"Science Stays True Here"
Journal of Mathematics and Statistical Science, 271-284 | Science Signpost Publishing

Generalized Chebyshev Polynomials Via Matrices and Combinatorial Forms of Their Derivatives

Inci GULTEKIN and Betul SAKIROGLU

Departmentof Mathematics, Faculty of Science, Atatürk University, 25240 Erzurum, Turkey.
igultekin@atauni.edu.tr
Departmentof Mathematics, Science Institute, Atatürk University, 25240 Erzurum, Turkey.
gulisbetul25@hotmail.com

Abstract

In this paper, we obtain the generalized Chebyshev polynomials $V_{n,m}(x)$ and $\Omega_{n,m}(x)$ via matrices and then we define new recurrence relation for the derivative of these polynomials. Also, we give combinatorial forms for the derivatives of these polynomials. Then we create tables for derivative polynomial with the help of combinatorial forms. Finally, we give examples showing how to write derivative polynomials easily.

Keywords: *Chebyshev polynomials, Vieta-Fibonacci polynomials, Vieta-Lucas polynomials, determinant, matrix.*

1. Introduction

Classes of Chebyshev polynomials, well known sequences of orthogonal polynomials, play a very important role in the studies on theoretical and applied mathematics, particularly approximation theory. Interesting properties of the Chebyshev's polynomials can be found in [3] and [4]. It is well known that the Chebyshev polynomials of the first kind and second kind are closely related to Vieta-Fibonacci and Vieta-Lucas polynomials.The recursive properties of Vieta-Fibonacci and Vieta-Lucas polynomials were given by Horadam [1]. Djordjevic studied the generalized Chebyshev polynomial $V_{n,m}(x)$ and $\Omega_{n,m}(x)$ [2]. He proved some properties of new polynomials and introduced the generalized Chebyshev polynomial $V_{n,m}(x)$ and $\Omega_{n,m}(x)$ as follows (x is a real variable):

$$V_{n,m}(x) = xV_{n-1,m}(x) - V_{n-m,m}(x); \qquad n \geq m \ , n, m \in \mathbb{N}$$

with $V_{n,m}(x) = x^n$, $n = 1,2,3 \dots, m-1$, $V_{m,m}(x) = x^m - 1$.

Moreover,

$$\Omega_{n,m}(x) = x\Omega_{n-1,m}(x) - \Omega_{n-m,m}(x); \quad n \geq m\ , \ n, m \in \mathbb{N}$$

Corresponding author: Inci GULTEKIN, Departmentof Mathematics, Faculty of Science, Atatürk University, 25240 Erzurum, TURKEY. igultekin@atauni.edu.tr.

with $\Omega_{n,m}(x) = x^n$, $n = 1, 2, 3 \ldots, m-1$, $\Omega_{m,m}(x) = x^m - 2$

Accordingly, the first few terms of $V_{n,m}(x)$ for $m = 1, 2, 3$ are the following.

$V_{1,1}(x) = x - 1$

$V_{2,1}(x) = x^2 - 2x + 1$

$V_{3,1}(x) = x^3 - 3x^2 + 3x - 1$

$V_{4,1}(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

$V_{5,1}(x) = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$

$V_{6,1}(x) = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

$V_{7,1}(x) = x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1$

$V_{1,2}(x) = x$

$V_{2,2}(x) = x^2 - 1$

$V_{3,2}(x) = x^3 - 2x$

$V_{4,2}(x) = x^4 - 3x^2 + 1$

$V_{5,2}(x) = x^5 - 4x^3 + 3x$

$V_{6,2}(x) = x^6 - 5x^4 + 6x^2 - 1$

$V_{7,2}(x) = x^7 - 6x^5 + 10x^3 - 4x$

$V_{1,3}(x) = x$

$V_{2,3}(x) = x^2$

$V_{3,3}(x) = x^3 - 1$

$V_{4,3}(x) = x^4 - 2x$

$V_{5,3}(x) = x^5 - 3x^2$

$V_{6,3}(x) = x^6 - 4x^3 + 1$

$V_{7,3}(x) = x^7 - 5x^4 + 3x$

In fact, it is easy to write the other polynomials with help of the obtained table from the written polynomials for $m = 1$. For example, the written polynomials for $m = 2$are obtained from the cross sum up of polynomials in the written table for $m = 1$. In this way, the written polynomials for $m = k$ are easily obtained from the tables of the written polynomial for $m = k - 1$.

