Fuzzy Almost $p$-Continuous and Almost $p^*$-Continuous Functions

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Abstract

This paper deals with two different types of fuzzy continuous functions, viz., fuzzy almost $p$-continuous and fuzzy almost $p^*$-continuous functions. It is shown that fuzzy almost $p^*$-continuous function is fuzzy almost $p$-continuous but the converse is true under certain condition only. Again it is proved that the composition of two fuzzy almost $p^*$-continuous functions is also so but this is not true for fuzzy almost $p$-continuous function. In the last section a new type of fuzzy regularity is introduced under which fuzzy almost $p$-continuous function is fuzzy almost $p^*$-continuous.

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1. Introduction and Preliminaries

Fuzzy preopen set is introduced by S. Nanda in [6]. Using this concept as a basic tool here we have first introduced an idempotent operator and using this operator two different types of fuzzy continuous-like functions are introduced and studied. Afterwards, a new type of fuzzy regularity is introduced in which these two functions coincide.

Throughout this paper, $(X, \tau)$ or simply by $X$ we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [3]. In 1965, L.A. Zadeh introduced fuzzy set [8] $A$ which is a function from a non-empty set $X$ into the closed interval $I = [0,1]$, i.e., $A \in I^X$. The support [8] of a fuzzy set

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A, denoted by \( \text{supp}A \) or \( A_0 \) and is defined by \( \text{supp}A = \{ x \in X : A(x) \neq 0 \} \). The fuzzy set with the singleton support \( \{x\} \subseteq X \) and the value \( t \) \( (0 < t \leq 1) \) will be denoted by \( x_t \). \( 0_x \) and \( 1_x \) are the constant fuzzy sets taking values 0 and 1 respectively in \( X \). The complement \([8]\) of a fuzzy set \( A \) in an fts \( X \) is denoted by \( 1_x \setminus A \) and is defined by \( (1_x \setminus A)(x) = 1 - A(x) \), for each \( x \in X \). For any two fuzzy sets \( A, B \) in \( X \), \( A \leq B \) means \( A(x) \leq B(x) \), for all \( x \in X \) \([8]\) while \( A \cap B \) means \( A \) is quasi-coincident (q-coincident, for short) \([7]\) with \( B \), i.e., there exists \( x \in X \) such that \( A(x) + B(x) > 1 \). The negation of these two statements will be denoted by \( A \not\leq B \) and \( A \cap B \) respectively. For a fuzzy set \( A \), \( \text{cl}A \) and \( \text{int}A \) will stand for fuzzy closure \([3]\) and fuzzy interior \([3]\) respectively.

A fuzzy set \( A \) in an fts \( (X, \tau) \) is called fuzzy regular open \([1]\) (resp., fuzzy preopen \([6]\)) if \( A = \text{int}A \) (resp., \( A \leq \text{int}A \)). The complement of a fuzzy preopen set is called fuzzy preclosed \([6]\). The union (intersection) of all fuzzy preopen (resp., fuzzy preclosed) sets contained in (resp., containing) a fuzzy set \( A \) is called fuzzy preinterior \([6]\) (resp., fuzzy preclosure \([6]\)) of \( A \), denoted by \( \text{pint}A \) (resp., \( \text{pcl}A \)). A fuzzy set \( A \) in \( X \) is called a fuzzy neighbourhood (nbd, for short) \([7]\) of a fuzzy point \( x_t \) if there exists a fuzzy open set \( G \) in \( X \) such that \( x_t \in G \leq A \). If, in addition, \( A \) is fuzzy open, then \( A \) is called fuzzy open nbd of \( x_t \). A fuzzy set \( A \) is said to be a fuzzy q-nbd of a fuzzy point \( x_t \) in an fts \( X \) if there is a fuzzy open set \( U \) in \( X \) such that \( x_t qU \leq A \). If, in addition, \( A \) is fuzzy open (resp., fuzzy preopen), then \( A \) is called a fuzzy open \([7]\) (resp., fuzzy preopen \([6]\)) q-nbd of \( x_t \). The collection of all fuzzy preopen (resp., fuzzy preclosed) sets in \( X \) is denoted by \( \text{FPO}(X) \) (resp., \( \text{FPC}(X) \)).
2. Fuzzy $p^*$-Open and $p^*$-Closed Sets: Some Properties

In this section we first introduce fuzzy $p^*$-open and fuzzy $p^*$-closed sets. Some important results are established here. Afterwards it is shown that $p^*$-closure operator is an idempotent operator. Again it is established that the collection of all fuzzy $p^*$-open sets in an fts $(X, \tau)$ is strictly weaker than that of fuzzy preopen sets.

**Definition 2.1.** A fuzzy set $A$ in an fts $(X, \tau)$ is called fuzzy $p^*$-open if $A \leq \text{int}(p\text{cl}A)$. The complement of this set is called fuzzy $p^*$-closed set.

The collection of fuzzy $p^*$-open (resp., fuzzy $p^*$-closed) sets in $(X, \tau)$ is denoted by $FP^O(X)$ (resp., $FP^C(X)$).

The union (resp., intersection) of all fuzzy $p^*$-open (resp., fuzzy $p^*$-closed) sets contained in (containing) a fuzzy set $A$ is called fuzzy $p^*$-interior (resp., fuzzy $p^*$-closure) of $A$, denoted by $p^\text{int}A$ (resp., $p^\text{cl}A$).

**Definition 2.2.** A fuzzy set $A$ in an fts $(X, \tau)$ is called fuzzy $p^*$-nbd of a fuzzy point $x_\alpha$ if there exists a fuzzy $p^*$-open set $U$ in $X$ such that $x_{\alpha} \leq U \leq A$. If, in addition, $A$ is fuzzy $p^*$-open, then $A$ is called fuzzy $p^*$-open nbd of $x_\alpha$.

**Definition 2.3.** A fuzzy set $A$ in an fts $(X, \tau)$ is called fuzzy $p^*$-$q$-nbd of a fuzzy point $x_\alpha$ if there exists a fuzzy $p^*$-open set $U$ in $X$ such that $x_{\alpha}qU \leq A$. If, in addition, $A$ is fuzzy $p^*$-open, then $A$ is called fuzzy $p^*$-open $q$-nbd of $x_\alpha$.

