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Stability Analysis of a Three Species Food Chain Model with Harvesting

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Abstract

The Present investigation is an analytical study of a three species syn-ecological model which comprises three species. Here first species (N_1) ammensal on the second (N_2), second species ammensal on the third (N_3). In this model first species (N_1) and third species (N_3) are neutral to each other. And first (N_1) and second species (N_2) are harvested at a rate proportional to their population sizes. All possible equilibrium points are identified and the stability of Interior equilibrium point is discussed by using Routh-Hurwitz criteria and the solutions are carried out. Further the global stability of the system is discussed by constructing a suitable Lyapunov function. The analytical results are supported by numerical simulation using Mat Lab.

Keywords: Ammensal, Neutral, Equilibrium Points, Lyapunov's function and Routh-Hurwitz criteria

1. Introduction

Ammensalism is a relationship in which a product of one organism has a negative effect on another organism. It is specifically a population interaction in which one organism is harmed, while the other is neither affected nor benefitted. Ever since research in the discipline of theoretical ecology was initiated by Lotka [1] and by Volterra [4], several mathematicians and ecologists contributed to the growth of this area of knowledge, which has been extensively reported in the treatises of Meyer [7], Cushing [2], Freedman [3], Kanpur [5, 6]. The ecological interactions can be broadly classified as prey-predation, competitions, neutralism, and mutualism and so on. Kondala Rao K and Lakshmi Narayan K [9, 15] studied the stability analysis of a three species syn-eco dynamical system with a limited alternative food for all the three species and a three species syn-ecological model with amensalism and neutralism and a three species syn-ecological model with ammensalism and neutralism. Lakshmi Narayan K and Papa Rao A V [10, 11] investigated a three species ecological model with a prey, predator and competitor to the predator and optimal harvesting of the prey and harvesting of the salmon fish on a three species fisheries model. Solution of a three species ecological model with harvesting of prey by homotopy analysis method

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[12], Stability analysis of two species ecological model with a strong prey and weak predator and a three species ecological model with a prey, predator and a competitor to both the prey and predator [13, 8] were carried out by Lakshmi Narayan K et al. Stability analysis of three species syn-eological model with pry-prediation and ammensalism, Stability analysis of amensal model comprising humans, plants and birds with harvesting [14, 16, 17, 18], discussed by Kondala Rao K and Lakshmi Narayan K.

Ammensalism is a ecological relationship between the species where N_1 and N_3 are neutral to each other and first species N_1 effect the second species N_2, second species N_2 effect third species N_3 without themselves being effected in any way. The model is represented by a system of three ordinary differential equations. All possible equilibrium points are identified and their stability was discussed using Routh-Hurwitz criteria. Further solutions of quasi-linearized equations and the results are simulated by numerical examples using MatLab.

2. Basic Equations

The model equations for a system of three interacting species are given by the following set of non-linear first order simultaneous differential equations.

$$\begin{aligned} \frac{dN_1}{dt} &= f_1(N_1, N_2, N_3) = a_1(1-k_1)N_1 - \alpha_{11}{N_1}^2; \\ \frac{dN_2}{dt} &= f_2(N_1, N_2, N_3) = a_2(1-k_2)N_2 - \alpha_{22}{N_2}^2 - \alpha_{21}N_1N_2; \\ \frac{dN_3}{dt} &= f_3(N_1, N_2, N_3) = a_3N_3 - \alpha_{33}{N_3}^2 - \alpha_{32}N_2N_3. \end{aligned} \tag{2.1}$$

with the following notation

N_i (t) : Population of the first, second and third species at time " t", i=1, 2, 3.

a_i : Natural growth rate of first, second and third species, i=1, 2, 3.

α_{ii} : Rate of decrease of first, second and third species due to internal competitions, i= 1, 2, 3.

α_{21} : Rate of decrease of the second species due to attacks of first species.

α_{32} : Rate of decrease of the third species due to attacks of second species.

k_1 : Harvesting rate of first species.

k_2 : Harvesting rate of second species.

Further the variables N_1, N_2, N_3 are non-negative and the model parameters $a_i, \alpha_{ii}, i = 1,2,3, \alpha_{21}, \alpha_{32}$ and α_{13} are assumed to be non negative constants.