2. The Generalized Chebyshev Polynomials Via Matrices

We give the generalized Chebyshev polynomial $V_{n,m}(x)$ with the following Theorem 2.1.

***Theorem* 2.1.**

(i) For $n < m$,

$$V_{n,m}(x) = det(A + xB)$$

where $A = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{nxn}$, B $= I_n = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}_{nxn}$ $, n = 1,2,3, \dots, m-1$.

(ii) For $n \geq m$,

$$V_{n,m}(x) = det(A + xB) + C$$

where

$$A = (a_{ij})_{(n-m+2)x(n-m+2)} = \begin{cases} a_{12} = 1 & , \quad n = m \\ a_{12} = n - m & , \quad n \neq m \\ a_{k,k-1} = 0 & , \quad k \geq 4 \\ a_{k,k-1} = 1 & , \quad k < 4 \\ a_{k-1,k} = 1 & , \\ 0 & , the\ other\ case \end{cases}$$

and

$$B = (b_{ij})_{(n-m+2)x(n-m+2)} = \begin{cases} b_{22} = x^{m-2} , & \\ b_{nn} = 1 & , \quad n \neq 2 \\ 0 & , the\ other\ case \end{cases}$$

i.e., $(n \neq m)$

$$A = \begin{pmatrix} 0 & n-m & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(n-m+2)x(n-m+2)}$$

and

$$B=\begin{pmatrix}1 & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & x^{m-2} & 0 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}_{(n-m+2)x(n-m+2)}$$

and

$$C=\begin{cases}\sum_{k=2}(-1)^k\binom{n-k(m-1)}{k}x^{n-km}, & n-k(m-1)\geq k\\ 0 & , \quad n-k(m-1)<k\end{cases}.$$

Proof.

(i)

$$V_{n,m}(x)=det(A+xB)=x^n$$

when $(A)_{n\times n}$ zero matrix, $(B)_{n\times n}$ identical matrix.

(ii) Firstly let us write C matrix. For $n-k(m-1)$ in C matrix when $n=m$;

$$n-k(n-1)=n(1-k)+k$$

and k is considered, including the minimum value when $k=2$

$$n(1-2)+2=-n+2.$$

It is clear that this expression will be small from 2 since $n\geq 1$.

Then it is seen that $C=0$. Thus we have

$$det(A+xB)=x\ (\ x^{m-1}\underbrace{(x.x\dots x)}_{(n-m-2)item}\ x^2-\underbrace{(x.x\dots x)}_{(n-m-1)item}\ )-\underbrace{(x.x\dots x)}_{(n-m)item}$$

$$=x.x^{m-1}.x^{m-m-2}.x^2-x^{m-m-1}.x-0$$

$$V_{m,m}(x)=x^m-1.$$

Now we consider $n\neq m$ and $n>m$, from the equality

$$V_{n,m}(x)=det(A+xB)+C,$$

$$det(A+xB)=x\ (\ x^{m-1}\underbrace{(x.x\dots x)}_{(n-m-2)item}\ x^2-\underbrace{(x.x\dots x)}_{(n-m-1)item}\ )-(n-m)\underbrace{(x.x\dots x)}_{(n-m)item}$$

$$=x.x^{m-1}.x^{n-m-2}.x^2-x^{n-m-1}.x-(n-m)x^{n-m}$$

$$=x^n-(n-m+1)x^{n-m}. \tag{1}$$

On the other hand,

$$C=\binom{n-2(m-1)}{2}x^{n-2m}-\binom{n-3(m-1)}{3}x^{n-3m}+\binom{n-4(m-1)}{4}x^{n-4m}-\cdots. \quad (2)$$