**Result 2.4.** Union (resp., intersection) of any two fuzzy $p^*$-open (resp., fuzzy $p^*$-closed) sets is also so.

**Proof.** Let $A, B$ be two fuzzy $p^*$-open (resp., fuzzy $p^*$-closed) sets in an fts $X$. Then
\[ A \leq \text{int}(\text{pcl}A), B \leq \text{int}(\text{pcl}B) \quad (\text{resp.,} \quad \text{cl}(\text{pint}A) \leq A, \text{cl}(\text{pint}B) \leq B). \quad \text{Now} \quad \text{int}(\text{pcl}(A \vee B)) = \text{int}(\text{pcl}A \vee \text{pcl}B) \geq \text{int}(\text{pcl}A) \vee \text{int}(\text{pcl}B) \geq A \vee B \quad (\text{resp.,} \quad \text{cl}(\text{pint}(A \wedge B)) = \text{cl}(\text{pint}A \wedge \text{pint}B) \leq \text{cl}(\text{pint}A) \wedge \text{cl}(\text{pint}B) \leq A \wedge B). \]

**Remark 2.5.** Intersection (resp., union) of two fuzzy \( p^* \)-open (resp., fuzzy \( p^* \)-closed) sets may not be so as it seen from the following example.

**Example 2.6.** Let \( X = \{a, b\} \), \( \tau = \{0_X, 1_X, A, B\} \) where \( A(a) = 0.5, A(b) = 0.4; B(a) = 0.5, B(b) = 0.55 \). Then \( (X, \tau) \) is an fts. Now \( FPO(X, \tau) = \{0_X, 1_X, U, V, W\} \) where \( 1_X \setminus B < U \leq B, V \geq 1_X \setminus A, W \leq A \) and \( FPC(X) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V, 1_X \setminus W\} \) where \( 1_X \setminus B \leq 1_X \setminus U < B, 1_X \setminus V < A, 1_X \setminus W \geq 1_X \setminus A \). Consider two fuzzy sets \( C, D \) in \( X \) defined by \( C(a) = 0.4, C(b) = 0.55, D(a) = 0.6, D(b) = 0.5 \). Now \( \text{int}(\text{pcl}C) = \text{int}(1_X \setminus A) = B \geq C, \text{int}(\text{pcl}D) = \text{int}(1_X) = 1_X \setminus D \Rightarrow C, D \) are fuzzy \( p^* \)-open sets in \( (X, \tau) \). Let \( E = C \wedge D \). Then \( E(a) = 0.4, E(b) = 0.5 \).

Now \( \text{int}(\text{pcl}E) = \text{int}(E) = 0_X \not\geq E \Rightarrow E \) is not fuzzy \( p^* \)-open in \( X \).

Here also \( 1_X \setminus C, 1_X \setminus D \) are fuzzy \( p^* \)-closed sets in \( (X, \tau) \). Now \( F = (1_X \setminus C) \vee (1_X \setminus D) \) is defined by \( F(a) = 0.6, F(b) = 0.5 \) and \( \text{cl}(\text{pint}F) = \text{cl}U \) where \( U(a) = U(b) = 0.5 \). Then \( \text{cl}U = 1_X \setminus A \not\leq F \Rightarrow F \) is not fuzzy \( p^* \)-closed in \( (X, \tau) \).

**Result 2.7.** \( x_a \in p^*\text{cl}A \) iff every fuzzy \( p^* \)-open \( q \)-nbd \( U \) of \( x_a, UqA \).

**Proof.** Let \( x_a \in p^*\text{cl}A \) for any fuzzy set \( A \) in an fts \( (X, \tau) \). Let \( U \in FP^*O(X) \) with \( x_a \cup U \). Then \( U(x) + \alpha > 1 \Rightarrow x_a \not\leq 1_X \setminus U \in FP^*C(X) \). By Definition 2.1, \( A \not\leq 1_X \setminus U \Rightarrow \) there exists \( y \in X \) such that \( A(y) > 1 - U(y) \Rightarrow A(y) + U(y) > 1 \Rightarrow UqA \).

Conversely, let the given condition hold. Let \( U \in FP^*C(X) \) with \( A \leq U \) \( \ldots \) (1). We have to show that \( x_a \in U \), i.e., \( U(x) \geq \alpha \). If possible, let \( U(x) < \alpha \). Then \( 1 - U(x) > 1 - \alpha \Rightarrow x_ac(1_X \setminus U) \) where
1. \( U \in FP^O(X) \). By hypothesis, \( (1_x \setminus U)qA \Rightarrow \) there exists \( y \in X \) such that \( 1 - U(y) + A(y) > 1 \Rightarrow A(y) > U(y) \), contradicts (1).

**Result 2.8.** \( p^*\text{cl}(p^*\text{cl}A) = p^*\text{cl}A \) for any fuzzy set \( A \) in an fts \( (X, \tau) \).

**Proof.** Let \( A \in I^X \). Then \( A \leq p^*\text{cl}A \Rightarrow p^*\text{cl}A \leq p^*\text{cl}(p^*\text{cl}A) \) ... (1).

Conversely, let \( x_a \in p^*\text{cl}(p^*\text{cl}A) \). If possible, let \( x_a \not\in p^*\text{cl}A \). Then there exists \( U \in FP^O(X) \), \( x_aqU, UqA \) ... (2). But as \( x_a \in p^*\text{cl}(p^*\text{cl}A) \), \( Uq(p^*\text{cl}A) \Rightarrow \) there exists \( y \in X \) such that \( U(y) + (p^*\text{cl}A)(y) > 1 \Rightarrow U(y) + t > 1 \) where \( t = (p^*\text{cl}A)(y) \). Then \( y \in p^*\text{cl}A \) and \( y, qU \) where \( U \in FP^O(X) \). Then by definition, \( UqA \), contradicts (2). So \( p^*\text{cl}(p^*\text{cl}A) \leq p^*\text{cl}A \) ... (3). Combining (1) and (3), we get the result.

**Note 2.9.** It is clear from above discussion that fuzzy \( p^* \)-open set is fuzzy preopen as \( p\text{cl}A \leq clA \) for any fuzzy set \( A \) in an fts. But not conversely, as it seen from the following example.

**Example 2.10.** Consider Example 2.6. Here \( E \) is not fuzzy \( p^* \)-open. But \( \text{int}(clE) = \text{int}(1_x \setminus A) = B > E \) and so \( E \) is fuzzy preopen.

**Remark 2.11.** It is clear from definition that every fuzzy open set is fuzzy \( p^* \)-open. But the converse may not be true follows from Example 2.6. Here \( C \) is fuzzy \( p^* \)-open, but \( C \) is not fuzzy open. So we have the following relation.

\( \text{fuzzy open} \Rightarrow \text{fuzzy } p^* \text{-open} \Rightarrow \text{fuzzy preopen} \)

### 3. Fuzzy Almost \( p \)-Continuous Function: Some Characterizations

In this section we first introduce a new type of fuzzy continuous-like function which implies fuzzy almost continuity but the converse need not be true.