3. Equilibrium Points

For the system under investigation, eight equilibrium points are identified. They are given bellow. The equilibrium points are identified by solving

$$\frac{dN_i}{dt} = 0, i = 1,2,3. \tag{3.1}$$

$$i.e., \; f_1(N_1, N_2, N_3) = 0; \; f_2(N_1, N_2, N_3) = 0; \; f_3(N_1, N_2, N_3) = 0 \tag{3.2}$$

The equilibrium points are given bellow.

(E_1) <u>Fully Extinct State</u>:

$$\overline{N}_1 = 0; \overline{N}_2 = 0; \overline{N}_3 = 0 \tag{3.3}$$

(E_2) <u>First and Second Species Extinct State</u>:

$$\overline{N}_1 = 0; \overline{N}_2 = 0; \overline{N}_3 = \frac{a_3}{\alpha_{33}} \tag{3.4}$$

(E_3) <u>First and Third Species Extinct State</u>:

$$\overline{N}_1 = 0; \overline{N}_2 = \frac{(1-k_2)a_2}{\alpha_{22}}; \overline{N}_3 = 0 \tag{3.5}$$

(E_4) <u>Second and Third Species Extinct State</u>:

$$\overline{N}_1 = \frac{(1-k_1)a_1}{\alpha_{11}}; \overline{N}_2 = 0; \overline{N}_3 = 0 \tag{3.6}$$

(E_5) <u>Second and Third Species Survived State</u>:

$$\overline{N}_1 = 0; \overline{N}_2 = \frac{(1-k_2)a_2}{\alpha_{22}}; \overline{N}_3 = \frac{a_3\alpha_{22} - \alpha_{32}(1-k_2)a_2}{\alpha_{22}\alpha_{33}} \tag{3.7}$$

This state is exists if

$$a_3\alpha_{22} - \alpha_{32}(1-k_2)a_2 > 0 \tag{3.8}$$

(E_6) <u>First and Third Species Survived State</u>:

$$\overline{N}_1 = \frac{(1-k_1)a_1}{\alpha_{11}}; \overline{N}_2 = 0; \overline{N}_3 = \frac{a_3}{\alpha_{33}} \tag{3.9}$$

(E_7) <u>First and Second Species Survived State</u>:

$$\overline{N}_1 = \frac{(1-k_1)a_1}{\alpha_{11}}; \overline{N}_2 = \frac{(1-k_2)a_2\alpha_{11} - \alpha_{21}(1-k_1)a_1}{\alpha_{11}\alpha_{22}}; \overline{N}_3 = 0 \tag{3.10}$$

This state would exist only when

$$\alpha_{11}(1-k_2)a_2 - \alpha_{21}(1-k_1)a_1 > 0 \tag{3.11}$$

(E_8) <u>Interior Equilibrium State</u> :

$$\begin{aligned}\overline{N}_1 &= \frac{(1-k_1)a_1}{\alpha_{11}};\ \overline{N}_2 = \frac{(1-k_2)a_2\alpha_{11} - \alpha_{21}(1-k_1)a_1}{\alpha_{11}\alpha_{22}};\\ \overline{N}_3 &= \frac{a_3\alpha_{11}\alpha_{22} - (1-k_2)a_2\alpha_{11}\alpha_{32} + (1-k_1)a_1\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}}\end{aligned} \tag{3.12}$$

This state would exist only when

$$(1-k_2)a_2\alpha_{11} - \alpha_{21}(1-k_1)a_1 > 0, \& a_3\alpha_{11}\alpha_{22} - (1-k_2)a_2\alpha_{11}\alpha_{32} + (1-k_1)a_1\alpha_{21}\alpha_{32} > 0 \tag{3.13}$$

4. Stability of the Interior Equilibrium Points

To examine the stability of the interior equilibrium state ($\overline{N}_1, \overline{N}_2, \overline{N}_3$) we consider a small perturbation (u_1, u_2, u_3) such that

$$N_1 = \overline{N}_1 + u_1, N_2 = \overline{N}_2 + u_2 \ \&\ N_3 = \overline{N}_3 + u_3. \tag{4.1}$$