From (1) and (2),

$$V_{n,m}(x)=x^n-(n-m+1)x^{n-m}+\binom{n-2m+2}{2}x^{n-2m}-$$

$$\binom{n-3m+3}{3}x^{n-3m}+\binom{n-4m+4}{4}x^{n-4m}-\cdots. \quad (3)$$

Similarly, from the equality

$$V_{n-1,m}(x)=det(A+xB)+C$$

We have

$$det(A+xB)=x\ (\ x^{m-1}\underbrace{(x.x\dots x)}_{(n-m-3)item}\ x^2-\underbrace{(x.x\dots x)}_{(n-m-2)item}\ )-(n-m-1)\ \underbrace{(x.x\dots x)}_{(n-m-1)item}$$

$$=x.x^{m-1}.x^{n-m-3}.x^2-x^{n-m-2}.x-(n-m-1)x^{n-m-1}$$

$$=x^{n-1}-(n-m)x^{n-m-1}. \quad (4)$$

and

$$C=\binom{n-1-2(m-1)}{2}x^{n-1-2m}-\binom{n-1-3(m-1)}{3}x^{n-1-3m}+\binom{n-1-4(m-1)}{4}x^{n-1-4m}-\cdots. \quad (5)$$

From (4) and (5),

$$V_{n-1,m}(x)=x^{n-1}-(n-m)x^{n-m-1}+\binom{n-2m+1}{2}x^{n-2m-1}-$$

$$\binom{n-3m+2}{3}x^{n-3m-1}+\binom{n-4m+3}{4}x^{n-4m-1}-\cdots. \quad (6)$$

And from the equality

$$V_{n-m,m}(x)=det(A+xB)+C,$$

$$det(A+xB)=x\ (x^{m-1}\ \underbrace{(x.x\dots x)}_{(n-2m-2)item}\ x^2-\underbrace{(x.x\dots x)}_{(n-2m-1)item}\ )-(n-2m)\ \underbrace{(x.x\dots x)}_{(n-2m)item}$$

$$=x.x^{m-1}.x^{n-2m-2}.x^2-x^{n-2m-1}.x-(n-2m)x^{n-2m}$$

$$=x^{n-m}-(n-2m+1)x^{n-2m}. \quad (7)$$

On the other hand;

$$C=\binom{n-m-2(m-1)}{2}x^{n-m-2m}-\binom{n-m-3(m-1)}{3}x^{n-m-3m}$$

$$+\binom{n-m-4(m-1)}{4}x^{n-m-4m}-\cdots. \tag{8}$$

Then,

$$V_{n-m,m}(x)=x^{n-m}-(n-2m+1)x^{n-2m}+\binom{n-3m+2}{2}x^{n-3m}-\binom{n-4m+3}{3}x^{n-4m}$$
$$+\binom{n-5m+4}{4}x^{n-5m}-\cdots. \tag{9}$$

In that case from (3), (6) and (9)

$$\begin{aligned}
xV_{n-1,m}(x)-V_{n-m,m}(x)\\
&=x\ \Big(x^{n-1}-(n-m)x^{n-m-1}+\binom{n-2m+1}{2}x^{n-2m-1}-\binom{n-3m+2}{3}x^{n-3m-1}\\
&+\binom{n-4m+3}{4}x^{n-4m-1}-\cdots\Big)-\big(\,x^{n-m}-(n-2m+1)x^{n-2m}\\
&+\binom{n-3m+2}{2}x^{n-3m}-\binom{n-4m+3}{3}x^{n-4m}+\binom{n-5m+4}{4}x^{n-5m}-\cdots\,\big)
\end{aligned}$$

