2n-4} &= \left(L_nL_{n+1} + (L_n)^2\right) - \left(L_{n-1}L_{n-2} + (L_{n-2})^2\right) \\ &= L_n(L_n + L_{n+1}) - L_{n-2}(L_{n-1} + L_{n-2}) \\ &= L_nL_{n+2} - L_{n-2}L_n \\ &= L_n(L_{n+2} - L_{n-2})\end{aligned}$$

$$= 5F_nL_n$$
$$= 5F_{2n}$$
$$= (L_{n+1})^2 - (L_n)^2$$
$$= L_{2n+2}^{(2)} - L_{2n}^{(2)}$$

Theorem 5. Let $n \in \{1, 2, \dots\}$. For fixed n, the generalized 2-Fibonacci numbers satisfy

$$F_{2n}^{(2)}\left(5F_{2n}^{(2)} + 3(-1)^n\right) = F_{2n+1}^{(2)}F_{2n+1} - F_{2n-1}^{(2)}F_{2n-1}\,.$$

Proof. We get from (7)

$$\begin{aligned}
F_{2n}^{(2)}\left(5F_{2n}^{(2)} + 3(-1)^n\right) &= 5\left(F_{2n}^{(2)}\right)^2 + 3(-1)^nF_{2n}^{(2)} \\
&= 5\left((F_n)^2\right)^2 + 3(-1)^n(F_n)^2 \\
&= 5(F_n)^4 + 3(-1)^n(F_n)^2 \\
&= F_n(5(F_n)^3 + 3(-1)^n(F_n) \\
&= F_nF_{3n} \\
&= F_n(F_{2n+1}F_{n+1} - F_{2n-1}F_{n-1}) \\
&= F_nF_{n+1}F_{2n+1} - F_nF_{n-1}F_{2n-1} \\
&= F_{2n+1}^{(2)}F_{2n+1} - F_{2n-1}^{(2)}F_{2n-1}
\end{aligned}$$

Theorem 6. Let $n \in \{1, 2, \dots\}$. For fixed n, we have the relation

$$L_{2n}^{(2)} - F_{2n}^{(2)} = 4(F_{2n-2}^{(2)} + F_{2n-1}^{(2)})$$

between the generalized 2-Fibonacci numbers and Lucas numbers.

Proof. Using (8), we can write

$$\begin{aligned}
4\left(F_{2n-2}^{(2)} + F_{2n-1}^{(2)}\right) &= 4[(F_{n-1})^2 + F_nF_{n-1}] \\
&= 4(F_{n-1}(F_{n-1} + F_n)) \\
&= 4F_{n-1}F_{n+1} \\
&= L_n^2 - F_n^2 \\
&= L_{2n}^{(2)} - F_{2n}^{(2)}
\end{aligned}$$

Theorem 7. Let $\boldsymbol{n} \in \{\mathbf{1}, \mathbf{2}, \ldots\}$. For fixed $\boldsymbol{n}$, we have the relation

$$\boldsymbol{L}_{2n+1}^{(2)} + \boldsymbol{L}_{2n}^{(2)} + 4(-1)^n = 15\boldsymbol{F}_{2n-1}^{(2)} + 10\boldsymbol{F}_{2n-2}^{(2)}$$

between the generalized $\mathbf{2}$-Fibonacci numbers and Lucas numbers.

Proof. By (9), we may write

$$\begin{aligned} L_{2n+1}^{(2)} + L_{2n}^{(2)} + 4(-1)^n &= L_nL_{n+1} + L_nL_n + 4(-1)^n \\ &= L_n(L_{n+1} + L_n) + 4(-1)^n \\ &= L_nL_{n+2} + 4(-1)^n \\ &= 5F_{n-1}F_{n+3} \\ &= 5F_{n-1}(2F_{n+1} + F_n) \\ &= 10F_{n-1}F_{n+1} + 5F_nF_{n-1} \\ &= 10F_{n-1}(F_{n-1} + F_n) + 5F_nF_{n-1} \\ &= 10F_{n-1}F_{n-1} + 10F_{n-1}F_n + 5F_nF_{n-1} \\ &= 10F_{2n-2}^{(2)} + 15F_{2n-1}^{(2)} \end{aligned}$$

Theorem 8. Let $\boldsymbol{n} \in \{\mathbf{1}, \mathbf{2}, \ldots\}$. For fixed $\boldsymbol{n}$, we have a relation among the generalized $\mathbf{2}$-Fibonacci numbers,

$$F_{4n+5}^{(4)} = \left(F_{2n}^{(2)}\right)^2 + F_{4n+1}^{(4)} + 2F_{4n-3}^{(4)} + 3F_{2n-1}^{(2)}F_{2n+3}^{(2)} + \left(F_{2n-2}^{(2)}\right)^2.$$

Proof. We get from (10),

$$\begin{aligned} F_{4n+5}^{(4)} &= (F_{n+1})^3(F_{n+2}) \\ &= (F_n^3 + F_{n-1}^3 + 3F_{n-1}F_nF_{n+1})F_{n+2} \\ &= F_n^3F_{n+2} + F_{n-1}^3F_{n+2} + 3F_{n-1}F_nF_{n+1}F_{n+2} \\ &= F_n^3(F_n + F_{n+1}) + F_{n-1}^3(2F_n + F_{n-1}) + 3F_{2n-1}^{(2)}F_{2n+3}^{(2)} \\ &= F_n^3F_n + F_n^3F_{n+1} + 2F_{n-1}^3F_n + F_{n-1}^3F_{n-1} + 3F_{2n-1}^{(2)}F_{2n+3}^{(2)} \\ &= (F_{2n}^{(2)})^2 + F_{4n+1}^{(4)} + 2F_{4n-3}^{(4)} + 3F_{2n-1}^{(2)}F_{2n+3}^{(2)} + (F_{2n-2}^{(2)})^2 \end{aligned}$$

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