After linearization we get

$$\frac{dU}{dt} = AU \tag{4.2}$$

Where

$$A = \begin{bmatrix} (1-k_1)a_1 - 2\alpha_{11}\overline{N}_1 & 0 & 0 \\ -\alpha_{21}\overline{N}_2 & (1-k_2)a_2 - 2\alpha_{22}\overline{N}_2 - \alpha_{21}\overline{N}_1 & 0 \\ 0 & -\alpha_{32}\overline{N}_3 & a_3 - 2\alpha_{33}\overline{N}_3 - \alpha_{32}\overline{N}_2 \end{bmatrix} \tag{4.3}$$

$$U = [u_1, u_2 u_3]^T \tag{4.4}$$

The characteristic equation for the system is

$$\left|A - \lambda I\right| = 0 \tag{4.5}$$

The equilibrium state is stable when the roots of the equation (4.5) are negative if they are real or have negative real parts if they are complex.

In the present model, we discussed the stability of E_8 (That is Interior Equilibrium point) states Linearized equations for the existence of all three species are

$$\frac{du_1}{dt} = -(1-k_1)a_1u_1; \frac{du_2}{dt} = -\alpha_{21}\overline{N}_2u_1 - \alpha_{22}\overline{N}_2u_2; \frac{du_3}{dt} = -\alpha_{32}\overline{N}_3u_2 - \alpha_{33}\overline{N}_3u_3. \tag{4.6}$$

The characteristic equation corresponding to the interior equilibrium point is $\left[\lambda+(1-k_1)a_1\right]\left(\lambda+\alpha_{22}\overline{N}_2\right)\left(\lambda+\alpha_{33}\overline{N}_3\right)=0$. The Eigen values of the characteristic equation are $\lambda_1=-(1-k_1)a_1,\ \lambda_2=-\alpha_{22}\overline{N}_2,\ \lambda_3=-\alpha_{33}\overline{N}_3$. Here clearly all Eigen values are negative. Hence Interior Equilibrium point is stable.

The solution of perturbation equations is

$$u_1=u_{10}e^{-(1-k_1)a_1 t};\ u_2=-\frac{\alpha_{21}\overline{N}_2 u_{10}}{\alpha_{21}\overline{N}_2-(1-k_1)a_1}e^{-(1-k_1)a_1 t}+u_{20}e^{-\alpha_{22}\overline{N}_2 t}+\frac{\alpha_{21}\overline{N}_2 u_{10}}{\alpha_{21}\overline{N}_2-(1-k_1)a_1}e^{-\alpha_{22}\overline{N}_2 t} \tag{4.7}$$

$$u_3=m_1e^{-(1-k_1)a_1 t}-\left(m_2+m_3\right)e^{-\alpha_{22}\overline{N}_2 t}+\left(u_{30}-m_1+m_2+m_3\right)e^{-\alpha_{33}\overline{N}_3 t}$$

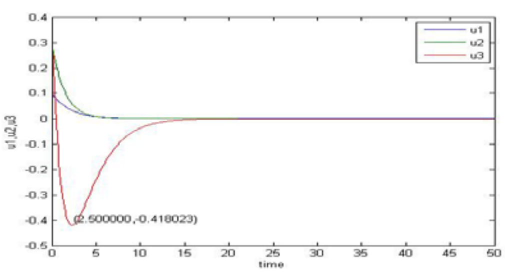
Where

$$m_1=\frac{\alpha_{21}\alpha_{33}\overline{N}_2\overline{N}_3u_{10}}{\left(\alpha_{22}\overline{N}_2-(1-k_1)a_1\right)\left(\alpha_{33}\overline{N}_3-(1-k_2)a_2\right)},\ m_2=\frac{\alpha_{33}\overline{N}_3}{\left(\alpha_{33}\overline{N}_3-\alpha_{22}\overline{N}_2\right)}\ \&\ m_3=\frac{\alpha_{21}\alpha_{33}\overline{N}_2\overline{N}_3u_{10}}{\left(\alpha_{22}\overline{N}_2-(1-k_1)a_1\right)\left(\alpha_{33}\overline{N}_3-\alpha_{22}\overline{N}_2\right)}$$

Numerical Example:

With Harvesting:

Let a_1=0.5, a_2=0.2, α_{21}= 0.06, α_{22}= 0.05, α_{33}= 0.03, u_{10}=10, u_{20}=20, u_{30}=11, N_2=20, N_3=15.