$$\begin{aligned}
&=x^{n}-(n-m)x^{n-m}+\binom{n-2m+1}{2}x^{n-2m}-\binom{n-3m+2}{3}x^{n-3m}+\binom{n-4m+3}{4}x^{n-4m}-\cdots\\
&\quad -x^{n-m}+(n-2m+1)x^{n-2m}-\binom{n-3m+2}{2}x^{n-3m}+\binom{n-4m+3}{3}x^{n-4m}\\
&\quad -\binom{n-5m+4}{4}x^{n-5m}-\cdots
\end{aligned}$$

$$\begin{aligned}
&=x^{n}-(n-m+1)x^{n-m}+\left(\binom{n-2m+1}{1}+\binom{n-2m+1}{2}\right)x^{n-2m}\\
&\quad -\left(\binom{n-3m+2}{2}+\binom{n-3m+2}{3}\right)x^{n-3m}+(\binom{n-4m+3}{3}+\binom{n-4m+3}{4})x^{n-4m}\\
&\quad -\cdots
\end{aligned}$$

$$\begin{aligned}
&=x^{n}-(n-m+1)x^{n-m}+\binom{n-2m+2}{2}x^{n-2m}-\binom{n-3m+3}{3}x^{n-3m}+\binom{n-4m+4}{4}x^{n-4m}\\
&\quad -\cdots
\end{aligned}$$

$$=V_{n,m}(x).$$

Then, we can give for Theorem 2.1 to the following example.

***Example* 2.1.** Let $n=4,\ m=2$, then we have the following matrices

$$A=\begin{pmatrix}0 & 4-2 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}_{(4-2+2)x(4-2+2)} \qquad B=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & x^{2-2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}_{(4-2+2)x(4-2+2)}$$

and

$$C = (-1)^2 \binom{4-2(2-1)}{2} x^{4-2.2} = 1.x^0 = 1.$$

Since $V_{4,2}(x) = det(A + xB) + C$

$$V_{4,2}(x) = det\left(\begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4x4} + x\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4x4}\right) + 1$$

$$= det\begin{pmatrix} x & 2 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 0 & x \end{pmatrix}_{4x4} + 1$$

$$= (x(\,x(x^2) - x) - 2(x^2)) + 1$$

$$= x^4 - 3x^2 + 1.$$

Now we give the generalized Chebyshev polynomial $\Omega_{n,m}(x)$ with the following theorem.

***Theorem* 2.2.**

(i) For$n < m$,

$$\Omega_{n,m}(x) = det(D + xE)$$

where $D = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{nxn}$, $E = I_n = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}_{nxn}$ $, n = 1,2,3, m-1$.

(ii) For$n \geq m$,

$$\Omega_{n,m}(x) = det(D + xE) + F$$

where

$$D = \begin{cases} d_{12} = 1 & , \ n = m \\ d_{12} = n - m + 1 & , \ n \neq m \\ d_{k,k-1} = 0 & , \ k \geq 4 \\ d_{k,k-1} = 1 & , \ k < 4 \\ d_{k-1,k} = 1 & , \\ 0 & , \textit{the other case} \end{cases}$$

and

$$E = \begin{cases} e_{22} = x^{m-2} & , \\ e_{nn} = 1 & , \ n \neq 2 \\ 0 & , \ \textit{the other case} \end{cases}$$

i.e.,

$$D=\begin{pmatrix}0 & n-m+1 & 0 & 0 & 0 & \cdots & 0\\ 1 & 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0\end{pmatrix}_{(n-m+2)x(n-m+2)}$$

and

$$E=\begin{pmatrix}1 & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & x^{m-2} & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}_{(n-m+2)x(n-m+2)}$$

(if $n=m$ in matrices D and $D+xE$, $a_{12}=1$.)