**Definition 3.1.** A function \( f : X \rightarrow Y \) is said to be fuzzy almost \( p \)-continuous if for each fuzzy
point $x_\alpha$ in $X$ and every fuzzy nbd $V$ of $f(x_\alpha)$ in $Y$, $pcl(f^{-1}(V))$ is a fuzzy nbd of $x_\alpha$ in $X$.

**Theorem 3.2.** For a function $f : X \rightarrow Y$, the following statements are equivalent:

(a) $f$ is fuzzy almost $p$-continuous,

(b) $f^{-1}(B) \leq int(pcl(f^{-1}(B)))$, for all fuzzy open set $B$ of $Y$,

(c) $f(clA) \leq cl(f(A))$, for all $A \in FPO(X)$.

**Proof** (a) $\Rightarrow$ (b). Let $B$ be any fuzzy open set in $Y$ and $x_\alpha \in f^{-1}(B)$. Then $f(x_\alpha) \leq B \Rightarrow B$ is a fuzzy nbd of $f(x_\alpha)$. By (a), $pcl(f^{-1}(B))$ is a fuzzy nbd of $x_\alpha$ in $X \Rightarrow x_\alpha \leq int(pcl(f^{-1}(B)))$.

Hence $f^{-1}(B) \leq int(pcl(f^{-1}(B)))$.

(b) $\Rightarrow$ (a). Let $x_\alpha$ be a fuzzy point in $X$ and $B$ be a fuzzy nbd of $f(x_\alpha)$ in $Y$. Then $x_\alpha \leq f^{-1}(B) \leq int(pcl(f^{-1}(B)))$ (by (b)) $\leq pcl(f^{-1}(B))$ $\Rightarrow pcl(f^{-1}(B))$ is a fuzzy nbd of $x_\alpha$ in $X$.

(b) $\Rightarrow$ (c). Let $A \in FPO(X)$. Then $1_Y \setminus cl(f(A))$ is a fuzzy open set in $Y$. By (b),

$$f^{-1}(1_Y \setminus cl(f(A))) \leq int(pcl(f^{-1}(1_Y \setminus cl(f(A))))) = int(pcl(1_x \setminus f^{-1}(cl(f(A))))).$$

$$int(pcl(1_x \setminus f^{-1}(f(A)))) \leq int(pcl(1_x \setminus A)) = 1_X \setminus cl(pintA) = 1_X \setminus clA.$$ Then

$$1_X \setminus f^{-1}(cl(f(A))) \leq 1_X \setminus clA \Rightarrow clA \leq f^{-1}(cl(f(A))) \Rightarrow f(clA) \leq cl(f(A)).$$

(c) $\Rightarrow$ (b). Let $B$ be any fuzzy open set in $Y$. Then $pint(f^{-1}(1_Y \setminus B)) \in FPO(X)$. By (c),

$$f(cl(pint(f^{-1}(1_Y \setminus B)))) \leq cl(f(pint(f^{-1}(1_Y \setminus B)))) \leq cl(f(f^{-1}(1_Y \setminus B))) \leq cl(1_Y \setminus B) =$$

$$1_Y \setminus B \Rightarrow f^{-1}(B) = 1_Y \setminus f^{-1}(1_Y \setminus B) \leq 1_X \setminus cl(pint(f^{-1}(1_Y \setminus B))) = 1_X \setminus cl(pint(1_x \setminus f^{-1}(B))) = int(pcl(f^{-1}(B))).$$

**Note 3.3.** It is clear from Theorem 3.2 that the inverse image under fuzzy almost $p$-continuous function of any fuzzy open set is fuzzy $p^*$-open and hence fuzzy preopen.
**Theorem 3.4.** For a function \( f : X \rightarrow Y \), the following statements are equivalent:

(a) \( f \) is fuzzy almost \( p \)-continuous,

(b) \( f^{-1}(B) \leq \text{int}(\text{pcl}(f^{-1}(B))) \), for all fuzzy open set \( B \) of \( Y \),

(c) for each fuzzy point \( x_\alpha \) in \( X \) and each fuzzy open nbd \( V \) of \( f(x_\alpha) \), there exists \( U \in \text{FP'}O(X) \) containing \( x_\alpha \) such that \( f(U) \leq V \),

(d) \( f^{-1}(F) \in \text{FP'}C(X) \), for all fuzzy closed sets \( F \) in \( Y \),

(e) for each fuzzy point \( x_\alpha \) in \( X \), the inverse image under \( f \) of every fuzzy nbd of \( f(x_\alpha) \) is a fuzzy \( p^* \)-nbd of \( x_\alpha \) in \( X \),

(f) \( f(p^*\text{cl}A) \leq \text{cl}(f(A)) \), for all \( A \in I^X \),

(g) \( p^*\text{cl}(f^{-1}(B)) \leq f^{-1}(\text{cl}B) \), for all \( B \in I^Y \),

(h) \( f^{-1}(\text{int}B) \leq p^*\text{int}(f^{-1}(B)) \), for all \( B \in I^Y \),

(i) for every basic open fuzzy set \( V \) in \( Y \), \( f^{-1}(V) \in \text{FP'}O(X) \).

**Proof** (a) \( \iff \) (b). Follows from Theorem 3.2 (a) \( \iff \) (b).

(b) \( \Rightarrow \) (c). Let \( x_\alpha \) be a fuzzy point in \( X \) and \( V \) be a fuzzy open nbd of \( f(x_\alpha) \). By (b),

\[
f^{-1}(V) \leq \text{int}(\text{pcl}(f^{-1}(V))) \quad \text{(1)}.
\]

Now \( f(x_\alpha) \leq V \Rightarrow x_\alpha \in f^{-1}(V) \) (\( = U \), say). Then \( x_\alpha \in U \) and by (1), \( U(= f^{-1}(V)) \in \text{FP'}O(X) \) and \( f(U) = f(f^{-1}(V)) \leq V \).

(c) \( \Rightarrow \) (b). Let \( V \) be a fuzzy open set in \( Y \) and let \( x_\alpha \leq f^{-1}(V) \). Then \( f(x_\alpha) \leq V \Rightarrow V \) is a fuzzy open nbd of \( f(x_\alpha) \). By (c), there exists \( U \in \text{FP'}O(X) \) containing \( x_\alpha \) such that \( f(U) \leq V \).

Then \( x_\alpha \leq U \leq f^{-1}(V) \). Now \( U \leq \text{int}(\text{pcl}U) \). Then \( U \leq \text{int}(\text{pcl}U) \leq \text{int}(\text{pcl}(f^{-1}(V))) \Rightarrow x_\alpha \leq U \leq \text{int}(\text{pcl}(f^{-1}(V))) \Rightarrow f^{-1}(V) \leq \text{int}(\text{pcl}(f^{-1}(V))) \).