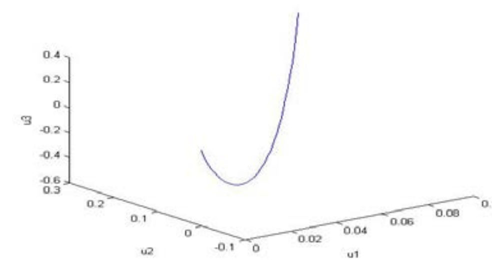
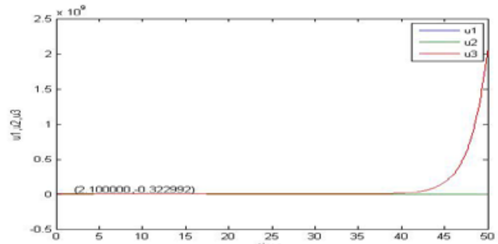


Fig 4.A **Fig 4.B**

Without Harvesting:



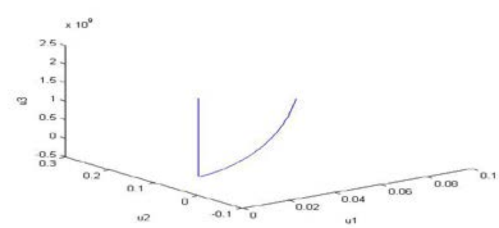


Fig 4.C **Fig 4.D**

Figures 4.C and 4.D show that the linearized state is asymptotically stable. Figures 4.A and 4.B show that harvesting strengthens slightly when comparing the equilibrium point.

5. Global Stability

Theorem:

Statement: The Co-existent state or Normal steady state is globally asymptotically stable.

Proof: Let us consider the Lyapunov's function for the Co-existing State or Normal steady state is

$$V(\overline{N}_1,\overline{N}_2,\overline{N}_3)=\left\{N_1-\overline{N}_1-\overline{N}_1\ln\left(\frac{N_1}{\overline{N}_1}\right)\right\}+\left\{N_2-\overline{N}_2-\overline{N}_2\ln\left(\frac{N_2}{\overline{N}_2}\right)\right\}+\left\{N_3-\overline{N}_3-\overline{N}_3\ln\left(\frac{N_3}{\overline{N}_3}\right)\right\} \quad (5.1)$$

Here $\overline{N}_1\neq 0,\overline{N}_2\neq 0,\overline{N}_3\neq 0$

Differentiating V with respect to't', we get

$$\frac{dV}{dt}=\left[\frac{N_1-\overline{N}_1}{N_1}\right]\frac{dN_1}{dt}+\left[\frac{N_2-\overline{N}_2}{N_2}\right]\frac{dN_2}{dt}+\left[\frac{N_3-\overline{N}_3}{N_3}\right]\frac{dN_3}{dt} \quad (5.2)$$

$$\frac{dV}{dt}=-\alpha_{11}(N_1-\overline{N}_1)^2-\alpha_{22}(N_2-\overline{N}_2)^2-\alpha_{33}(N_3-\overline{N}_3)^2<0\Rightarrow\frac{dV}{dt}<0 \quad (5.3)$$

Therefore the Co-existing state or Normal steady state is Globally Asymptotically stable.

6. Numerical Examples

Example 1:

Let a_1=0.02, a_2=0.05, a_3=0.08, α_{11}=0.04, α_{22}=0.4, α_{33}=0.4, α_{21}=0.09, α_{32}=0.09, N_{10}=5, N_{20}=10, N_{30}=15, k_1 = 0.8 and k_2 = 0.6.