And

$$F=\begin{cases}\sum_{k=2}(-1)^k\frac{n-(m-2)k}{n-(m-1)k}\binom{n-k(m-1)}{k}x^{n-km}, & n-k(m-1)\geq k\\ 0 & , \quad n-k(m-1)<k\end{cases}.$$

Proof of Theorem 2.2 is obtained a similar to Theorem 2.1.

Then we can give the following example for Theorem 2.2.

***Example* 2.2.** Let $n=6, m=2$ then,

$$D=\begin{pmatrix}0 & 6-2+1 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}_{6x6}, \quad E=\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & x^{2-2} & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}_{6x6}$$

and

$$F=(-1)^2\frac{6-2(2-2)}{6-2(2-1)}\binom{6-2(2-1)}{2}x^{6-2.2}+(-1)^3\frac{6-3(2-2)}{6-3(2-1)}\binom{6-3(2-1)}{3}x^{6-3.2}$$

$$=\frac{6}{4}\binom{4}{2}x^2-\frac{6}{3}\binom{3}{3}x^0=9x^2-\ 2$$

$$\Omega_{6,2}(x)=det\left(\begin{pmatrix}0 & 5 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}_{6x6}+x\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}_{6x6}\right)+9x^2-\ 2$$

$$= det\begin{pmatrix} x & 5 & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 & 0 \\ 0 & 1 & x & 1 & 0 & 0 \\ 0 & 0 & 0 & x & 1 & 0 \\ 0 & 0 & 0 & 0 & x & 1 \\ 0 & 0 & 0 & 0 & 0 & x \end{pmatrix}_{6x6} + 9x^2 - \ 2$$

$$= x(x(x(x(x^2)) - (x(x^2))) - 5(x(x(x^2) + 9x^2 - \ 2$$

$$= x^6 - 6x^4 + 9x^2 - 2.$$

3. Derivative Polynomials of the Generalized Chebyshev Polynomials and Combinatorial Forms

We show derivative of the generalized Chebyshev polynomials $V_{n,m}(x)$ and $\Omega_{n,m}(x)$ respectively $V'_{n,m}(x)$ and $\Omega'_{n,m}(x)$. Firstly, polynomial we define derivative of the generalized Chebyshev polynomials $V_{n,m}(x)$ (Vieta- Fibonnacci polynomial) (x is a real variable and $n \geq 2m$).

***Definition* 3.1.** Derivative of the generalized Chebyshev polynomials $V_{n,m}(x)$ is

$$V'_{n,m}(x) = xV'_{n-1,m}(x) - 2V'_{n-m,m}(x) + x^{n-1} + \sum_{j=1}(-1)^j \ H(n,j)x^{n-(j+1)m-1}$$

where

$$V'_{n,n}(x) = nx^{n-1}, V'_{n,m}(x) = nx^{n-1} - 2\binom{n-m+1}{2}x^{n-m-1}, \ (n < 2m)$$

and

$$H(n,j) = \binom{n-2m+1}{j+1}A_j \quad , \qquad \left(1 \leq j \leq \left[\frac{n-1}{m}\right]\right) (\text{For } j = 1 \ A_1 = 1)$$

$$A_j = \frac{(n-(j+1)m)(n-(j+1)m+1)\ldots(n-(j+1)m+j)}{\underbrace{(n-2m+1)(n-2m)(n-2m-1)\ldots}_{j+1}}j, \ j > 1 .$$

If we select $m = 1$ for $H(n,j)$, it is obtained the following table.