(b) \( \iff \) (d). Obvious.
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(b) $\Rightarrow$ (e). Let $W$ be a fuzzy nbd of $f(x_\alpha)$. Then there exists a fuzzy open set $V$ in $Y$ such that $f(x_\alpha) \leq V \leq W \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$. Then by (b), $f^{-1}(V) \in FP^O(X)$ and $x_\alpha \leq f^{-1}(V) \leq f^{-1}(W) \Rightarrow f^{-1}(W)$ is a fuzzy $p^*$-nbd of $x_\alpha$.

(e) $\Rightarrow$ (b). Let $V$ be a fuzzy open set in $Y$ and $x_\alpha \leq f^{-1}(V)$. Then $f(x_\alpha) \leq V \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$. By (e), $f^{-1}(V)$ is a fuzzy $p^*$-nbd of $x_\alpha$. Then there exists $U \in FP^O(X)$ containing $x_\alpha$ such that $U \leq f^{-1}(V) \Rightarrow x_\alpha \leq U \leq \text{int}(pclU) \leq \text{int}(pcl(f^{-1}(V))) \Rightarrow f^{-1}(V) \leq \text{int}(pcl(f^{-1}(V)))$.

(d) $\Rightarrow$ (f). Let $A \in I^X$. Then $\text{cl}(f(A))$ is a fuzzy closed set in $Y$. By (d), $f^{-1}(cl(f(A))) \in FP^C(X)$ containing $A$. Therefore, $p^*clA \leq p^*\text{cl}(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A))) \Rightarrow f(p^*clA) \leq cl(f(A))$.

(f) $\Rightarrow$ (d). Let $B$ be a fuzzy closed set in $Y$. Then $f^{-1}(B) \in I^X$. By (f), $f(p^*\text{cl}(f^{-1}(B))) \leq \text{cl}(f(f^{-1}(B))) \leq clB = B \Rightarrow p^*\text{cl}(f^{-1}(B)) \leq f^{-1}(B) \Rightarrow f^{-1}(B) \in FP^C(X)$.

(f) $\Rightarrow$ (g). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (f), $f(p^*\text{cl}(f^{-1}(B))) \leq \text{cl}(f(f^{-1}(B))) \leq clB \Rightarrow p^*\text{cl}(f^{-1}(B)) \leq f^{-1}(clB)$.

(g) $\Rightarrow$ (f). Let $A \in I^X$. Let $B = f(A)$. Then $B \in I^Y$. By (g), $p^*\text{cl}A = p^*\text{cl}(f^{-1}(B)) \leq f^{-1}(clB) = f^{-1}(cl(f(A))) \Rightarrow f(p^*\text{cl}A) \leq cl(f(A))$.

(b) $\Rightarrow$ (h). Let $B \in I^Y$. Then $\text{int}B$ is a fuzzy open set in $Y$. By (b), $f^{-1}(\text{int}B) \leq \text{int}(pcl(f^{-1}(\text{int}B))) \Rightarrow f^{-1}(\text{int}B) \in FP^O(X) \Rightarrow f^{-1}(\text{int}B) = p^*\text{int}(f^{-1}(\text{int}B)) \leq p^*\text{int}(f^{-1}(B))$.

(h) $\Rightarrow$ (b). Let $A$ be any fuzzy open set in $Y$. Then $f^{-1}(A) = f^{-1}(\text{int}A) \leq p^*\text{int}(f^{-1}(A))$ (by (h)) $\Rightarrow f^{-1}(A) \in FP^O(X)$. 

(b) \(\Rightarrow\) (i). Obvious.

(i) \(\Rightarrow\) (b). Let \(W\) be any fuzzy open set in \(Y\). Then there exists a collection \(\{W_{\alpha}: \alpha \in \Lambda\}\) of fuzzy basic open sets in \(Y\) such that \(W = \bigvee_{\alpha \in \Lambda} W_{\alpha}\). Now \(f^{-1}(W) = f^{-1}(\bigvee_{\alpha \in \Lambda} W_{\alpha}) = \bigvee_{\alpha \in \Lambda} f^{-1}(W_{\alpha}) \in FP^*O(X)\) (by (i) and by Result 2.4). Hence (b) follows.

**Theorem 3.5.** A function \(f: X \to Y\) is fuzzy almost \(p\)-continuous iff for each fuzzy point \(x_{\alpha}\) in \(X\) and each fuzzy open \(q\)-nbd \(V\) of \(f(x_{\alpha})\) in \(Y\), there exists a fuzzy \(p^*\)-open set \(W\) in \(X\) with \(x_{\alpha}qW\) such that \(f(W) \leq V\).

**Proof.** Let \(f\) be fuzzy almost \(p\)-continuous function and \(x_{\alpha}\) be a fuzzy point in \(X\) and \(V\) be a fuzzy open set in \(Y\) with \(f(x_{\alpha})qV\). Let \(f(x) = y\). Then \(V(y) + \alpha > 1 \Rightarrow V(y) > 1 - \alpha \Rightarrow V(y) > \beta > 1 - \alpha\), for some real number \(\beta\). Then \(V\) is a fuzzy open nbd of \(y_{\beta}\). By Theorem 3.4 (a) \(\Rightarrow\) (c), there exists \(W \in FP^*O(X)\) containing \(x_{\beta}\), i.e., \(W(x) \geq \beta\) such that \(f(W) \leq V\). Then \(W(x) \geq \beta > 1 - \alpha \Rightarrow x_{\alpha}qW\) and \(f(W) \leq V\).