<u>Without Harvesting</u>:

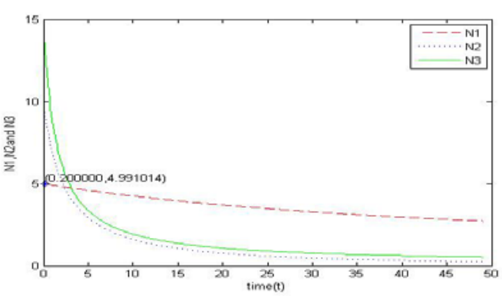


Fig (6.1.A)

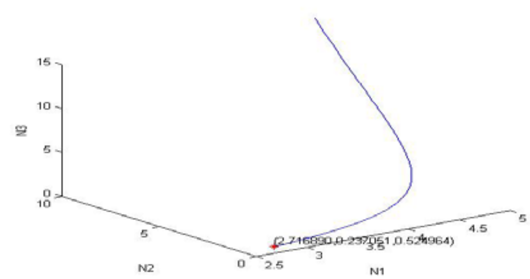
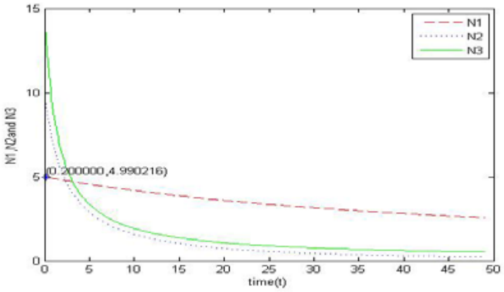


Fig (6.1.B)

With Harvesting:



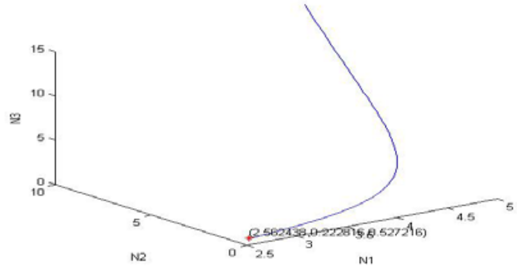


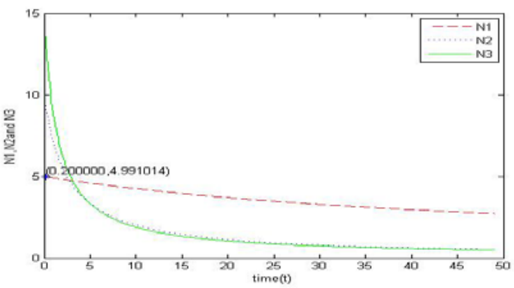
Fig (6.1.C) **Fig (6.1.D)**

In these figures 6.1.A & 6.1.B show that the trajectories and phase portrait in which the interacting coefficients of two species i.e., α_{21} and α_{32} are same (0.09) and the equilibrium point is (2.716890, 0.237051, 0.524964). Fig 6.1.C, 6.1.D represents graphs with harvesting proportionalities k_1=0.8, k_2=0.6 with stability point (2.5672438, 0.222818, 0.527216) and harvesting strengthened the third species considerably.

Example 2:

Let a_1=0.02, a_2=0.05, a_3=0.08, α_{11}=0.04, α_{22}=0.4, α_{33}=0.4, α_{21}=0.009, α_{32}=0.09, N_{10}=5, N_{20}=10,N_{30}=15, k_1 = 0.8 and k_2 = 0.6.

Without Harvesting:



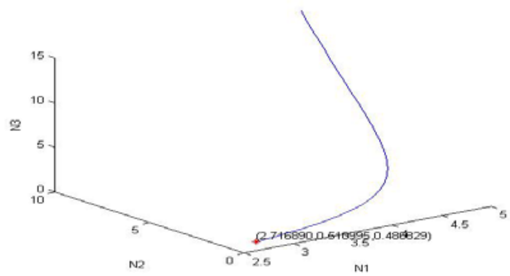
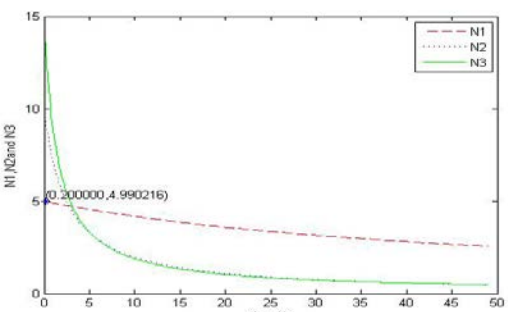