1				
3	2			
6	8	3		
10	20	15	4	
15	40	45	25	5
⋮	⋮	⋮	⋮	⋮

Table 1

The first few terms of $V'_{n,m}(x)$ for $m = 1, 2, 3$ are the following.

$$V'_{1,1}(x) = 1$$
$$V'_{2,1}(x) = 2x - 2$$
$$V'_{3,1}(x) = 3x^2 - 6x + 3$$
$$V'_{4,1}(x) = 4x^3 - 12x^2 + 12x - 4$$
$$V'_{5,1}(x) = 5x^4 - 20x^3 + 30x^2 - 20x + 5$$
$$V'_{6,1}(x) = 6x^5 - 30x^4 + 60x^3 - 60x^2 + 30x - 6$$
$$V'_{7,1}(x) = 7x^6 - 42x^5 + 105x^4 - 140x^3 + 105x^2 - 42x + 7$$
$$V'_{8,1}(x) = 8x^7 - 56x^6 + 168x^5 - 280x^4 + 280x^3 - 168x^2 + 56x - 8$$
$$\vdots$$
$$V'_{1,2}(x) = 1$$
$$V'_{2,2}(x) = 2x$$
$$V'_{3,2}(x) = 3x^2 - 2$$
$$V'_{4,2}(x) = 4x^3 - 6x$$
$$V'_{5,2}(x) = 5x^4 - 12x^2 + 3$$
$$V'_{6,2}(x) = 6x^5 - 20x^3 + 12x$$
$$V'_{7,2}(x) = 7x^6 - 30x^4 + 30x^2 - 4$$
$$V'_{8,2}(x) = 8x^7 - 42x^5 + 60x^3 - 20x$$
$$\vdots$$
$$V'_{1,3}(x) = 1$$
$$V'_{2,3}(x) = 2x$$
$$V'_{3,3}(x) = 3x^2$$
$$V'_{4,3}(x) = 4x^3 - 2$$
$$V'_{5,3}(x) = 5x^4 - 6x$$
$$V'_{6,3}(x) = 6x^5 - 12x^2$$
$$V'_{7,3}(x) = 7x^6 - 20x^3 + 3$$
$$V'_{8,3}(x) = 8x^7 - 30x^4 + 12x$$

It is seen that written polynomial up with the cross member of polynomial $V'_{n,m}(x)$ gives polynomial $V'_{n,m+1}(x)$.

Now we define derivative of the generalized Chebyshev polynomials $\Omega_{n,m}(x)$ (Vieta- Lucas polynomial) (x is a real variable and $n \geq 2m$).

***Definition* 3.2.** Derivative of the generalized Chebyshev polynomials $\Omega_{n,m}(x)$ is

$$\Omega'_{n,m}(x) = x\Omega'_{n-1,m}(x) - 2\Omega'_{n-m,m}(x) + x^{n-1} - x^{n-m-1} + \sum_{j=1} (-1)^j \; h(n,j)x^{n-(j+1)m-1}$$

where

$$\Omega'_{n,n}(x) = nx^{n-1}, \Omega'_{n,m}(x) = nx^{n-1} - \left[2\binom{n-m+1}{2} + \binom{n-m}{1}\right]x^{n-m-1}, \quad (n < 2m)$$

and

$$h(n,j) = \binom{n-2m+1}{j+1} a_j \;, \; \left(1 \leq j \leq \left[\frac{n-1}{m}\right]\right), \text{(for } j = 1 \,,\; a_1 = 1),$$

with

$$a_j = \left((n-m)j - (m-2)j^2 - 1\right)\frac{\overbrace{(n-2m-j)(n-2m-j-1)\cdots}^{[m+(j-2)(m-1)-1]}}{\underbrace{(n-m-(m-1)j)(n-m-(m-1)j+1)\cdots}_{[m+(j-2)(m-1)]}} \quad ,j > 1.$$

If we select $m = 1$ for $h(n,j)$, the following table is obtained.

1				
3	3			
6	11	5		
10	26	23	7	
15	50	65	39	9
⋮	⋮	⋮	⋮	⋮

Table 2

The first few terms of $\Omega'_{n,m}(x)$ for $m = 1,2,3$ are the followings.