Conversely, let the given condition hold and let \(V\) be a fuzzy open set in \(Y\). Put \(W = f^{-1}(V)\). If \(W = 0_X\), then we are done. Suppose \(W \neq 0_X\). Then for any \(x \in W_0\), let \(y = f(x)\). Then \(W(x) = [f^{-1}(V)](x) = V(f(x)) = V(y)\). Let us choose \(m \in \mathcal{N}\) where \(\mathcal{N}\) is the set of all natural numbers such that \(1/m \leq W(x)\). Put \(\alpha_n = 1 + 1/n - W(x)\), for all \(n \in \mathcal{N}\). Then for \(n \in \mathcal{N}\) and \(n \geq m\), \(1/n \leq 1/m \Rightarrow 1 + 1/n \leq 1 + 1/m \Rightarrow \alpha_n = 1 + 1/n - W(x) \leq 1 + 1/m - W(x) \leq 1\). Again \(\alpha_n > 0\), for all \(n \in \mathcal{N}\) \(\Rightarrow 0 < \alpha_n \leq 1\) so that \(V(y) + \alpha_n > 1 \Rightarrow y_{\alpha_n}qV \Rightarrow V\) is a fuzzy open \(q\)-nbd of \(y_{\alpha_n}\). By the given condition, there exists \(U_{\alpha_n}^x \in FP^*O(X)\) such that \(x_{\alpha_n}qU_{\alpha_n}^x\) and \(f(U_{\alpha_n}^x) \leq V\), for all \(n \geq m\). Let \(U^x = \bigvee\{U_{\alpha_n}^x: n \in \mathcal{N}, n \geq m\}\). Then \(U^x \in FP^*O(X)\) (by Result 2.4) and \(f(U^x) \leq V\). Again
n \geq m \implies U^x_n(x) + \alpha_n > 1 \implies U^x_n(x) + 1 + 1/n - W(x) > 1 \implies U^x_n(x) > W(x) - 1/n \implies U^x_n(x) \geq W(x), \text{ for each } x \in W_0. \text{ Then } W \leq U^x_n, \text{ for all } n \geq m \text{ and for all } x \in W_0 \implies W \leq U^x_n, \text{ for all } x \in W_0 \implies W \leq \bigvee_{x \in W_0} U^x_n = U \text{ (say) } \cdots (1) \text{ and } f(U^x) \leq V, \text{ for all } x \in W_0 \implies f(U) \leq V = U \leq f^{-1}(f(U)) \leq f^{-1}(V) = W \cdots (2). \text{ By (1) and (2), } U = W = f^{-1}(V) \implies f^{-1}(V) \in FP'O(X). \text{ Hence by Theorem 3.2, } f \text{ is fuzzy almost } p\text{-continuous.}

**Remark 3.6.** If \( f : X \to Y \) is fuzzy almost \( p \)-continuous, then by Note 3.3, inverse image of every fuzzy regular open set is fuzzy \( p^* \)-open. But the converse may not be true, as it seen from the following example.

**Example 3.7.** Let \( X = \{a, b\}, \ \tau = \{0_X, 1_X, A\}, \ \tau_1 = \{0_X, 1_X, B\} \) where \( A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.6 \). Then \( (X, \tau) \) and \( (X, \tau_1) \) are fts’s. Now \( FPO(X, \tau) = \{0_X, 1_X, U, V\} \) where \( U \leq A, \ V \leq 1_X \setminus A \) and \( FPC(X, \tau) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\} \) where \( 1_X \setminus U \geq 1_X \setminus A, \ 1_X \setminus V \geq A \). Consider the identity function \( i : (X, \tau) \to (X, \tau_1) \). The fuzzy regular open sets in \( (X, \tau_1) \) are only \( 0_X \) and \( 1_X \). So obviously inverse image of every fuzzy regular open set in \( (X, \tau_1) \) is fuzzy \( p^* \)-open in \( (X, \tau) \). But \( B \in \tau_1, \ i^{-1}(B) = B \leq int_c(pcl_c(i^{-1}(B))) = int_c(pcl_c(B)) = int_c(1_X \setminus A) = A \implies i \) is not fuzzy almost \( p \)-continuous function.

**Remark 3.8.** The inverse image of a fuzzy preopen set under fuzzy almost \( p \)-continuous function may not be fuzzy \( p^* \)-open follows from the following example.

**Example 3.9.** Let \( X = \{a, b\}, \ \tau = \{0_X, 1_X, A, B\}, \ \tau_1 = \{0_X, 1_X, C\} \) where \( A(a) = 0.5, A(b) = 0.3, B(a) = 0.5, B(b) = 0.4, C(a) = 0.5, C(b) = 0.4 \). Then \( (X, \tau) \) and \( (X, \tau_1) \) are fts’s. Consider the identity function \( i : (X, \tau) \to (X, \tau_1) \). Clearly \( i \) is fuzzy almost \( p \)-continuous. Indeed, \( FPO(X, \tau) = \{0_X, 1_X, U, V\} \) where \( U \leq B, V \leq 1_X \setminus A \) and \( FPC(X, \tau) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\} \).
where \( 1 \chi \setminus U \geq 1 \chi \setminus B, 1 \chi \setminus V \geq A \). Then \( i^{-1}(C) = C = \text{int}_\tau(\text{pcl}_\tau(i^{-1}(C))) = \text{int}_\tau(\text{pcl}_\tau(C)) = \text{int}_\tau(1 \chi \setminus B) = B = C \). Consider a fuzzy set \( D \) in \( X \) defined by \( D(a) = 0.5, D(b) = 0.7 \). Then \( \text{int}_\tau(\text{cl}_\tau D) = \text{int}_\tau 1 \chi = 1 \chi > D \Rightarrow D \in \text{FPO}(X, \tau_i) \). Now \( i^{-1}(D) = D \). \( \text{int}_\tau(\text{pcl}_\tau(i^{-1}(D))) = \text{int}_\tau(\text{pcl}_\tau D) = \text{int}_\tau D = B \geq D \Rightarrow D \not\in \text{FP''O}(X) \).

Let us now recall the following definition and theorem from \cite{5} for ready references.

**Definition 3.10** \cite{5}. A function \( f : X \rightarrow Y \) is said to be fuzzy almost continuous if for each fuzzy point \( x_\alpha \) in \( X \) and each fuzzy nbd \( V \) of \( f(x_\alpha) \) in \( Y \), \( \text{cl}(f^{-1}(V)) \) is a fuzzy nbd of \( x_\alpha \) in \( X \).

**Theorem 3.11** \cite{5}. A function \( f : X \rightarrow Y \) is fuzzy almost continuous iff \( f^{-1}(B) \leq \text{int}(\text{cl}(f^{-1}(B))) \), for all fuzzy open set \( B \) in \( Y \).

**Remark 3.12.** Since for any fuzzy set \( A \) in an fts \( (X, \tau) \), \( \text{pcl} A \leq \text{cl} A \), it is immediate that every fuzzy almost \( p \)-continuous function is fuzzy almost continuous. But the converse is not true, in general, follows from the following example.

**Example 3.13.** Let \( X = \{a\}, \ \tau = \{0_X, 1_X, B\}, \ \tau_i = \{0_X, 1_X, A\} \) where \( A(a) = 1/3, B(a) = 0.4 \). Then \( (X, \tau) \) and \( (X, \tau_i) \) are fts’s. Now \( \text{FPO}(X, \tau) = \{0_X, 1_X, U, V\} \) where \( U \leq B, V > 1_X \setminus B \) and \( \text{FPC}(X, \tau) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\} \) where \( 1_X \setminus U \geq 1_X \setminus B, 1_X \setminus V < B \). Consider the identity function \( i : (X, \tau) \rightarrow (X, \tau_i) \). Clearly \( i \) is fuzzy almost continuous. Indeed, other than \( 0_X \) and \( 1_X \), \( A \) is the only fuzzy open set in \( (X, \tau_i) \). Now \( \text{int}_\tau(\text{cl}_\tau(i^{-1}(A))) = \text{int}_\tau(\text{cl}_\tau A) = \text{int}_\tau(1_X \setminus B) = B \geq A = i^{-1}(A) \). But \( i \) is not fuzzy almost \( p \)-continuous. Because, \( \text{int}_\tau(\text{pcl}_\tau(i^{-1}(A))) = \text{int}_\tau(\text{pcl}_\tau A) = \text{int}_\tau A = 0_X \geq A \).