Fig (6.2.A) **Fig (6.2.B)**

With Harvesting:



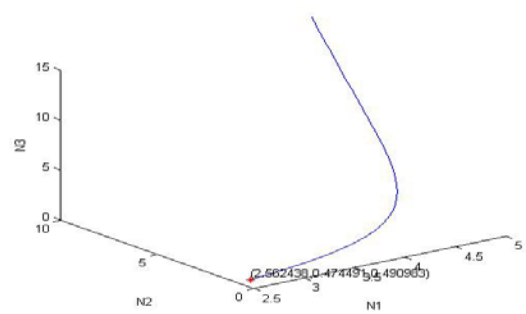


Fig (6.2.C) **Fig (6.2.D)**

Figures 6.2.A & 6.2.B represent that α_{21} is decreased (0.009) and α_{32} is kept at same value (0.09) and the equilibrium point is (2.716890, 0.510995, 0.488629), which shows this one has more asymptotic stable than previous case. Fig 6.2.C, 6.2.D represent graphs with harvesting proportionalities k_1=0.8, k_2=0.6 with the stability point (2.562438, 0.477491, 0.490983).

Example 3:

Let a_1=0.02, a_2=0.05, a_3=0.08, α_{11}=0.04, α_{22}=0.4, α_{33}=0.4, α_{21}=0.09, α_{32}=0.009, N_{10}=15, N_{20}=20, N_{30}=25, k_1 = 0.8 and k_2 = 0.6.

Without Harvesting:

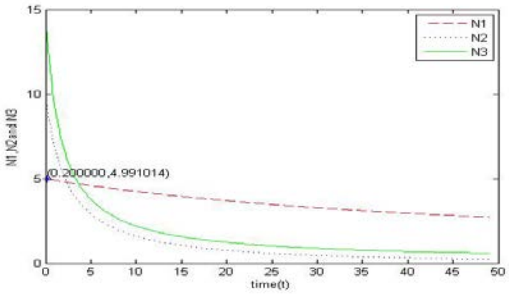


Fig (6.3.A)

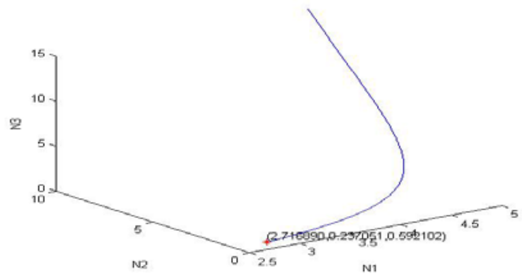


Fig (6.3.B)

With Harvesting:

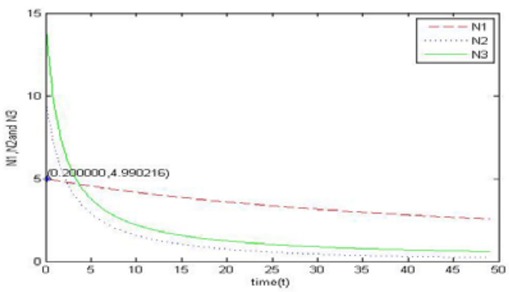


Fig (6.3.C)

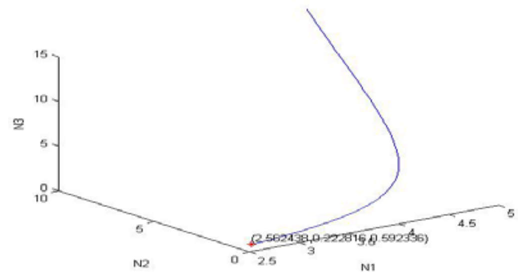


Fig (6.3.D)

Fig 6.3.A & 6.3.B show deflections when α_{21} is kept at same (0.09) and α_{32} is decreased (0.009) and the equilibrium point is (2.716890, 0.237051, 0.592102). Fig 6.1.C, 6.1.D represent graphs with harvesting proportionalities k_1=0.7, k_2 =0.8 and stability point is (2.562438, 0.222816, 0.92336) which shows that harvesting strengthened when we decreased interacting coefficient of second and third species.