$\Omega'_{1,1}(x) = 1$

$\Omega'_{2,1}(x) = 2x - 3$

$\Omega'_{3,1}(x) = 3x^2 - 8x + 5$

$\Omega'_{4,1}(x) = 4x^3 - 15x^2 + 18x - 7$

$\Omega'_{5,1}(x) = 5x^4 - 24x^3 + 42x^2 - 32x + 9$

$\Omega'_{6,1}(x) = 6x^5 - 35x^4 + 80x^3 - 90x^2 + 50x - 11$

$\Omega'_{7,1}(x) = 7x^6 - 48x^5 + 135x^4 - 200x^3 + 165x^2 - 72x + 13$

$\vdots$

$\Omega'_{1,2}(x) = 1$

$\Omega'_{2,2}(x) = 2x$

$\Omega'_{3,2}(x) = 3x^2 - 3$

$\Omega'_{4,2}(x) = 4x^3 - 8x$

$\Omega'_{5,2}(x) = 5x^4 - 15x^2 + 5$

$\Omega'_{6,2}(x) = 6x^5 - 24x^3 + 18x$

$\Omega'_{7,2}(x) = 7x^6 - 35x^4 + 42x^2 - 7$

$\Omega'_{8,2}(x) = 8x^7 - 48x^5 + 80x^3 - 32x$

$\Omega'_{9,2}(x) = 9x^8 - 63x^6 + 135x^4 - 90x^2 + 9$

$\Omega'_{1,3}(x) = 1$

$\Omega'_{2,3}(x) = 2x$

$\Omega'_{3,3}(x) = 3x^2$

$\Omega'_{4,3}(x) = 4x^3 - 3$

$\Omega'_{5,3}(x) = 5x^4 - 8x$

$\Omega'_{6,3}(x) = 6x^5 - 15x^2$

$\Omega'_{7,3}(x) = 7x^6 - 24x^3 + 5$

$\Omega'_{8,3}(x) = 8x^7 - 35x^4 + 18x$

$\Omega'_{9,3}(x) = 9x^8 - 48x^5 + 42x^2$

$\Omega'_{10,3}(x) = 10x^9 - 63x^6 + 80x^3 - 7$

$\Omega'_{11,3}(x) = 11x^{10} - 80x^7 + 135x^4 + 32x$

It is clear that written polynomial above with the cross member of polynomial $\Omega'_{n,m}(x)$ gives polynomial $\Omega'_{n,m+1}(x)$.

***Definition* 3.3.** Combinatorial formsof derivative polynomial of $V'_{n,m}(x)$ and $\Omega'_{n,m}(x)$ are $(n \geq 1)$ respectively

$$B'(n,j) = (n-(m-1)j)\binom{n-j-((m-2)j+1)}{j}, \qquad \left(1 \leq j \leq \left[\frac{n-1}{m}\right]\right)$$

and

$$b'(n,j) = \frac{(n-(m-2)j)(n-mj)}{(n-(m-1)j)}\binom{n-j-(m-2)j}{j}, \qquad \left(1j \leq \left[\frac{n-1}{m}\right]\right).$$

The first few terms for $m = 1$ are given in the following tables.

1							
2	2						
3	6	3					
4	12	12	4				
5	20	30	20	5			
6	30	60	60	30	6		
7	42	105	140	105	42	7	
…	…	…	…	…	…	…	…

Table 3 Array for $B'(n,j)$

1							
2	3						
3	5	8					
4	15	18	7				
5	24	42	32	9			
6	35	80	90	50	11		
7	48	135	200	165	71	13	
…	…	…	…	…	…	…	…

Table 4 Array for $b'(n,j)$

Table3 and 4 will be called derivative polynomial of the generalized Chebyshev polynomials $V_{n,m}(x)$ and $\Omega_{n,m}(x)$. These tables will change according to value of m. In general terms, n k, the column is obtained by writing the elements of the tables from the line $((k-1)m+1)$ th. Tables of the derivative of the generalized Chebyshev polynomials $V_{n,m}(x)$ and $\Omega_{n,m}(x)$ consist of the components of this polynomial coefficients so derivative polynomial $V'_{n,m}(x)$ and $\Omega'_{n,m}(x)$ can be obtained without using the definition of polynomials