**Lemma 3.14** \cite{2}. Let \( Z, X, Y \) be fts’s and \( f_1 : Z \rightarrow X \) and \( f_2 : Z \rightarrow Y \) be functions. Let \( f : Z \rightarrow X \times Y \) be defined by \( f(z) = (f_1(z), f_2(z)) \) for \( z \in Z \), where \( X \times Y \) is provided with the
product fuzzy topology. Then if $B, U_1, U_2$ are fuzzy sets in $Z, X, Y$ respectively such that $f(B) \leq U_1 \times U_2$, then $f_1(B) \leq U_1$ and $f_2(B) \leq U_2$.

**Theorem 3.15.** Let $Z, X, Y$ be fts’s. For any functions $f_1: Z \to X, f_2: Z \to Y$, if $f: Z \to X \times Y$, defined by $f(x) = (f_1(x), f_2(x))$, for all $x \in Z$, is fuzzy almost $p$-continuous, so are $f_1$ and $f_2$.

**Proof.** Let $U_1$ be any fuzzy open $q$-nbd of $f_1(x_\alpha)$ in $X$ for any fuzzy point $x_\alpha$ in $Z$. Then $U_1 \times 1_Y$ is a fuzzy open $q$-nbd of $f(x_\alpha)$, i.e., $(f(x))_\alpha$ in $X \times Y$. Since $f$ is fuzzy almost $p$-continuous, there exists $V \in FP^*O(Z)$ with $x_\alpha \in V$ such that $f(V) \leq U_1 \times 1_Y$. By Lemma 3.14, $f_1(V) \leq U_1$, $f_2(V) \leq 1_Y$. Consequently, $f_1$ is fuzzy almost $p$-continuous.

Similarly, $f_2$ is fuzzy almost $p$-continuous.

**Lemma 3.16.** [1]. Let $X, Y$ be fts’s and let $g: X \to X \times Y$ be the graph of a function $f: X \to Y$. Then if $A, B$ are fuzzy sets in $X$ and $Y$ respectively, $g^{-1}(A \times B) = A \cap f^{-1}(B)$.

**Theorem 3.17.** Let $f: X \to Y$ be a function from an fts $X$ to an fts $Y$ and $g: X \to X \times Y$ be the graph function of $f$. If $g$ is fuzzy almost $p$-continuous, then $f$ is so.

**Proof.** Let $g$ be fuzzy almost $p$-continuous and $B$ be a fuzzy set in $Y$. Then by Lemma 3.16, $f^{-1}(B) = 1_X \cap f^{-1}(B) = g^{-1}(1_Y \times B)$. Now if $B$ is fuzzy open in $Y$, then $1_Y \times B$ is fuzzy open in $X \times Y$. Again, $g^{-1}(1_Y \times B) = f^{-1}(B) \in FP^*O(X)$ (by hypothesis) $\Rightarrow f$ is fuzzy almost $p$-continuous.
4. Fuzzy Almost $p^*$-Continuous Function: Some Characterizations

In this section a new type of function, viz., fuzzy almost $p^*$-continuous function is introduced and shown that inverse image under this function of any fuzzy $p^*$-open set is fuzzy $p^*$-open. It is shown that fuzzy almost $p^*$-continuous function is fuzzy almost $p$-continuous and the converse is true under certain condition.

**Definition 4.1.** A function $f : X \rightarrow Y$ is called fuzzy almost $p^*$-continuous if the inverse image of every fuzzy $p^*$-open set in $Y$ is fuzzy $p^*$-open in $X$.

**Theorem 4.2.** For a function $f : X \rightarrow Y$, the following statements are equivalent:

(a) $f$ is fuzzy almost $p^*$-continuous,

(b) for each fuzzy point $x_\alpha$ in $X$ and each fuzzy $p^*$-open nbd $V$ of $f(x_\alpha)$, there exists a fuzzy $p^*$-open nbd $U$ of $x_\alpha$ in $X$ and $f(U) \leq V$,

(c) $f^{-1}(F) \in FP^*C(X)$, for all $F \in FP^*C(Y)$,

(d) for each fuzzy point $x_\alpha$ in $X$, the inverse image under $f$ of every fuzzy $p^*$-open nbd of $f(x_\alpha)$ is a fuzzy $p^*$-open nbd of $x_\alpha$ in $X$,

(e) $f(p^*clA) \leq p^*cl(f(A))$, for all $A \in I^X$,

(f) $p^*cl(f^{-1}(B)) \leq f^{-1}(p^*clB)$, for all $B \in I^Y$,

(g) $f^{-1}(p^*intB) \leq p^*int(f^{-1}(B))$, for all $B \in I^Y$.

**Proof.** The proof is similar to that of Theorem 3.4 and hence is omitted.

**Theorem 4.3.** A function $f : X \rightarrow Y$ is fuzzy almost $p^*$-continuous iff for each fuzzy point $x_\alpha$ in $X$ and corresponding to any fuzzy $p^*$-open $q$-nbd $V$ of $f(x_\alpha)$ in $Y$, there exists a fuzzy $p^*$-open $q$-nbd $W$ of $x_\alpha$ in $X$ such that $f(W) \leq V$. 
Proof. The proof is similar to that of Theorem 3.5 and hence is omitted.

Remark 4.4. It is clear from definition that composition of two fuzzy almost $p^*$-continuous functions is fuzzy almost $p^*$-continuous.

Theorem 4.5. If $f : X \rightarrow Y$ is fuzzy almost $p^*$-continuous and $g : Y \rightarrow Z$ is fuzzy almost $p$-continuous, then $g \circ f : X \rightarrow Z$ is fuzzy almost $p$-continuous.

Proof. Obvious.

Remark 4.6. Every fuzzy almost $p^*$-continuous function is fuzzy almost $p$-continuous, but the converse is not true, in general, follows from the following example.