Example 4: Without Harvesting:

Let a_1=0.02, a_2=0.05, a_3=0.08, α_{11}=0.04, α_{22}=0.4, α_{33}=0.4, α_{21}=0.009, α_{32}=0.009, N_{10}=15, N_{20}=20, N_{30}=25, k_1 = 0.8 and k_2 = 0.6.

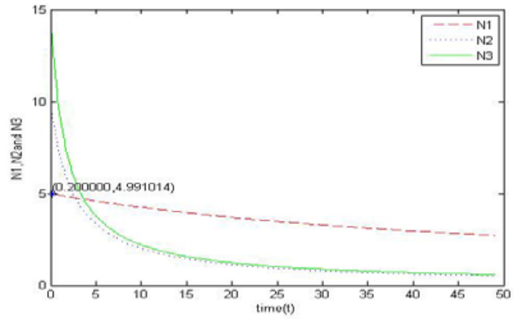


Fig (6.4.A)

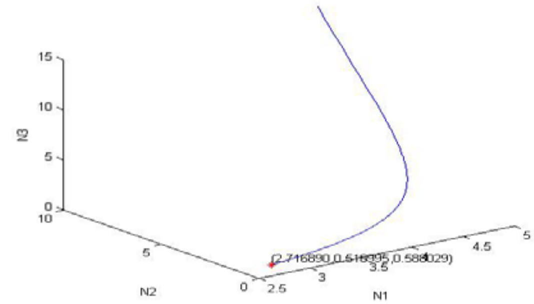


Fig (6.4.B)

With Harvesting:

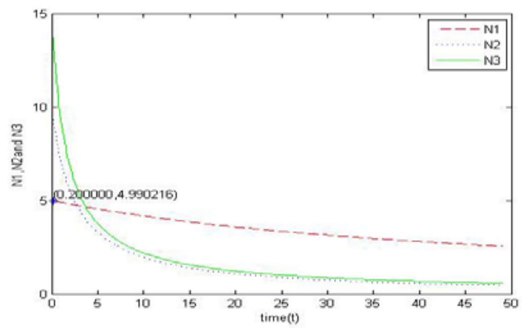


Fig (6.4.C)

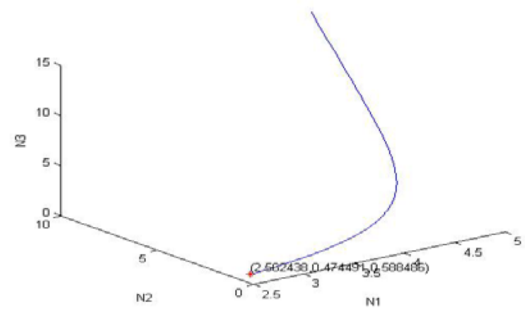


Fig (6.4.D)

Figures 6.4.A & 6.4.B show that the trajectories and phase portrait in which the interacting coefficients of two species i.e., α_{21} and α_{32} are same (0.009) both are decreasing then the equilibrium point is (2.716890, 0.510995, 0.588029). Figures 6.4.C and 6.4.D represents graphs with harvesting proportionalities k_1=0.8, k_2=0.6 with stability point (2.562438, 0.474491, 0.588486) and harvesting strengthened the third species. Finally when we decrease the interacting coefficient between first and second species (α_{21}) and second and third species (α_{32}), first and second species are weaken and third species are strengthens and harvesting strengthens more the third species.

7. Conclusion

In this present investigation, we studied a three species syn-ecological model in which the first species ammensol the second species and second species amensal the third species. Here first species and third species are neutral to each other. All possible equilibrium points were identified and stability the of interior equilibrium point (E_8) was discussed analytically. The analytical results were supported by the numerical simulation. We observed that the interior equilibrium point was globally asymptotically stable. The results show that harvesting strengthens the global asymptotical stability of the system. From the interior equilibrium point we observed that harvesting stabilize the unstable equilibrium point and unstabilized the stable equilibrium point. And finally from the numerical simulation we observed that when the interacting coefficient between the first species and the second species (α_{31}) and second species and third species (α_{32}) both were decreased, first and second species were weakened and third species strengthened considerably.

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