Then, we can write $V'_{n,m}(x)$ and $\Omega'_{n,m}(x)$ from definition 3.3,

$$V'_{n,m}(x) = \sum_{k=0}^{\left\lfloor\frac{n-1}{m}\right\rfloor}(-1)^k(n-(m-1)k)\binom{n-k-((m-2)k+1)}{k}x^{n-mk-1}$$

And

$$\Omega'_{n,m}(x) = \sum_{k=0}^{\left\lfloor\frac{n-1}{m}\right\rfloor}(-1)^k\frac{(n-(m-2)k)(n-mk)}{(n-(m-1)k)}\binom{n-k-((m-2)k}{k}x^{n-mk-1}.$$

***Example* 3.1.** Derivative polynomial of $V'_{10,3}(x)$is

$$V'_{10,3}(x) = \sum_{k=0}^{3}(-1)^k(10-(3-1)k)\binom{10-k-((3-2)k+1)}{k}x^{10-3k-1}$$

$$= (-1)^0(10-(3-1)0)\binom{10-0-((3-2)0+1)}{0}x^{10-3.0-1}$$

$$+(-1)^1(10-(3-1)1)\binom{10-1-((3-2)1+1)}{1}x^{10-3.1-1}$$

$$+(-1)^2(10-(3-1)2)\binom{10-2-((3-2)2+1)}{2}x^{10-3.2-1}$$

$$+(-1)^3(10-(3-1)3)\binom{10-3-((3-2)3+1)}{3}x^{10-3.3-1}$$

$$= 10x^9 - 8.7x^6 + 6.10x^3 - 4.1x^0$$

$$= 10x^9 - 56x^6 + 60x^3 - 4\,.$$

***Example* 3.2.** Derivative polynomial of $\Omega'_{7,2}(x)$ is

$$\Omega'_{7,2}(x) = \sum_{k=0}^{3}(-1)^k\frac{(7-(2-2)k)(7-2k)}{(7-(2-1)k)}\binom{7-k-((2-2)k}{k}x^{7-2k-1}$$

$$= (-1)^0\frac{(7-(2-2)0)(7-2.0)}{(7-(2-1)0)}\binom{7-0-((2-2)0}{0}x^{7-2.0-1}$$

$$= (-1)^1\frac{(7-(2-2)1)(7-2.1)}{(7-(2-1)1)}\binom{7-1-((2-2)1}{1}x^{7-2.1-1}$$

$$= (-1)^2\frac{(7-(2-2)2)(7-2.2)}{(7-(2-1)2)}\binom{7-2-((2-2)2}{2}x^{7-2.2-1}$$

$$= (-1)^3\frac{(7-(2-2)3)(7-2.3)}{(7-(2-1)3)}\binom{7-3-((2-2)3}{3}x^{7-2.3-1}$$

$$= 7x^6 - \frac{35}{6}\binom{6}{1}x^4 + \frac{21}{5}\binom{5}{2}x^2 - \frac{7}{4}\binom{4}{3}x^0$$

$$= 7x^6 - 35x^4 + 42x^2 - 7\,.$$

References

[1]. A.F. Horadam,Vieta polynomials, Fibonacci Quartely, 40 (3) (2002), 223-232.

[2]. G.B.D jordjevic, Polynomials Related to Generalized Chebyshev Polynomials.Filomat, 23 (3) (2009), 279-290.

[3]. M.A. Snyder, Chebyshev Methods in Numerical Approximation, Prentice Hall, Englewood Cliffs, NN, 1966.

[4]. T.J. Rivlin, Chebyshev Polynomials, second edition, Pure and Applied Mathematics, Wiley, New York, 1990.

Published: Volume 2017, Issue 9 / September 25, 2017