Example 4.7. Let $X = \{a\}$, $\tau = \{0_X, 1_X, A, B\}$, $\tau_1 = \{0_X, 1_X, C\}$ where $A(a) = 0.52$, $B(a) = 0.45$, $C(a) = 0.52$. Then $(X, \tau)$ and $(X, \tau_1)$ are fts’s. Now $FPO(X, \tau) = \{0_X, 1_X, U, V, W\}$ where $1_X \setminus A < U \leq A, V > 1_X \setminus B, W \leq B$ and $FPC(X, \tau) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V, 1_X \setminus W\}$ where $1_X \setminus A \leq 1_X \setminus U < A, 1_X \setminus V < B, 1_X \setminus W \geq 1_X \setminus B$. $FPO(X, \tau_1) = \{0_X, 1_X, T\}$ where $T > 1_X \setminus C$ and $FPC(X, \tau_1) = \{0_X, 1_X, 1_X \setminus T\}$ where $1_X \setminus T < C$. Consider the identity function $i : (X, \tau) \rightarrow (X, \tau_1)$. Other than $0_X$ and $1_X$, $C$ is the only fuzzy open set in $(X, \tau_1)$. Now $i^{-1}(C) = C = int_\tau(pcl_\tau(C)) = int_\tau(1_X \setminus W)$ (where $(1_X \setminus W)(a) \geq 0.55 = A = C \Rightarrow i$ is fuzzy almost $p$-continuous. Let $D$ be a fuzzy set in $(X, \tau)$, defined by $D(a) = 0.53$. Now $pcl_\tau(D) = 1_X$ and so $int_\tau(pcl_\tau(D)) = 1_X > D \Rightarrow D \in FP^*(O(X, \tau_1))$. Then $i^{-1}(D) = D$. But $int_\tau(pcl_\tau(i^{-1}(D))) = int_\tau(pcl_\tau(D)) = int_\tau(1_X \setminus B) = A < D \Rightarrow i$ is not fuzzy almost $p^*$-continuous.

Note 4.8. From Remark 3.12 and Remark 4.6, we say that every fuzzy almost $p^*$-continuous function is fuzzy almost $p$-continuous and hence fuzzy almost continuous. But the converses are not true follow from Example 4.7. Here also $i^{-1}(C) = C = int_\tau(cl_\tau(C)) = int_\tau(1_X \setminus B) = A = C \Rightarrow i$ is fuzzy almost continuous.
To achieve the converse of Remark 4.6, we have to introduce some sort of fuzzy open-like function, as follows.

**Definition 4.9.** A function \( f : X \to Y \) is said to be fuzzy preopen if \( f(U) \) is fuzzy preopen in \( Y \) for every fuzzy preopen set \( U \) in \( X \).

**Lemma 4.10.** If \( f : X \to Y \) is fuzzy preopen function, then \( f^{-1}(pcl(U)) \leq pcl(f^{-1}(U)) \), for any fuzzy set \( U \) in \( Y \).

**Proof.** Let \( x_\alpha \leq pcl(f^{-1}(U)) \) for some fuzzy set \( U \) in \( Y \). Then there exists \( W \in FPO(X) \) such that \( x_\alpha qW \), \( W \subseteq f^{-1}(U) \Rightarrow \) \( f(W) \subseteq gU \). As \( f \) is fuzzy preopen function, \( f(W) \in FPO(Y) \). Now \( x_\alpha qW \Rightarrow f(x_\alpha) qf(W) \Rightarrow f(W) \) is a fuzzy preopen \( q \)-nbd of \( f(x_\alpha) \) in \( Y \), but \( f(W) \notin pclU \Rightarrow x_\alpha \notin pclU \).

**Theorem 4.11.** If \( f : X \to Y \) is fuzzy almost \( p \)-continuous and fuzzy preopen function, then \( f \) is fuzzy almost \( p^* \)-continuous function.

**Proof.** Let \( V \in FPO(Y) \). Then \( V \leq int(pclV) \). Since \( f \) is fuzzy almost \( p \)-continuous, \( f^{-1}(V) \leq f^{-1}(int(pclV)) \leq int(pcl(f^{-1}(int(pclV)))) \) (by Theorem 3.4 (a) \( \iff \) (b)) \( \leq int(pcl(f^{-1}(pclV))) \leq int(pcl(pcl(f^{-1}(V)))) \) (by Lemma 4.10) \( = int(pcl(f^{-1}(V))) \) \( \Rightarrow f^{-1}(V) \in FPO(X) \Rightarrow f \) is fuzzy almost \( p^* \)-continuous.

### 5. Fuzzy \( p^* \)-Regular Space

In this section a new type of fuzzy regularity is introduced and shown that in this space fuzzy closed (resp., fuzzy open) set and fuzzy \( p^* \)-closed (resp., fuzzy \( p^* \)-open) set coincide.

**Definition 5.1.** An fts \( (X, \tau) \) is said to be fuzzy \( p^* \)-regular if for each fuzzy \( p^* \)-closed set \( F \) in \( X \) and each fuzzy point \( x_\alpha \) in \( X \) with \( x_\alpha q(1_X \setminus F) \), there exist a fuzzy open set \( U \) in \( X \) and a
fuzzy $p^*$-open set $V$ in $X$ such that $x_\alpha qU$, $F \leq V$ and $UqV$.

**Theorem 5.2.** For an fts $(X, \tau)$, the following statements are equivalent:

(a) $X$ is fuzzy $p^*$-regular,

(b) for each fuzzy point $x_\alpha$ in $X$ and each fuzzy $p^*$-open set $U$ in $X$ with $x_\alpha qU$, there exists a fuzzy open set $V$ in $X$ such that $x_\alpha qV \leq p^*clV \leq U$,

(c) for each fuzzy $p^*$-closed set $F$ in $X$, $\bigcap\{clV: F \leq V, V \in FP^*O(X)\} = F$,

(d) for each fuzzy set $G$ in $X$ and each fuzzy $p^*$-open set $U$ in $X$ such that $GqU$, there exists a fuzzy open set $V$ in $X$ such that $GqV$ and $p^*clV \leq U$.

**Proof** (a) $\Rightarrow$ (b). Let $x_\alpha$ be a fuzzy point in $X$ and $U$, a fuzzy $p^*$-open set in $X$ with $x_\alpha qU$. By (a), there exist a fuzzy open set $V$ and a fuzzy $p^*$-open set $W$ in $X$ such that $x_\alpha qV$, $1_X \setminus W \leq W \setminus U$, $VqW$. Then $x_\alpha qV \leq 1_X \setminus W \leq U \Rightarrow x_\alpha qV$ and $p^*clV \leq p^*cl(1_X \setminus W) = 1_X \setminus W \leq U \Rightarrow x_\alpha qV \leq p^*clV \leq U$.

(b) $\Rightarrow$ (a). Let $F$ be a fuzzy $p^*$-closed set in $X$ and $x_\alpha$ be a fuzzy point in $X$ with $x_\alpha q(1_X \setminus F)$. Then $1_X \setminus F \in FP^*O(X)$. By (b), there exists a fuzzy open set $V$ in $X$ such that $x_\alpha qV \leq p^*clV \leq 1_X \setminus F$. Put $U = 1_X \setminus p^*clV$. Then $U \in FP^*O(X)$ and $x_\alpha qV$, $F \leq U$ and $UqV$.

(b) $\Rightarrow$ (c). Let $F$ be fuzzy $p^*$-closed set in $X$. It is clear that $F \leq \bigcap\{clV: F \leq V, V \in FP^*O(X)\}$.

Conversely, let $x_\alpha \not\in F$. Then $F(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus F)$ where $1_X \setminus F \in FP^*O(X)$. By (b), there exists a fuzzy open set $U$ in $X$ such that $x_\alpha qU \leq p^*clU \leq 1_X \setminus F$. Put $V = 1_X \setminus p^*clU$. Then
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$F \leq V$ and $UqV \Rightarrow x_{\alpha} \not\subseteq clV \Rightarrow \bigcap\{clV : F \leq V, V \in FP^*O(X)\} \leq F$.

(c) $\Rightarrow$ (b). Let $V$ be any fuzzy $p^*$-open set in $X$ and $x_{\alpha}$ be any fuzzy point in $X$ with $x_{\alpha}qV$. Then $V(x) + \alpha > 1 \Rightarrow x_{\alpha} \not\subseteq (1_x \setminus V)$ where $1_x \setminus V \in FP^*C(X)$. By (c), there exists $G \in FP^*O(X)$ such that $1_x \setminus V \leq G$ and $x_{\alpha} \not\subseteq clG \Rightarrow$ there exists a fuzzy open set $U$ in $X$ with $x_{\alpha}qU$, $UqG \Rightarrow U \setminus 1_x \setminus V \leq G \Rightarrow x_{\alpha}qU \leq p^*clU \leq p^*cl(1_x \setminus G) = 1_x \setminus G \leq V$.

(c) $\Rightarrow$ (d). Let $G$ be any fuzzy set in $X$ and $U$ be any fuzzy $p^*$-open set in $X$ with $GqU$. Then there exists $x \in X$ such that $G(x) + U(x) > 1$. Let $G(x) = \alpha$. Then $x_{\alpha}qU \Rightarrow x_{\alpha} \not\subseteq 1_x \setminus U$ where $1_x \setminus U \in FP^*C(X)$. By (c), there exists $W \in FP^*O(X)$ such that $1_x \setminus U \leq W$ and $x_{\alpha} \not\subseteq clW \Rightarrow (clW)(x) < \alpha \Rightarrow x_{\alpha}q(1_x \setminus clW)$. Let $V = 1_x \setminus clW$. Then $V$ is fuzzy open in $X$ and $V(x) + \alpha > 1 \Rightarrow V(x) + G(x) > 1 \Rightarrow VqG$ and $p^*clV = p^*cl(1_x \setminus clW) \leq p^*cl(1_x \setminus W) = 1_x \setminus W \leq U$.

(d) $\Rightarrow$ (b). Obvious.

**Note 5.3.** It is clear from Theorem 5.2 that in a fuzzy $p^*$-regular space, every fuzzy $p^*$-closed set is fuzzy closed and hence every fuzzy $p^*$-open set is fuzzy open. As a result, in a fuzzy $p^*$-regular space, the collection of all fuzzy closed (resp., fuzzy open) sets and fuzzy $p^*$-closed (resp., fuzzy $p^*$-open) sets coincide.

**Theorem 5.4.** If $f : X \rightarrow Y$ is fuzzy almost $p$-continuous function and $Y$ is fuzzy $p^*$-regular space, then $f$ is fuzzy almost $p^*$-continuous.

**Proof.** Let $x_{\alpha}$ be a fuzzy point in $X$ and $V$ be any fuzzy $p^*$-open $q$-nbd of $f(x_{\alpha})$ in $Y$ where $Y$ is fuzzy $p^*$-regular space. By Theorem 5.2 (a) $\Rightarrow$ (b), there exists a fuzzy open set $W$ in $Y$ such that $f(x_{\alpha})qW \leq p^*clW \leq V$. Since $f$ is fuzzy almost $p$-continuous, by Theorem 3.5, there exists $U \in FP^*O(X)$ with $x_{\alpha}qU$ and $f(U) \leq W \leq V$. By Theorem 4.3, $f$ is fuzzy almost $p^*$-continuous.
-continuous function.

Let us now recall following definitions from [3,4] for ready references.

**Definition 5.5** [3]. A collection $\mathcal{U}$ of fuzzy sets in an fts $X$ is said to be a fuzzy cover of $X$ if $\bigcup \mathcal{U} = 1_X$. If, in addition, every member of $\mathcal{U}$ is fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of $X$.

**Definition 5.6** [3]. A fuzzy cover $\mathcal{U}$ of an fts $X$ is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = 1_X$.

**Definition 5.7** [4]. An fts $(X, \tau)$ is said to be fuzzy almost compact if every fuzzy open cover $\mathcal{U}$ of $X$ has a finite proximate subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{clU : U \in \mathcal{U}_0\}$ is again a fuzzy cover of $X$.

**Theorem 5.8.** If $f : X \rightarrow Y$ is a fuzzy almost $p$-continuous surjective function and $X$ is fuzzy $p^*$-regular and fuzzy almost compact space, then $Y$ is fuzzy almost compact space.

**Proof.** Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of $Y$. Then as $f$ is fuzzy almost $p$-continuous, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy $p^*$-open cover and hence fuzzy open cover of $X$ as $X$ is fuzzy $p^*$-regular space. Since $X$ is fuzzy almost compact, there are finitely many members $U_1, U_2, \ldots, U_n$ of $\mathcal{U}$ such that $\bigcup_{i=1}^n cl(f^{-1}(U_i)) = 1_X$. Since $X$ is fuzzy $p^*$-regular space, by Theorem 5.2, $clA = p^*clA$ and so $1_X = \bigcup_{i=1}^n p^*cl(f^{-1}(U_i)) \Rightarrow 1_Y = f(\bigcup_{i=1}^n p^*cl(f^{-1}(U_i))) = \bigcup_{i=1}^n f(p^*cl(f^{-1}(U_i))) \leq \bigcup_{i=1}^n cl(f(f^{-1}(U_i)))$ (by Theorem 3.4 (a) $\Rightarrow$ (f)) $\leq \bigcup_{i=1}^n cl(U_i) \Rightarrow \bigcup_{i=1}^n cl(U_i) = 1_Y \Rightarrow Y$ is fuzzy almost compact space.
References


